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Math 4567. Homework Set # VI

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Chapter 5, page 113, problem 1), (page 122, problem 1), (page 128, problem 2), (page 133, problem 4), (page 136, problem 1). (page 146, problem 1), Chapter 8 (page 209, problem 1)

Chapter 5 page 113, Problem 1 The initial temperature of a slab $0 \leq x \leq \pi$ is everywhere 0 and the face $x = 0$ is kept at that temperature. Heat is supplied through the face $x = \pi$ at a constant rate $ku_x(\pi, t) = A > 0$. Write $u(x,t) = U(x,t) + \Phi(x)$ and use the solution to the problem

$$(*)\ \ U_t = kU_{xx}, \quad 0 < x < \pi, t > 0,$$
$$U(0,t) = 0,\ U_x(\pi, t) = 0, \quad 0 < x < \pi,$$

and $U(x, 0) = F(x)$ where

$$F(x) = \begin{cases} f(x) & \text{when } 0 < x < \pi \\ f(2\pi - x) & \text{when } \pi < x < 2\pi, \end{cases}$$

which is

$$U(x,t) = \sum_{n=1}^{\infty} B_n \exp(-\frac{n^2 k}{4} t) \sin \frac{nx}{2},$$
$$B_n = \frac{1 - (-1)^n}{\pi} \int_0^{\pi} f(x) \sin \frac{nx}{2}\, dx.$$

to derive the final solution $u(x,t)$

Solution: We first find a function $u = \Phi(x)$ that satisfies the nonhomogeneous condition $ku_x(\pi, t) = A$ and the homogeneous condition $u(0,t) = 0$. The differential equation is $\Phi''(x) = 0$, so $\Phi(x) = Bx + C$. The nonhomogeneous boundary condition says $KB = A$ and the homogeneous condition says $C = 0$ thus $\Phi(x) = \frac{A}{k} x$. Then, setting

$u(x,t) = U(x,t) + \Phi(x)$ we see that $u(x,t)$ will be a solution of our original problem, provided $U(x,t)$ satisfies problem $(*)$ where $U(x,0) = f(x) = -\Phi(x) = -\frac{A}{k}x$ for $0 < x < \pi$. Thus

$$u(x,t) = \frac{A}{k}x + \sum_{n=1}^{\infty} B_n \exp(-\frac{n^2 k}{4}t) \sin \frac{nx}{2},$$

where

$$B_n = -\frac{A}{k}(\frac{1-(-1)^n}{\pi}) \int_0^{\pi} x \sin \frac{nx}{2}\ dx.$$

Note that $B_n = 0$ unless $n = 2m - 1$ is odd. Since

$$\int_0^{\pi} x \sin \frac{nx}{2}\ dx = \frac{2}{n}\left\{-x \cos \frac{nx}{2}|_0^{\pi} + \int_0^{\pi} \cos \frac{nx}{2}\ dx\right\} = \frac{2}{n}[-\pi \cos \frac{n\pi}{2} + \frac{2}{n} \sin \frac{n\pi}{2}]$$

$$= (\frac{2}{2m-1})^2 (-1)^{m+1},$$

we get the solution

$$u(x,t) = \frac{A}{k}\left\{x + \frac{8}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^m}{(2m-1)^2} \exp\left[-\frac{(2m-1)^2 k}{4}t\right] \sin \frac{(2m-1)x}{2}\right\}.$$

Chapter 5, page 122, Problem 1 The faces and edges $x = 0$ and $x = \pi$, $(0 < y < \pi)$ of a square plate $0 \le x \le \pi,\ 0 \le y \le \pi$ are insulated. The edges $y = 0$ and $y = \pi$, $(0 < x < \pi)$ are kept at temperatures 0 and $f(x)$, respectively. Let $u(x,y)$ be the steady state temperature distribution in the plate. Show that

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh ny \cos nx,$$

$$A_0 = \frac{1}{\pi^2}\int_0^{\pi} f(x)dx,\ A_n = \frac{2}{\pi \sinh n\pi}\int_0^{\pi} f(x) \cos nx\ dx, \quad n = 1, 2, \cdots.$$

Find $u(x,y)$ if $f(x) = u_0$.

Solution: The problem is

$$u_{xx} + u_{yy} = 0, \quad 0 < x < \pi,\ 0 < y < \pi,$$

$$u_x(0,y)=0,\ u_x(\pi,y)=0,\quad 0<y<\pi,$$
$$u(x,0)=0,\quad 0<x<\pi$$
$$u(x,\pi)=f(x),\quad 0<x<\pi.$$

We use separation of variables $u = X(x)Y(y)$ to find solutions satisfying the homogeneous conditions. The Sturm-Liouville eigenvalue problem is

$$X''(x)+\lambda X(x)=0,\ X'(0)=X'(\pi)=0.$$

From past work we know that the eigenvalues are $\lambda_n = n^2$, $n = 1, 2, \cdots$ with eigenfunctions $X_n(x) = \cos nx$, and $\lambda_0 = 0$ with eigenfunction $X_0(x) = 1$. The corresponding equations for $Y(y)$ are

$$Y''(y)-\lambda Y(y)=0,\ Y(0)=0.$$

As has been shown earlier, for $\lambda_n = n^2$ we have $Y_n(y) = \sinh ny$ and for $\lambda_0 = 0$ we have $Y_0(y) = y$. Thus we can write

$$u(x,y)=A_0 y+\sum_{n=1}^{\infty} A_n \cos nx \sinh ny,$$

where

$$u(x,\pi)=f(x)=A_0\pi+\sum_{n=1}^{\infty} A_n \cos nx \sinh n\pi.$$

Thus

$$A_n \sinh n\pi = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\ dx,\quad n=1,2,\cdots,$$
$$A_0\pi=\frac{1}{\pi}\int_0^{\pi} f(x)dx.$$

If $f(x) = u_0$ then $A_n = 0$ $A_0\pi = u_0$, so the solution is $u(x, y) = \frac{u_0}{\pi}y$.

Chapter 5, page 128, Problem 2 Let the faces of a wedge shaped plate $0 \le \rho \le a$, $0 \le \phi \le \alpha$ be insulated Find the steady temperature $u(\rho, \phi)$ in the plate when $u = 0$ on the rays $\phi = 0$, $\phi = \alpha$ $(0 < \rho < \alpha)$ and $u = f(\phi)$ on the arc $\rho = a$ $(0 < \phi < \alpha)$. Assume f is piecewise smooth and u is bounded.

Solution: Our problem in polar coordinates is to find a function $u(\rho, \phi)$ for

$$\rho^2 u_{\rho\rho} + \rho u_\rho + u_{\phi\phi} = 0, \quad 0 < \rho < a, \ 0 < \phi < \alpha,$$

$$u(\rho, 0) = u(\rho, \alpha) = 0, \quad 0 < \rho < \alpha,$$

$$u(a, \phi) = f(\phi), \quad 0 < \phi < \alpha,$$

where f is piecewise smooth and $|u| < M$, i.e., u is bounded.

Separating variables, $u = R(\rho)\Phi(\phi)$ satisfies the homogeneous conditions if Φ satisfies the Sturm-Liouville problem

$$\Phi'' + \lambda\Phi = 0, \ \Phi(0) = \Phi(\alpha) = 0,$$

and R satisfies

$$\rho^2 R'' + \rho R' - \lambda R = 0$$

and R is bounded. It is straightforward to show that the eigenvalues are $\lambda_n = \frac{n^2\pi^2}{\alpha^2}$ with eigenfunctions $\Phi_n(\phi) = \sin\frac{n\pi\phi}{\alpha}$, $n = 1, 2, \cdots$.. The change of variable $\rho = e^s$ gives the corresponding equation for R as $R_{ss} - \lambda_n R = 0$. The general solutions are

$$R_n(\rho) = A\rho^{n\pi/\alpha} + B\rho^{-n\pi/\alpha},$$

and the boundedness requirement yields $R_n(\rho) = \rho^{n\pi/\alpha}$. Thus

$$u(\rho, \phi) = \sum_{n=1}^{\infty} B_n \rho^{n\pi/\alpha} \sin\frac{n\pi\phi}{\alpha},$$

and

$$u(a, \phi) = f(\phi) = \sum_{n=1}^{\infty} B_n a^{n\pi/\alpha} \sin\frac{n\pi\phi}{\alpha},$$

where

$$B_n a^{n\pi/\alpha} = \frac{2}{\alpha}\int_0^\alpha f(\psi)\sin\frac{n\pi\psi}{\alpha}\, d\psi.$$

Thus

$$u(\rho, \phi) = \frac{2}{\alpha}\sum_{n=1}^{\infty}(\frac{\rho}{a})^{n\pi/\alpha}\sin\frac{n\pi\phi}{\alpha}\int_0^\alpha f(\psi)\sin\frac{n\pi\psi}{\alpha}\, d\psi.$$

Chapter 5, page 133, Problem 4 A string is stretched between points 0 and π on x-axis and, initially at rest, is released from the position $y = f(x)$. The equation of motion is

$$y_{tt} = y_{xx} - 2\beta y_t, \quad 0 < x < \pi,\ t > 0,$$

where $0 < \beta < 1$ and β is constant. Show that

$$y(x,t) = e^{-\beta t} \sum_{n=1}^{\infty} B_n \left(\cos \alpha_n t + \frac{\beta}{\alpha_n} \sin \alpha_n t \right) \sin nx,$$

$$\alpha_n = \sqrt{n^2 - \beta^2},\ B_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\ dx, \quad n = 1, 2, \cdots.$$

Solution: Set $z(x,t) = e^{\beta t} y(x,t)$. Then $z(x,t)$ satisfies

$$z_{tt} = z_{xx} + \beta^2 z, \quad 0 < \pi,\ t > 0,$$

$$z(0,t) = z(\pi,t) = 0, \quad t > 0$$

$$z_t(x,0) = \beta f(x),\ z(x,0) = f(x), \quad 0 < x < \pi, .$$

where $f(0) = f(\pi) = 0$. We look for a solution of the form

$$z(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin nx,\ A_n(t) = \frac{2}{\pi} \int_0^{\pi} z(x,t) \sin nx\ dx.$$

Then

$$A_n''(t) = \frac{2}{\pi} \int_0^{\pi} z_{tt}(x,t) \sin nx\ dx = \frac{2}{\pi} \int_0^{\pi} (z_{xx}(x,t) + \beta^2 z(x,t)) \sin nx\ dx$$

$$= \frac{2}{\pi} \int_0^{\pi} z_{xx}(x,t) \sin nx\ dx + \frac{2\beta^2}{\pi} \int_0^{\pi} z(x,t) \sin nx\ dx.$$

$$= \frac{2}{\pi} [z_x(x,t) \sin nx|_0^{\pi} - n \int_0^{\pi} z_x(x,t) \cos nx\ dx] + \beta^2 A_n(t)$$

$$= \frac{2}{\pi} [-nz(x.t) \cos nx|_0^{\pi} - n^2 \int_0^{\pi} z(x,t) \sin nx\ dx] + \beta^2 A_n(t)$$

$$= (-n^2 + \beta^2) A_n(t).$$

Thus

$$A_n''(t) + (n^2 - \beta^2)A_n(t) = 0,$$

so

$$A_n(t) = B_n \cos \alpha_n t + C_n \sin \alpha_n t$$

where $\alpha_n = \sqrt{n^2 - \beta^2}$, $n = 1, 2, \cdots$. The condition

$$z(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin nx$$

gives

$$B_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \; dx, \quad n = 1, 2, \cdots.$$

The condition

$$z_t(x,0) = \beta f(x) = \sum_{n=1}^{\infty} C_n \alpha_n \sin x$$

gives $C_n = \beta B_n / \alpha_n$. (Here we are assuming that it is permissible to differentiate the sum term-by-term. This assumption could be avoided by taking $z_t(x,t) = \sum_{n=1}^{\infty} E(t) \sin nx$ and obtaining $E_n(t)$ by integration by parts, just as we did for $A_n(t)$.) Thus we obtain the formal solution

$$y(x,t) = e^{-\beta t} z(x,t) = e^{-\beta t} \sum_{n=1}^{\infty} B_n \left(\cos \alpha_n t + \frac{\beta}{\alpha_n} \sin \alpha_n t \right) \sin nx,$$

Chapter 5, page 136, Problem 1 Solve the problem

$$y_{tt} = a^2 y_{xx} + Ax \sin \omega t, \quad 0 < x < c, \; t > 0,$$

$$y(0,t) = y(c,t) = 0, \; y(x,0) = y_t(x,0) = 0.$$

Show that resonance occurs for $\omega = \omega_n$, where

$$\omega_n = \frac{n\pi a}{c}, \quad n = 1, 2, \cdots.$$

Solution: We look for a solution in the form

$$y(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin \frac{n\pi x}{c},$$

$$B_n(t) = \frac{2}{c}\int_0^c y(x,t)\sin\frac{n\pi x}{c}\ dx.$$

Then

$$B_n''(t) = \frac{2}{c}\int_0^c y_{tt}(x,t)\sin\frac{n\pi x}{c}\ dx = \frac{2}{c}\int_0^c [a^2 y_{xx}(x,t)+Ax\sin\omega t]\sin\frac{n\pi x}{c}\ dx$$

$$= \frac{2A(-1)^{n+1}}{n\pi}\sin\omega t - \frac{a^2n^2\pi^2}{c^2}B_n(t),$$

Where we have integrated by parts several times and applied the boundary conditions. Thus

$$(*)\ B_n''(t) + \frac{a^2n^2\pi^2}{c^2}B_n(t) = \frac{2A(-1)^{n+1}}{n\pi}\sin\omega t.$$

This is a nonhomogeneous equation. We need only find one solution and then add to it the general solution $H_n\cos\frac{an\pi x}{c} + K_n\sin\frac{an\pi x}{c}$ of the homogeneous equation to get the general solution. We look for a solution of the form $B_n(t) = C_n\sin\omega t$. Substituting this into equation $(*)$ and setting $\omega_n = an\pi/c$ we find a solution if $C_n = 2A(-1)^{n+1}/(\omega_n^2 - \omega^2)$. Thus

$$B_n(t) = \frac{2A(-1)^{n+1}}{\omega_n^2-\omega^2}\sin\omega t,$$

and the general solution is

$$B_n(t) = A_n\cos\frac{an\pi t}{c} + C_n\sin\frac{an\pi t}{c} + \frac{2A(-1)^{n+1}}{\omega_n^2-\omega^2}\sin\omega t.$$

The boundary conditions are $B_n(0) = B_n'(0) = 0$, and these are satisfied for $A_n = 0$ and $C_n = 2(-1)^n\omega/[\omega_n(\omega_n^2-\omega^2)]$. Thus the final solution is

$$B_n(t) = \frac{2(-1)^n\omega}{\omega_n(\omega_n^2-\omega^2)}\sin\frac{an\pi t}{c} + \frac{2A(-1)^{n+1}}{\omega_n^2-\omega^2}\sin\omega t,$$

unless $\omega = \omega_n$ for some n. In that case we have resonance and the solution becomes unbounded.

To see this, we look for a particular solution of $(*)$ in the case $\omega = \omega_n$. Take the trial solution $B_n(t) = D_n t\cos\omega_n t$. Then we find a solution provided $D_n = (-1)^n A/n\pi\omega_n$:

$$(\dagger)\qquad B_n(t) = \frac{(-1)^nA}{n\pi\omega_n}t\cos\omega_n t.$$

To this solution we can add a general solution of the homogeneous equation, but the resonant solution quickly dominates the bounded solution of the wave equation as t gets large.

Chapter 5, page 146, Problem 1 Write $\lambda = -\alpha^2$, $\alpha > 0$ and show that the Sturm-Liouville problem

$$X'' + \lambda X = 0,\ X(-\pi) = X(\pi),\ X'(-\pi) = X'(\pi),$$

has no solutions.

Solution: The general solution of the differential equation is

$$X(x) = Ae^{\alpha x} + Be^{-\alpha x},$$

so

$$X'(x) = \alpha(Ae^{\alpha x} - Be^{-\alpha x}).$$

The conditions $X(-\pi) = X(\pi),\ X'(-\pi) = X'(\pi)$ can be written as

$$A \sinh \alpha\pi = B \sinh \alpha\pi,\ A \sinh \alpha\pi = -B \sinh \alpha\pi,$$

respectively. Since $\sinh \alpha\pi \neq 0$ for $\alpha \neq 0$, we have $A = -B = B$, so $A = B = 0$. Thus there are no negative eigenvalues.

Chapter 8, page 209, Problem 1 Find the eigenvalues and eigenfunctions:

$$X'' + \lambda X = 0,\ X(0) = 0,\ X'(1) = 0.$$

Solution: If $\lambda = 0$ then $X(x) = Ax + B$ and $X'(x) = A$. Thus the boundary conditions are $B = 0$, $A = 0$ and $\lambda = 0$ is not an eigenvalue.

If $\lambda = -\alpha^2$, $\alpha > 0$ then $X(x) = Ae^{\alpha x} + Be^{-\alpha x}$, $X'(x) = \alpha(Ae^{\alpha x} - Be^{-\alpha x})$. Thus the boundary conditions are $A + B = 0$ and $Ae^{\alpha} - e^{-\alpha}$, or $B = -A$ where $A \cosh \alpha = 0$. Since $\cosh \alpha \neq 0$ we have $A = B = 0$ and $\lambda = -\alpha^2$ is not an eigenvalue.

If $\lambda = \alpha^2$, $\alpha > 0$ then $X(x) = A \cos \alpha x + B \sin \alpha x$, $X'(x) = \alpha(-A \sin \alpha x + B \cos \alpha x)$, and the boundary conditions can be read as

$$A = 0, \quad \alpha(B \cos \alpha) = 0,$$

or $\cos\alpha = 0$, so $\lambda_n = \alpha_n^2$ where

$$\alpha_n = (2n-1)\frac{\pi}{2}, \;\; X_n(x) = \sin\alpha_n x \quad n = 1, 2, \cdots.$$

Since $\int_0^1 X_n^2(x)dx = \frac{1}{2}\int_0^1 (1 - \cos\pi(2n-1)x)dx = \frac{1}{2}$ the normalized eigenfunctions are $\phi_n(x) = \sqrt{2}\sin\alpha_n x$.