Name: ______________________

Math 4567. Final Exam (take home)

Due by May 12, 2010

There are a total of 180 points and 8 problems on this take home exam.

Problem	Score
1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
Total:	

1. (15 points) Chapter 6, page 168, Problem 8

   A semi-infinite string, with one end fixed at the origin, is stretched along the positive x-axis and released at rest from a position $y = f(x)$, $x \geq 0$. Derive the expression

   $$y(x,t) = \frac{2}{\pi}\int_o^\infty \cos(\alpha\ at)\sin\alpha x \int_0^\infty f(s)\sin\alpha s\ ds\ d\alpha.$$

   If $F(x)$, $-\infty < x < \infty$, is the odd extension of $f(x)$, show that this result reduces to the form

   $$y(x,t) = \frac{1}{2}[F(x+at) + F(x-at)].$$

2. (15 points) Chapter 6, page 168, Problem 10

   Find the bounded harmonic function $u(x,y)$ in the horizontal semi-infinite strip $x > 0$, $0 < y < 1$, that satisfies the conditions

   $$u_x(0,y) = 0, \quad u_y(x,1) = -u(x,1), \quad u(x,0) = f(x).$$

   where

   $$f(x) = \begin{cases} 1 & \text{when} \quad 0 < x < 1, \\ 0 & \text{when} \quad x > 1. \end{cases}$$

   Interpret this problem physically, in terms of heat conduction.

   Show that the answer is:

   $$u(x,y) = \frac{2}{\pi}\int_0^\infty \frac{\alpha\cosh\alpha(1-y) + \sinh\alpha(1-y)}{\alpha^2\cosh\alpha + \alpha\sinh\alpha}\sin\alpha\cos\alpha x\ d\alpha.$$

3. (15 points) Verify directly that for $t > 0$ and fixed s the function

   $$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\exp\left[-\frac{(x-s)^2}{4kt}\right]$$

   satisfies the heat equation $u_t - ku_{xx} = 0$. For $s = 0$ this is known as the fundamental solution of the heat equation.

4. (20 points) Find the eigenvalues and normalized eigenfunctions of the Sturm-Liouville system

   $$-x^2(x^2y')' = \lambda y,\ y'(1) = 0,\ y'(2) = 0, \quad 1 \leq x \leq 2.$$

   What are the orthogonality relations for the eigenfunctions?

5. **a.** (15 points) Determine a formal eigenfunction series expansion for the solution $y(x)$ of

$$-y'' - \mu y = f(x), \ y(0) = 0, \ y'(1) = 0, \quad 0 \le x \le 1,$$

where f is a given continuous function on $[0, 1]$.

b. (10 points) What happens if the parameter μ is an eigenvalue?

6. Laplace's equation in polar coordinates is

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0.$$

a. (10 points) Use separation of variables to find the solution $u(r, \theta)$ of this equation **outside** the circle $r = a$ and satisfying the boundary condition

$$u(a, \theta) = f(\theta)$$

on the circle. Require that $u(r, \theta)$ is bounded and continuous for $r \ge a$. To make u single-valued, require that $u(r, \theta) = u(r, \theta + 2\pi)$. Here, $f(\theta)$ is a continuous function with piecewise continuous derivative such that $f(0) = f(2\pi)$.

b. (5 points) Show that formally the solution is

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^{-n} \left(a_n \cos n\theta + b_n \sin n\theta\right), \tag{1}$$

and compute the coefficients a_n, b_n.

c. (5 points) Show that your formal solution is an actual solution of Laplace's equation satisfying the boundary conditions.

d. (15 points) By interchanging the order of summation and integration in (1), derive the Poisson integral formula for the solution:

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\psi) \frac{1 - \rho^2}{[1 + \rho^2 - 2\rho \cos(\theta - \psi)]} d\psi,$$

where $\rho = a/r < 1$.

7. In the next two problems we use the following definition of the complex Fourier integral transform and its inversion:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\lambda)e^{i\lambda x}d\lambda, \quad \hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx.$$

Fourier transforms on $(-\infty, \infty)$ and Fourier series have interesting relations between them. Here is one. The periodization of a function f on $(-\infty, \infty)$ is defined as

$$P[f](x) = \sum_{m=-\infty}^{\infty} f(x + 2\pi m).$$

This is a way to produce a 2π-periodic function from a general function with no periodicity. However, for many functions $f(x)$ this infinite sum will not converge. To guarantee convergence of the infinite sum we restrict ourselves to functions that decay rapidly at infinity. (An example of such a function is $f(x) = e^{-x^2}$.)

a. (10 points) Show that if f and f' are continuous on $(-\infty, \infty)$ and $|f(x)| \leq C_1 e^{-C_2|x|}$ for some postive constants $C_1,\ C_2$ and all x then its periodization is well defined and has period 2π. (You can assume the true fact that $P[f](x)$ is continuous and continuously differentiable.)

b. (10 points) Expand $P[f](x)$ into a complex Fourier series

$$P[f](x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

and show that the Fourier coefficients

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} P[f](t)e^{-int}dt$$

are given by

$$c_n = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-int}dt = \frac{1}{2\pi}\hat{f}(n)$$

where $\hat{f}(\lambda)$ is the complex Fourier transform of $f(x)$.

c. (5 points) Conclude that

$$\sum_{n=-\infty}^{\infty} f(x+2\pi n) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}, \tag{2}$$

so $P[f](x)$ tells us the value of $\hat{f}$ at the integer points $\lambda = n$, but not in general at the non-integer points. (For $x = 0$, equation (2) is known as the *Poisson summation formula.*)

d. (5 points) Apply the Poission summation formula to the function $f(x) = \exp(-sx^2)$ for $s > 0$. The Fourier transform of this function is $\hat{f}(\lambda) = \sqrt{\pi/s}\exp(-\lambda^2/4s)$. Derive the famous relation

$$\sum_{n=-\infty}^{\infty} \exp(-4s\pi^2 n^2) = \sqrt{\frac{1}{4\pi s}} \sum_{n=-\infty}^{\infty} \exp(-\frac{n^2}{4s}).$$

8. Let $f(x) = \frac{a}{x^2+a^2}$ for $a > 0$.

a. (10 points) Show that $\hat{f}(\lambda) = \pi e^{-a|\lambda|}$. Hint: It is easier to work backwards.

b. (5 points) Use the Poisson summation formula to derive the identity

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a} \frac{1+e^{-2\pi a}}{1-e^{-2\pi a}}.$$

c. (10 points) What happens as $a \to 0+$? (Look at the $n = 0$ term on the left hand side.) Can you obtain the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ from this?