Name: ______________________

Math 4567. Midterm Exam III (take home)

Due April 23, 2010

There are a total of 100 points and 6 problems on this take home exam.

Problem	Score
1.	______
2.	______
3.	______
4.	______
5.	______
6.	______
Total:	______

1. **Chapter 5, page 113, Problem 2. (20 points).** A solid spherical body 40 cm in diameter, initially at 100^o C throughout, is cooled by keeping its surface at 0^o C. Use the temperature formula derived in class and in the text,

$$u(r,t) = \frac{1}{r}\sum_{n=1}^{\infty} B_n \exp(-\frac{n^2\pi^2 k}{a^2}t)\sin\frac{n\pi r}{a}, \; B_n = \frac{2}{a}\int_0^a r f(r)\sin\frac{n\pi r}{a}\,dr,$$

to show formally that

$$u(0+,t) = 200\sum_{n=1}^{\infty}(-1)^{n+1}\exp(-\frac{n^2\pi^2 k}{400}t).$$

Find the approximate temperature at the center of the sphere 10 min after cooling begins if (a) $k = 0.15$ cgs unit ; and (b) $k = 0.005$ cgs unit. Make sure that your answer is accurate to within 1/10th of a degree Celsius and justify your reasoning.

2. **Chapter 5, page 117, Problem 1. (20 points)** Solve the boundary value problem

$$u_t = u_{xx} + xp(t), \quad 0 < x < 1, \; t > 0,$$

$$u(0,t) = 0, \; u(1,t) = 0, \; u(x,0) = 0.$$

where $p(t)$ is a continuous function for all $t \geq 0$ and nonzero only in the bounded interval $0 \leq t < M$. Verify that the solution is

$$u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi x \int_0^t e^{-n^2\pi^2(t-\tau)} p(\tau) d\tau.$$

Why does this series converge?

3. **Chapter 8, page 215, problem 6. (10 points)** Use the normalized eigenfunctions in Problem 2, page 209 to derive

$$x\left(\frac{2+h}{1+h}-x\right)=4h\sum_{n=1}^{\infty}\frac{1-\cos\alpha_n}{\alpha_n^3(h+\cos^2\alpha_n)}\sin\alpha_n x,\quad 0<x<1,$$

where $\tan\alpha_n=-\alpha_n/h$, $\alpha>0$.

4. **Chapter 8, page 215, problem 7.** (10 points) Use the normalized eigenfunctions in Problem 1, page 209 to derive

$$\sin \omega x = 2\omega \cos \omega \sum_{n=1}^{\infty} \frac{(-1)^n}{\omega^2 - \omega_n^2} \sin \omega_n x, \quad 0 < x < 1,$$

where

$$\omega_n = \frac{(2n-1)\pi}{2}, \text{ and } \omega \neq \omega_n, \text{ for any n.}$$

5. **Chapter 6, page 157, Problem 3. (20 points)**

(a) Show that the function

$$f(x) = \begin{cases} 0 & \text{when } x < 0, \\ \exp(-x) & \text{when } x > 0, \\ \frac{1}{2} & \text{when } x = 0, \end{cases}$$

satisfies the conditions of the Fourier integral pointwise convergence theorem. Establish

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{\cos \alpha x + \alpha \sin \alpha x}{1 + \alpha^2} \, d\alpha, \quad -\infty < x < \infty.$$

(b) Verify this directly at the point $x = 0$.

6. **(20 points)** Use the real form of the Fourier transform pair for the real-valued function $f(x)$,

$$f(x) = \int_0^\infty [A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x]\ d\alpha,$$

where

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\alpha t\ dt, \quad B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\alpha t\ dt, \tag{1}$$

with Parseval formula

$$\frac{1}{\pi}\int_{-\infty}^{\infty} f^2(x)\ dx = \int_0^\infty \Big(A^2(\alpha) + B^2(\alpha)\Big)\ d\alpha, \tag{2}$$

to derive the complex form of the transform pair for $f(x)$:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\lambda)e^{i\lambda x}d\lambda,$$

where

$$\hat{f}(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx, \tag{3}$$

with Parseval formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f^2(x)\ dx = \int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2\ d\lambda. \tag{4}$$

The similar computation relating real and complex forms of the Fourier series in problem 8, Chapter 2, page 42 should prove helpful.

How would the formulas change if $f(x)$ was a complex valued function?