

SYMMETRY TECHNIQUES FOR q -SERIES: ASKEY-WILSON POLYNOMIALS

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ABSTRACT. We advocate the exploitation of symmetry (recurrence relation) techniques for the derivation of properties associated with families of basic hypergeometric functions, in analogy with the local Lie theory techniques for ordinary hypergeometric functions. Here these ideas are applied to the (continuous) Askey-Wilson polynomials, introduced by Askey and Wilson, to obtain a strikingly simple derivation of their orthogonality relations.

1. Introduction. In [1] the authors and A.K. Agarwal introduced symmetry techniques for the study of families of basic hypergeometric functions, in analogy with the local Lie theory techniques for ordinary hypergeometric functions. The fundamental objects in this study are the recurrence relations obeyed by the families: generating functions and identities for each family are characterized in terms of the recurrence relations. The functions studied in [1] are of the form

$$r\phi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k z^k}{(b_1; q)_k \cdots (b_s; q)_k (q; q)_k}$$

and many-variable extension. Here

$$(a; q)_k = \begin{cases} 1 & \text{if } k = 0 \\ (1-a)(1-qa) \cdots (1-q^{k-1}a) & \text{if } k = 1, 2, \dots \end{cases}$$

and $|q| < 1$. In this paper we study the Askey-Wilson polynomials

$$(1.1) \quad \phi_n^{(a,b,c,d)}(x) = 4\phi_3 \left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, a/z \\ ab, ac, ad \end{matrix}; q, q \right)$$

where $n = 0, 1, 2, \dots$, a, b, c, d, q real and $(|a|, |b|, |c|, |d|) < 1$. Initially we also require $abcd \neq 0$. These functions are orthogonal polynomials of order n in $x = (z + z^{-1})/2$ where frequently $z = e^{i\theta}$, $0 \leq \theta \leq \pi$.

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They were introduced by Askey and Wilson in their impressive Memoir [2]. Askey and Wilson computed the orthogonality relation and demonstrated that these polynomials are natural generalizations of the classical orthogonal polynomials, indeed the most extensive generalization known. However, the proof of the orthogonality relations in [2] is very complicated. Simplifications in the proof have since been made by Askey [3], by Rahman [4] and by Ismail, Stanton and Viennot [5, 6]. However, it is fair to say that these orthogonality proofs all rest on the explicit summation of certain special q -series, are verifications of the stated result, and are not readily comprehensible by nonexperts in the field. The approach to these polynomials through their recurrence relations, however, leads to a transparent, elementary orthogonality proof which is a derivation rather than a verification. Furthermore, an important q -series identity, Sears transformation for a balanced $4\phi_3$ [2, 7], arises as a by-product of the proof.

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2. The results. Two fundamental recurrence relations for the Askey-Wilson polynomials are

$$(2.1A) \quad \tau^{(a,b,c,d)} \phi_n^{(a,b,c,d)} = \frac{aq^{1/2}(1-q^{-n})(1-q^{n-1}abcd)}{(1-ab)(1-ac)(1-ad)} \phi_{n-1}^{(aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2})}$$

$$(2.1B) \quad \mu^{(a,b,c,d)} \phi_n^{(a,b,c,d)} = \left(1 - \frac{ab}{q}\right) \phi_n^{(aq^{-1/2}, bq^{-1/2}, cq^{1/2}, dq^{1/2})},$$

where

$$(2.2) \quad \tau^{(a,b,c,d)} = (z - z^{-1})^{-1} (E_z^{1/2} - E_z^{-1/2})$$

and

$$\begin{aligned} \mu^{(a,b,c,d)} = (z - z^{-1})^{-1} & \left(-z^{-1} \left(1 - \frac{az}{q^{1/2}}\right) \left(1 - \frac{bz}{q^{1/2}}\right) E_z^{1/2} \right. \\ & \left. + z \left(1 - \frac{a}{q^{1/2}z}\right) \left(1 - \frac{b}{q^{1/2}z}\right) E_z^{-1/2} \right). \end{aligned}$$

Here $E_z^\alpha f(z) = f(q^\alpha z)$. Relation (2.1A) follows from

$$\tau(az; q)_k \left(\frac{a}{z}; q\right)_k = \frac{a}{q^{1/2}} (1 - q^k) (aq^{1/2}z; q)_{k-1} \left(\frac{aq^{1/2}}{z}; q\right)_{k-1}$$

and (2.1B) follows from

$$\mu(az; q)_k \left(\frac{a}{z}; q\right)_k = (1 - abq^{k-1}) \left(\frac{az}{q^{1/2}}; q\right)_k \left(\frac{a}{q^{1/2}z}; q\right)_k.$$

The first relation was pointed out by Askey and Wilson [2]; we have not found (2.1B) in the literature.

Consider relation (2.1B); it suggests that there should be an operator μ^* mapping $\phi_n^{(aq^{-1/2}, bq^{-1/2}, cq^{1/2}, dq^{1/2})}$ to $\phi_n^{(a,b,c,d)}$. Indeed we find

$$(2.3) \quad \begin{aligned} & \mu^{(cq^{1/2}, dq^{1/2}, aq^{-1/2}, bq^{-1/2})} \phi_n^{(aq^{-1/2}, bq^{-1/2}, cq^{1/2}, dq^{1/2})} \\ &= \frac{q^{-n} (1 - q^n cd) \left(1 - \frac{ab}{q} q^n\right)}{\left(1 - \frac{ab}{q}\right)} \phi_n^{(a,b,c,d)}. \end{aligned}$$

This follows from

$$\begin{aligned} & \mu^{(cq^{1/2}, dq^{1/2}, aq^{-1/2}, bq^{-1/2})} \left(\frac{az}{q^{1/2}}; q\right)_k \left(\frac{a}{zq^{1/2}}; q\right)_k \\ &= (q^{-k} - cd) (az; q)_k \left(\frac{a}{z}; q\right)_k \\ &+ (1 - q^{-k}) (1 - acq^{k-1}) \cdot (1 - adq^{k-1}) (az; q)_{k-1} \left(\frac{a}{z}; q\right)_{k-1}. \end{aligned}$$

QUESTION. Can we introduce a pre-Hilbert space structure such that $\mu^* \equiv \mu^{(cq^{1/2}, dq^{1/2}, aq^{-1/2}, bq^{-1/2})}$ is the adjoint operator to $\mu \equiv \mu^{(a,b,c,d)}$? (Here μ^* and μ act like raising and lowering operators. We would like $\mu^* \mu$ to be self-adjoint.)

EDUCATED GUESS. Let $w_{a,b,c,d}(z) = w_{a,b,c,d}(z^{-1})$ be analytic as a function of the complex variable z in a neighborhood of the unit circle $|z| = 1$ and such that $w_{a,b,c,d}(e^{i\theta}) > 0$. Define an inner product

$$(2.4) \quad (g_1, g_2)_{a,b,c,d} = \frac{1}{2\pi i} \oint_C g_1(x) g_2(x) w_{a,b,c,d}(z) \frac{dz}{z},$$

where C is a deformation of the unit circle and g_1, g_2 are real polynomials in $x = \frac{1}{2}(z + z^{-1})$. Let $S_{a,b,c,d}$ be the space of such polynomials with inner product (2.4). Then we have

$$\begin{aligned}\mu &: S_{a,b,c,d} \rightarrow S_{aq^{-1/2}, bq^{-1/2}, cq^{1/2}, dq^{1/2}} \\ \mu^* &: S_{aq^{-1/2}, bq^{-1/2}, cq^{1/2}, dq^{1/2}} \rightarrow S_{a,b,c,d}.\end{aligned}$$

We seek a weight function $w_{a,b,c,d}$ such that

$$(2.5) \quad (f, \mu g)_{aq^{-1/2}, bq^{-1/2}, cq^{1/2}, dq^{1/2}} = (\mu^* f, g)_{a,b,c,d}$$

for all polynomials $f \in S_{aq^{-1/2}, bq^{-1/2}, cq^{1/2}, dq^{1/2}}$ and $g \in S_{a,b,c,d}$. A simple computation yields the necessary and sufficient condition

$$\frac{w_{a,b,c,d}(qz)}{w_{a,b,c,d}(z)} = \frac{(1-az)(1-bz)(1-cz)(1-dz)\left(1 - \frac{1}{z^2q}\right)\left(1 - \frac{1}{q^2z^2}\right)}{\left(1 - \frac{a}{qz}\right)\left(1 - \frac{b}{qz}\right)\left(1 - \frac{c}{qz}\right)\left(1 - \frac{d}{qz}\right)(1-z^2)(1-qz^2)}$$

with the solution

$$(2.6) \quad w_{a,b,c,d}(z) = \frac{(z^2; q)_\infty (z^{-2}; q)_\infty}{(az; q)_\infty \left(\frac{a}{z}; q\right)_\infty (bz, -q)_\infty \left(\frac{b}{z}; q\right)_\infty (cz; q)_\infty \left(\frac{c}{z}; q\right)_\infty (dz; q)_\infty \left(\frac{d}{z}; q\right)_\infty}$$

unique up to multiplication by a positive constant. Here

$$(a; q)_\infty = \prod_{k=1}^{\infty} (1 - aq^{k-1}).$$

With the weight function (2.6) the adjoint relation (2.5) holds.

It follows immediately that $\mu^* \mu : S_{a,b,c,d} \rightarrow S_{a,b,c,d}$ is formally self-adjoint:

$$(2.7) \quad (\mu^* \mu g_1, g_2)_{a,b,c,d} = (g_1, \mu^* \mu g_2)_{a,b,c,d}.$$

From (2.1B) and (2.3) we see that the Askey-Wilson polynomials are eigenfunctions of $\mu^* \mu$:

$$(2.8) \quad \mu^* \mu \phi_n^{(a,b,c,d)} = \lambda_n \phi_n^{(a,b,c,d)}, \quad \lambda_n = q^{-n} (1 - q^n cd) \left(1 - \frac{ab}{q} q^n\right).$$

Note that $\lambda_n = \lambda_m$ if and only if $n = m$. Since the eigenfunctions corresponding to distinct eigenvalues are orthogonal, we have

$$(\phi_n^{(a,b,c,d)}, \phi_m^{(a,b,c,d)})_{a,b,c,d} = 0 \text{ if } n \neq m.$$

The operator $\mu^* \mu$ and the weight function are symmetric with respect to the interchange $a \leftrightarrow b$. Thus the polynomials $\{\phi_n^{(b,a,c,d)}\}$ are also orthogonal in $S_{a,b,c,d}$ and eigenfunctions of $\mu^* \mu$. This means that there exists a constant K_n such that

$$\phi_n^{(b,a,c,d)}(x) = K_n \phi_n^{(a,b,c,d)}(x).$$

Equating coefficients of x^n on both sides of this expression to obtain K_n , we find

$$(2.9) \quad \begin{aligned} & 4\varphi_3 \left(\begin{matrix} q^{-n}, q^{n-1}bacd, bz, b/z \\ ba, bc, bd \end{matrix} ; q, q \right) \\ &= \frac{(b/a)^n (ac; q)_n (ad; q)_n}{(bc; q)_n (bd; q)_n} 4\varphi_3 \left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, a/z \\ ab, ac, ad \end{matrix} ; q, q \right). \end{aligned}$$

This is Sears' formula for a balanced $4\varphi_3$ [2, 7]. (Although we shall not do so, it is clear that we could renormalize the Askey-Wilson polynomials so that they are symmetric in a, b, c, d .)

Equation (2.5) gives a useful relationship between the norms on $S_{aq^{-1/2}, bq^{-1/2}, cq^{1/2}, dq^{1/2}}$ and $S_{a,b,c,d}$. Setting $f(x) = g(x) = 1$ in (2.5) and using (2.1B), (2.3) we find

$$(2.10) \quad \|1\|_{aq^{-1/2}, bq^{-1/2}, cq^{1/2}, dq^{1/2}}^2 = \frac{(1 - cd)}{\left(1 - \frac{ab}{q}\right)} \|1\|_{a,b,c,d}^2.$$

Since $w_{a,b,c,d}$ is symmetric in a, b, c, d it is now obvious that we can "lower" any pair of parameters a, b, c, d and "raise" the remaining pair to obtain a condition analogous to (2.10). These six recurrences imply

$$(2.11) \quad \|1\|_{a,b,c,d}^2 = \frac{h(abcd, q)}{(ab; q)_\infty (ac; q)_\infty (ad; q)_\infty (bc; q)_\infty (bd; q)_\infty (cd; q)_\infty}$$

for some analytic function h .

Now we turn our attention to the recurrence (2.1A):

$$\tau^{(a,b,c,d)} : S_{a,b,c,d} \rightarrow S_{aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}}.$$

We seek the adjoint τ^* to $\tau \equiv \tau^{(a,b,c,d)}$:

$$(2.12) \quad (f, \tau g)_{aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}} = (\tau^* f, g)_{a,b,c,d}$$

for all $f \in S_{aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2}}$ and $g \in S_{a,b,c,d}$. A simple computation yields

$$(2.13) \quad \begin{aligned} \tau^* &\equiv \tau^{*(aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2})} \\ &= q^{-1/2} (z - z^{-1})^{-1} \left(\frac{(1 - az)(1 - bz)(1 - cz)(1 - dz)}{z^2} E_z^{1/2} \right. \\ &\quad \left. - z^2 \left(1 - \frac{a}{z}\right) \left(1 - \frac{b}{z}\right) \left(1 - \frac{c}{z}\right) \left(1 - \frac{d}{z}\right) E_z^{-1/2} \right). \end{aligned}$$

From (2.1A) and the orthogonality relations we see that

$$(2.14) \quad \tau^* \phi_{n-1}^{(aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2})} = H_n \phi_n^{(a,b,c,d)}.$$

Comparing coefficients of x^n on both sides of (2.14) we find

$$H_n = -\frac{1}{aq^{1/2}} (1 - ab)(1 - ac)(1 - ad).$$

Thus, $\tau^* \tau$ is self-adjoint on $S_{a,b,c,d}$ and the eigenvalue equation is

$$(2.15) \quad \tau^* \tau \phi_n^{(a,b,c,d)} = -(1 - q^{-n})(1 - abcdq^{n-1}) \phi_n^{(a,b,c,d)}.$$

Furthermore we have the "Rodrigues formula"

$$(2.16) \quad \begin{aligned} \phi^{(a,b,c,d)} &= J_n \tau^{*(aq^{1/2}, \dots, dq^{1/2})} \tau^{*(aq^{2/2}, \dots, dq^{2/2})} \dots \tau^{*(aq^{n/2}, \dots, dq^{n/2})} (1), \\ J_n &= \frac{(-1)^n (abq; q)_n (acq; q)_n (adq; q)_n}{a^n q^{n(n+1)/4}}. \end{aligned}$$

(This is just an example of the Infeld-Hull-Inoui factorization method [8, 9]. To obtain the eigenvalues and eigenfunctions of the "factorized" operators $\tau^{*(aq^{(k+1)/2}, \dots, dq^{(k+1)/2})} \tau^{(aq^{k/2}, \dots, dq^{k/2})}$ we go to the bottom of

the weight ladder where τ “annihilates” the eigenfunction and move up one step at a time.)

From relation (2.12) with $f = \phi_{n-1}^{(aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2})}$ and $g = \phi_n^{(a,b,c,d)}$ we have

$$(2.17) \quad \begin{aligned} \|\phi_n^{(a,\dots,d)}\|_{a,\dots,d}^2 &= \frac{-a^2q(1-q^{-n})(1-q^{n-1}abcd)}{(1-ab)^2(1-ac)^2(1-ad)^2} \\ &\cdot \left\| \phi_{n-1}^{(aq^{1/2}, \dots, dq^{1/2})} \right\|_{aq^{1/2}, \dots, dq^{1/2}}^2 \end{aligned}$$

which permits us to compute recursively the norm of any Askey-Wilson polynomial once we know $\|1\|_{a,\dots,d}^2$ for all a, b, c, d . To determine this norm we note that

$$(\phi_1^{(a,\dots,d)}, \phi_0^{(a,\dots,d)})_{a,\dots,d} = 0.$$

Substituting the explicit expansions (1.1) into this relation for $n = 0, 1$ and making use of the property

$$(g_m^a, 1)_{a,b,c,d} = \|1\|_{aq^m,b,c,d}^2, \quad g_m^a(x) = (az; q)_m \left(\frac{a}{z}; q\right)_m$$

for $m = 1$ we find

$$(2.18) \quad \|1\|_{a,b,c,d}^2 = \frac{(1-abcd)}{(1-ab)(1-ac)(1-ad)} \|1\|_{aq,b,c,d}^2$$

It follows from this recurrence and (2.11) that

$$h(qabcd, q)(1-abcd) = h(abcd, q).$$

Hence $h = K(q)(abcd; q)_\infty$, where $K(q)$ is to be determined. In the special case $a = 1, b = q^{1/2}, c = -1, d = -q^{1/2}$ (suggested by Askey) the norm can be computed directly:

$$(2.19) \quad \|1\|_{1,q^{1/2},-1,-q^{1/2}}^2 = \frac{1}{2\pi} \int_0^{2\pi} d\theta = 1.$$

However

$$\begin{aligned} &\|1\|_{1,q^{1/2},-1,-q^{1/2}}^2 \\ &= \frac{K(q)(q; q)_\infty}{(q^{1/2}; q)_\infty (-1; q)_\infty (-q^{1/2}; q)_\infty (-q^{1/2}; q)_\infty (-q, q)_\infty (q^{1/2}; q)_\infty} \\ &= \frac{K(q)(q; q)_\infty}{2}. \end{aligned}$$

We conclude that [2]

$$(2.20) \quad \frac{1}{2\pi i} \int_C \frac{(z^2; q)_\infty (z^{-2}; q)_\infty dz/z}{(az; q)_\infty \left(\frac{a}{z}; q\right)_\infty (bz; q)_\infty \left(\frac{b}{z}; q\right)_\infty (cz; q)_\infty \left(\frac{c}{z}; q\right)_\infty (dz; q)_\infty \left(\frac{d}{z}; q\right)_\infty} = \frac{2(abcd; q)_\infty}{(q, -q)_\infty (ab; q)_\infty (ac; q)_\infty (ad; q)_\infty (bc; q)_\infty (bd; q)_\infty (cd; q)_\infty}.$$

A final note: Nassrallah and Rahman [10] have recently obtained an integral analog of the recurrence (2.1B).

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