TABLE I. The exact phase shifts are $\delta_l = K^2g^22\pi S_1$, phase shifts to the zeroth order in $\hbar^2$ are $\delta_l^{(0)} = K^2g^22\pi S_2$, and phase shifts to the first order in $\hbar^2$ are $\delta_l^{(1)} = K^2g^22\pi S_3$.

l	$S_1 = \dfrac{1}{(2l+1)(2l-1)(2l+3)}$	$S_2 = \dfrac{1}{8[l(l+1)]^{3/2}}$	$S_3 = S_2\cdot\left(1+\dfrac{5}{8l(l+1)}\right)$
1	0.066667	0.044194	0.058005
2	0.009524	0.008505	0.009391
3	0.003175	0.003007	0.003164
4	0.001443	0.001398	0.001441
5	0.000777	0.000761	0.000777

$$\sigma_S = |f(\theta)|^2, \tag{17b}$$

we obtain, for $l \geq 1$, the following expression for the phase shifts in the small q limit

$$\delta_l = 2\pi K^2g^2/[(2l+1)(2l-1)(2l+3)], \tag{18}$$

where we keep only terms in q^2 by omitting the higher powers. By comparison with the exact result at its small energy limit, we can determine how good the WKB approximation is by simply expanding the WKB results to the small limit. So that, to the zeroth in $\hbar^2$, we have, for $l \geq 1$,

$$\delta_l^{(o)} = K(r-S) = \frac{K^2g^2}{[l(l+1)]^{3/2}}\frac{\pi}{4}, \tag{19}$$

which was obtained by keeping only terms to $q^2 = 2K^2g^2$, such that

$$F(\tfrac{1}{2}\pi, k^2) = \tfrac{1}{2}\pi(1 + \tfrac{1}{4}k^2 + \cdots), \tag{20}$$

$$E(\tfrac{1}{2}\pi, k^2) = \tfrac{1}{2}\pi(1 - \tfrac{1}{4}k^2 + \cdots), \tag{21}$$

and

$$k^2 = r_2^2/(r_1^2 + r_2^2) = 2g^2K^2/\hbar^2[l(l+1)]^2. \tag{22}$$

To the first order in $\hbar^2$, in addition to the same expansion formula as given in Eqs. (20), (21), and (22), we use the expansion formula, which is given in 906.05 of Ref. 6:

$$J_1 = \frac{\pi}{2}\sum_{m=0}^{\infty}\sum_{j=0}^{m}\frac{(2m)!(2j)!}{4^m4^j(m!)^2(j!)^2}k^{2j}(\alpha^2)^{m-j}.$$

Now from Eq. (22) and $\alpha^2 = -2g^2K^2/[l(l+1)]^2\hbar^2$ to the second order in q, we get

$$\delta_l^{(1)} = K(r-S) = \{K^2g^2\pi/4[l(l+1)]^{3/2}\}(1 + 5/8l(l+1)). \tag{23}$$

This comparison in the small q limit is shown in Table I. We see that it is agreeing in this limit. And the agreement improves as l becomes larger.

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Clebsch-Gordan Coefficients and Special Function Identities. I. The Harmonic Oscillator Group

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It is shown that by constructing explicit realizations of the Clebsch–Gordan decomposition for tensor products of irreducible representations of a group G, one can derive a wide variety of special function identities with physical interest. In this paper, the representation theory of the harmonic oscillator group is used to give elegant derivations of identities involving Hermite, Laguerre, Bessel, and hypergeometric functions.

1. INTRODUCTION

In two recent papers Armstrong[1] and Cunningham[2] have employed Lie algebraic techniques to compute some integrals which are useful in the quantum mechanical treatment of the hydrogen atom. An advantage of such techniques is that they allow one to compute desired matrix elements for a quantum mechanical system directly from the symmetry properties of the system. There is no need to appeal to special function theory for an independent derivation. Moreover, the corresponding special function identities themselves can be more simply and elegantly derived on the basis of group theoretic considerations. The identities useful in quantum mechanics tend to be exactly those which are derivable from a study of the symmetry groups of quantum mechanical systems.

In this paper we extend the single example of Armstrong and Cunningham to a general method for the derivation of special function identities. The method is simple to describe. Let $\{\nu_j\}$ be a family of irreducible representations of the Lie algebra G and suppose the tensor product $\nu_k \otimes \nu_l$ can be decomposed into a direct sum of representations

$$\nu_k \otimes \nu_l \cong \sum_j \oplus\, n_j(k,l)\nu_j, \tag{1.1}$$

where the multiplicity $n_j(k,l)$ is either one or zero. Let $\{h_m^{(j)}\}$ be a suitably chosen basis (which we call *canonical*) for the representation space of ν_j. Then the vectors $\{h_m^{(k)} \otimes h_n^{(l)}\}$ form a basis for the representation space V of $\nu_k \otimes \nu_l$. On the other hand, from expression (1.1) we see that for each j such that $n_j(k,l) \neq 0$, we can find vectors $\{H_p^{(j)}\}$ which form a canonical basis for that subspace V_j of V which transforms irreducibly under ν_j. As is well known, the vectors $\{H_p^{(j)}\}$ can be expressed as linear combinations of the $\{h_m^{(k)} \otimes h_n^{(l)}\}$ via the Clebsch–Gordan (CG) coefficients

$$H_p^{(j)} = \sum_{m,n} C(k,m;l,n \mid j,p) h_m^{(k)} \otimes h_n^{(l)}. \tag{1.2}$$

Also, relations (1.2) can be inverted to express the $\{h_m^{(k)} \otimes h_n^{(l)}\}$ as linear combinations of the $\{H_p^{(j)}\}$. We suppose that the coefficients $C(\cdot)$ are known.

Consider a realization (model) of $\nu_k \otimes \nu_l$ such that V is a function space. Then the $h_m^{(k)} \otimes h_n^{(l)}$ are functions and (1.2) shows us how to construct the functions $H_p^{(j)}$. If, however, we can compute the functions $H_p^{(j)}$ directly from our model, we can view (1.2) as an identity relating two families of functions.

Armstrong and Cunningham considered an example where G was $sl(2,R)$, $\{\nu_j\}$ was the discrete series of representations, and V was a functional Hilbert space such that the basis vectors $h_m^{(k)} \otimes h_n^{(l)}, H_p^{(j)}$ were computable in terms of Laguerre polynomials. Substituting these results into (1.2) and using the known CG coefficients for the discrete series, they obtained an identity obeyed by Laguerre polynomials. (Actually these authors computed the matrix elements

$$\langle H_p^{(j)}, h_m^{(k)} \otimes h_n^{(l)} \rangle, \tag{1.3}$$

where $\langle \cdot , \cdot \rangle$ is the inner product on V but that is equivalent to a knowledge of (1.2). Rather than study (1.3) via the Wigner–Eckart theorem, we choose to examine the sums (1.2). This is because (1.2) makes sense in many cases where there is no convenient inner product space structure on V.)

The key to obtaining a variety of useful identities is in the construction of models of $\nu_k \otimes \nu_l$. Once a model is constructed the identity follows automatically. The author's works[3–6] contain a classification of these models for many of the symmetry groups of physics, in which the representation acts via differential and difference operators. Thus, choosing appropriate models from these papers we can substitute into (1.2) and obtain a wide variety of special function identities.

In this paper we consider the Lie algebra of the harmonic oscillator group S, a group which arises in the study of the harmonic oscillator problem in quantum mechanics. The irreducible representations and CG coefficients for S are computed in Ref. 3. In particular, some of the CG coefficients are expressible as hypergeometric functions and some as Laguerre polynomials. By choosing appropriate models we obtain identities involving Hermite, Laguerre, Bessel, and hypergeometric functions.

The identity (5.14) may be new. All results are obtained with a minimum of computation. We do not attempt to list all possible models but only a few which lead to especially interesting formulas.

In a subsequent paper we shall apply this method to the Lie algebras $su(2)$ and $sl(2,R)$, the latter related to the hydrogen atom problem. The CG coefficients and special function identities for these algebras are considerably more complicated than those presented here.

Unless otherwise stated, all variables appearing in this paper are real.

2. THE HARMONIC OSCILLATOR GROUP

We designate by S the real four-parameter group of matrices

$$g\{w,\alpha,\delta\} = \begin{pmatrix} 1 & \frac{1}{2}e^{-i\alpha}\overline{w} & i\delta - \frac{1}{8}w\overline{w} & \alpha \\ 0 & e^{-i\alpha} & -\frac{1}{2}w & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.1}$$

where $w = x + iy \in \mathbb{C}$ and α, δ are real. The group multiplication law is

$$g\{w,\alpha,\delta\} \cdot g\{w',\alpha',\delta'\} = g\{w + e^{-i\alpha}w',\ \alpha + \alpha',\ \delta + \delta' + \tfrac{1}{8}(\overline{w}w'e^{-i\alpha} - w\overline{w}'e^{i\alpha}\}. \tag{2.2}$$

In particular, $g\{0,0,0\}$ is the identity and the inverse of a group element is given by

$$g^{-1}\{w,\alpha,\delta\} = g\{-e^{i\alpha}w, -\alpha, \delta\}. \tag{2.3}$$

As a basis for the Lie algebra $\mathcal{S}$ of S we choose the matrices $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{I}$ such that

$$g\{iy,0,0\} = \exp y\,\mathcal{J}_1, \quad g\{x,0,0\} = \exp x\,\mathcal{J}_2, \quad g\{0,\alpha,0\} = \exp\alpha\,\mathcal{J}_3, \quad g\{0,0,\delta\} = \exp\delta\,\mathcal{I}. \tag{2.4}$$

It is easy to verify that these matrices satisfy the commutation relations

$$[\mathcal{J}_1, \mathcal{J}_2] = \tfrac{1}{2}\mathcal{I}, \quad [\mathcal{J}_3, \mathcal{J}_1] = \mathcal{J}_2, \quad [\mathcal{J}_3, \mathcal{J}_2] = -\mathcal{J}_1, \quad [\mathcal{J}_k, \mathcal{I}] = \mathcal{O}, \quad k = 1,2,3. \tag{2.5}$$

where $\mathcal{O}$ is the zero matrix. For many purposes a more convenient basis is provided by the matrices

$$\mathcal{J}^{\pm} = \mp\mathcal{J}_2 + i\mathcal{J}_1, \ \mathcal{J}^3 = i\mathcal{J}_3, \ \mathcal{E} = -i\mathcal{S}$$

in the complexification of $\mathcal{S}$. Here,

$$[\mathcal{J}^3, \mathcal{J}^{\pm}] = \pm\mathcal{J}^{\pm}, \quad [\mathcal{J}^+, \mathcal{J}^-] = -\mathcal{E}, \quad [\mathcal{E}, \mathcal{J}^{\pm}] = [\mathcal{E}, \mathcal{J}^3] = \mathcal{O}. \tag{2.6}$$

The unitary irreducible representations of S were determined in Refs. 3 and 7. We list the results as given in Ref. 3. (In this reference, representations of the factor group S/D are computed where D is the cyclic group generated by $\exp 2\pi\mathcal{J}_3$. However, the modification of these results to compute representations of S is trivial.)

There are four classes of unitary irreducible representations. The first class consists of one-dimensional representations and is of no concern to us. The second class consists of representations (λ, l) where both λ and $l > 0$ are real numbers. Each (λ, l) can be defined on a Hilbert space $\mathcal{H}$ with ON basis $\{h_n : n = 0,1,2,\cdots\}$. Indeed, the defining relations are

$$J^3 h_n = (n-\lambda)h_n, \quad E h_n = l h_n$$
$$J^+ h_n = [l(n+1)]^{1/2} h_{n+1}, \quad J^- h_n = (ln)^{1/2} h_{n-1}, \quad n = 0,1,2,\ldots, \tag{2.7}$$

where $J^{\pm}, J^3, E$ are the linear operators on $\mathcal{H}$ corresponding to $\mathcal{J}^{\pm}, \mathcal{J}^3, E$ in the Lie algebra representation induced by (λ, l). The unitary operators $\mathbf{U}(g)$ which define the representation on $\mathcal{H}$ have matrix elements:

$$U_{n,m}(g) = \langle h_n, U(g)h_m\rangle = \exp[i\alpha(\lambda - m) + il\delta + i(n-m)\theta] \exp\left(-\frac{lr^2}{2}\right)\left(\frac{n!}{m!}\right)^{1/2} \times (rl^{1/2})^{m-n}L_n^{(m-n)}(lr^2), \quad (2.8)$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathcal{H}$ and $L_n^{(m)}(x)$ is an associated Laguerre polynomial (See Ref. 8, Vol. II). We have introduced polar coordinates $re^{i\theta} = \frac{1}{2}w$.

The third class consists of representations $(\lambda, -l)$ where again $l > 0$ and λ are real numbers. The representations are defined on the same Hilbert space $\mathcal{H}$, but the defining relations are now

$$J^3h_n = (-\lambda - n - 1)h_n, \quad Eh_n = -lh_n,$$
$$J^+h_n = (ln)^{1/2}h_{n-1}, \quad J^-h_n = [l(n+1)]^{1/2}h_{n+1}, \quad (2.9)$$
$$n = 0, 1, 2, \cdots.$$

The matrix elements are

$$V_{n,m}(g) = \langle h_n, \mathbf{V}(g)h_m\rangle = \exp[i\alpha(\lambda + m + 1) - il\delta + i(m-n)\theta] \quad \exp(-\tfrac{1}{2}lr^2) \times (n!/m!)^{1/2}(-l^{1/2}r)^{m-n}L_n^{(m-n)}(lr^2), \quad (2.10)$$

where $2re^{i\theta} = w$.

The fourth class contains representations of the form $[\rho, s]$ where $\rho^2 > 0$ and s are real numbers with $0 \leq s < 1$. There is an equivalence $[\rho, s] \cong [-\rho, s]$, but all other pairs of representations are inequivalent. Each $[\rho, s]$ can be defined on a Hilbert space $\mathcal{H}$ with ON basis $\{k_m: m = 0, \pm 1, \pm 2, \cdots\}$. The defining relations are

$$J^3k_m = (m+s)k_m, \quad Ek_m = 0$$
$$J^+k_m = \rho k_{m+1}, \quad J^-k_m = \rho k_{m-1}, \quad (2.11)$$
$$m = 0, \pm 1, \pm 2, \ldots,$$

and the matrix elements are given by

$$W_{n,m}(g) = \langle k_n, \mathbf{W}(g)k_m\rangle = (-i)^{n-m}e^{i[(m-n)\theta + (m+s)\alpha]}J_{n-m}(\rho r), \quad (2.12)$$

where $g = g\{2re^{i\theta}, \alpha, \delta\}$ and $J_n(x)$ is a Bessel function (Ref. 8, Vol. II).

Of special interest to us will be the Clebsch–Gordan series for the decomposition of a tensor product of two irreducible representations of S into a direct sum of such representations. Again we quote the results from Ref. 3. First we have the decomposition

$$(\lambda, l) \otimes (\lambda', l') \cong \sum_{a=0}^{\infty} \oplus (\lambda + \lambda' - a, l + l'). \quad (2.13)$$

A natural basis for the Hilbert space $\mathcal{H} \otimes \mathcal{H}'$ corresponding to the left-hand side of this expression is given by $\{h_{n,p} = h_n \otimes h_p': n, p = 0, 1, 2, \cdots\}$, while a canonical basis for the subspace transforming according to $(\lambda + \lambda' - a, l + l')$ is denoted $\{h_m^{(\lambda+\lambda'-a, l+l')}: m = 0, 1, \cdots\}$. The CG coefficients relating these bases are

$$K[l, n; l', p \mid a, m] = \langle h_{n,p}, h_m^{(\lambda+\lambda'-a, l+l')}\rangle', \quad (2.14)$$

where $\langle\cdot,\cdot\rangle'$ is the inner product on $\mathcal{H} \otimes \mathcal{H}'$.

Explicitly,

$$\exp\left(\frac{l^{1/2}(zu + xv) + l'^{1/2}(wu - zv)}{(l + l')^{1/2}}\right) = \sum_{a,m,n,p=0}^{\infty} K[l, n; l', p \mid a, m] \frac{z^a w^m v^n u^p}{(a!m!n!p!)^{1/2}}. \quad (2.15)$$

It follows that these coefficients are zero unless $a + m = n + p$. Furthermore,

$$K[l, n; l', a + m - n \mid a, m] = (-1)^{m-n} \times \left(\frac{a!(l'/l)^{n-m}n!}{m!(a+m-n)!(1 + l'/l)^{a+m}}\right)^{1/2} \times \frac{F(-m, n-a-m; n-m+1; -l'/l)}{\Gamma(n-m+1)}, \quad (2.16)$$

where $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function and $\Gamma(z)$ is the gamma function (Ref. 8, Vol. I). In Ref. 3, several identities are derived for these coefficients based on relations (2.14) and (2.15).

The CG coefficients for the decomposition

$$(\lambda, -l) \otimes (\lambda', -l') \cong \sum_{a=0}^{\infty} \oplus (\lambda + \lambda' + a + 1, -l - l') \quad (2.17)$$

are given by

$$\langle h_{n,p}, h_m^{(\lambda+\lambda'+a+1, -l-l')}\rangle' = K[l, n; l', p \mid a, m], \quad (2.18)$$

identical with (2.14).

If $l > l' > 0$, we have

$$(\lambda, l) \otimes (\lambda', -l') \cong \sum_{a=0}^{\infty} \oplus (\lambda + \lambda' + a + 1, l - l') \quad (2.19)$$

with CG coefficients

$$\langle h_{n,j}, h_m^{(\lambda+\lambda'+a+1, l-l')}\rangle' = G[l, n; l', j \mid a, m]. \quad (2.20)$$

Here,

$$G[l, n; l', j \mid a, m] = (1 - l'/l)^{1/2}K[l - l', n; l', a \mid j, m]. \quad (2.21)$$

The representation $[\rho, s] \otimes (\lambda, l)$ can be defined on the Hilbert space $\mathcal{H} \otimes \mathcal{H}$. The Clebsch–Gordan series is

$$[\rho, s] \otimes (\lambda, l) \cong \sum_{a=-\infty}^{\infty} \oplus (\lambda - s + a, l) \quad (2.22)$$

and the CG coefficients are

$$\langle k_n \otimes h_j, h_m^{(\lambda-s+a, l)}\rangle' = E(n, j; a, m; \rho^2/l), \quad (2.23)$$

where $\{h_m^{(\lambda-s+a, l)}; m = 0, 1, 2, \cdots\}$ is a canonical basis for $(\lambda - s + a, l)$ and $\langle\cdot,\cdot\rangle'$ is the inner product on $\mathcal{H} \otimes \mathcal{H}$. These coefficients are zero unless $m - a = n + j$, in which case

$$E(n, j; a, m; \rho^2/l) = E(n + a, j; o, m; \rho^2/l) = E(m - j, j; o, m; \rho^2/l) = (j!/m!)^{1/2} \exp(-\rho^2/2l)(\rho/l^{1/2})^{m-j}L_j^{(m-j)}(\rho^2/l). \quad (2.24)$$

The CG coefficients for the decomposition

$$[\rho, s] \otimes (\lambda, -l) \cong \sum_{a=-\infty}^{\infty} \oplus (\lambda - s + a, -l) \qquad (2.25)$$

are essentially identical to (2.23) so we omit them. Finally, the representations $(\lambda, l) \otimes (\lambda', -l)$ and $[\rho, s] \otimes [\rho', s']$ have direct integral rather than direct sum decompositions and we will not consider them here.

It follows immediately from their definitions that the CG coefficients satisfy unitarity relations. For example, from (2.14) we have

$$\sum_{a,m=0}^{\infty} K[l, n_1; l', p_1 | a, m] K[l, n_2; l', p_2 | a, m] = \delta_{n_1 n_2} \delta_{p_1 p_2}, \qquad (2.26)$$

$$\sum_{n,p=0}^{\infty} K[l, n; l', p | a_1, m_1] K[l, n; l', p | a_2, m_2] = \delta_{a_1 a_2} \delta_{m_1 m_2}.$$

(Note that these coefficients are all real.) Similar relations hold for the other CG coefficients.

3. IDENTITIES FOR THE MATRIX ELEMENTS OF S

As our first application of the preceding results we consider models of the representation $(\lambda, l) \otimes (\lambda', l')$ in terms of functions on the group S. Let $\mathfrak{F}$ be the space of all functions $f(g)$, $g \in S$, defined on S. The operators $\mathbf{P}(g)$,

$$[\mathbf{P}(g)f](g') = f(g'g), \quad g, g' \in S, \qquad (3.1)$$

determine a representation of S on $\mathfrak{F}$, the *left regular representation*. Let $U_{b,n}^{(\lambda,l)}(g)$ be the matrix element (2.8) corresponding to (λ, l) and (for fixed b) define functions $h_n(g) = U_{b,n}^{(\lambda,l)}(g)$, $n = 0, 1, 2, \ldots$, in $\mathfrak{F}$. Then

$$[\mathbf{P}(g)h_n](g') = U_{b,n}^{(\lambda,l)}(g'g) = \sum_{j=0}^{\infty} U_{j,n}^{(\lambda,l)}(g) h_j(g') \qquad (3.2)$$

so that the $\{h_n(g')\}$ form an ON basis for a Hilbert subspace of $\mathfrak{F}$ which transforms according to the irreducible representation (λ, l). The last equality in (3.2) follows from the group multiplication property

$$U_{b,n}^{(\lambda,l)}(g'g) = \sum_{j=0}^{\infty} U_{b,j}^{(\lambda,l)}(g')\, U_{j,n}^{(\lambda,l)}(g), \quad g, g' \in S, \qquad (3.3)$$

of the matrix elements.

It follows that (for fixed b, c) the functions

$$h_{n,p}(g, g') = U_{b,n}^{(\lambda,l)}(g) U_{c,p}^{(\lambda',l')}(g'), \quad n, p = 0, 1, 2, \ldots, \qquad (3.4)$$

on the group $S \times S$ form a natural basis for the representation $(\lambda, l) \otimes (\lambda', l')$ under the left regular representation. Using (2.13) and (2.14), we see that the functions

$$h_m^{(\lambda+\lambda'-a, l+l')}(g, g') = \sum_{n,p=0}^{\infty} K[l, n; l', p | a, m] \times U_{b,n}^{(\lambda,l)}(g) U_{c,p}^{(\lambda',l')}(g'), \quad m = 0, 1, 2, \cdots, \qquad (3.5)$$

form a canonical basis for a model of $(\lambda + \lambda' - a, l + l')$. (Note that $K[\cdot]$ is zero unless $n + p = a + m$.)

We shall obtain an identity for the matrix elements by computing the functions $h_m^{(\lambda+\lambda'-a, l+l')}{}_{(g,g')} \equiv h_m^a(g, g')$ in an alternate manner. This computation makes use of the obvious properties:

$$h_m^a(hh', kh') = \sum_{j=0}^{\infty} U_{j,m}^{(\lambda+\lambda'-a, l+l')}(h') h_j^a(h, k) \quad h, h', k \in S, \qquad (3.6)$$

and

$$U_n^{(\lambda,l)}(e) = \delta_{n,m}, \qquad (3.7)$$

where e is the identity element of S. Setting $g' = e$ in (3.5), we find

$$h_m^a(g, e) = K[l, a + m - c; l', c | a, m] U_{b,a+m-c}^{(\lambda,l)}(g).$$

Substituting this result in (3.6) with $k = e$, $h' = g'$, $h = g(g')^{-1}$, we obtain

$$h_m^a(g, g') = \sum_{j=0}^{\infty} K[l, a + j - c; l', c | a, j] \times U_{j,m}^{(\lambda+\lambda'-a, l+l')}(g') U_{b,a+j-c}^{(\lambda,l)}(g(g')^{-1}). \qquad (3.8)$$

The desired identity follows from a comparison of (3.5) and (3.8). In particular, if $g = g'$, we find the familiar identity:

$$\sum_{n,p} K[l, n; l', p | a, m] U_{b,n}^{(\lambda,l)}(g) U_{c,p}^{(\lambda',l')}(g) = K[l, b; l', c | a, b + c - a] U_{b+c-a,m}^{(\lambda+\lambda'-a, l+l')}(g). \qquad (3.9)$$

This identity can be written in several equivalent forms by making use of the unitarity of the K coefficients. Substitution of relations (2.8) and (2.16) into (3.9) leads to a special function identity. Similar identities can be derived in the same manner corresponding to each of the coefficients $G[\cdot]$ and $E[\cdot]$. Some of these are listed in Ref. 3.

4. IDENTITIES FOR HERMITE POLYNOMIALS

We now search for additional models of the representations $(\lambda, \pm l)$ and $[\rho, s]$. Many such models have been classified in [3]–[6] in terms of Lie algebras of differential and difference operators. We select a few of particular interest.

As shown in Ref. 3, the operators

$$J_x^+ = -\frac{d}{dx} + lx, \quad J_x^- = \frac{d}{dx}, \quad E = l, \qquad J_x^3 = -l^{-1}\frac{d^2}{dx^2} + \frac{xd}{dx} - \lambda, \qquad (4.1)$$

and basis functions

$$h_n(x) = 2^{-n/2}(n!)^{-1/2} H_n(x\sqrt{l/2}), \quad n = 0, 1, 2, \ldots, \qquad (4.2)$$

determine a model of (λ, l), where $H_n(x)$ are Hermite polynomials (Ref. 8, Vol. II). Another model is given by the operators

$$J^+ = e^{i\Theta}\left(-\frac{\partial}{\partial x} + \frac{lx}{2}\right), \quad J^- = e^{i\Theta}\left(\frac{\partial}{\partial x} + \frac{lx}{2}\right) \quad E = l, \qquad J^3 = -i\frac{\partial}{\partial\Theta} \qquad (4.3)$$

and basis functions

$$h_n(x, \Theta) = 2^{-n/2}(n!)^{-1/2} \exp(-lx^2/4) H_n(x\sqrt{l/2}) \times e^{i(n-\lambda)\Theta}. \qquad (4.4)$$

Suppose the operators $J_x^{\pm}, J_x^3, E_x$ and $J_y'^{\pm}, J_y'^3, E_y'$ are given by (4.1) and define models of the representations $(\lambda, l), (\lambda', l')$, respectively. Then the operators

$$J^{\pm} = J_x^{\pm} + J_y'^{\pm}, \quad J^3 = J_x^3 + J_y'^3, \quad E = E_x + E_y' \qquad (4.5)$$

and basis functions

$$h_{n,p}(x,y) = h_n(x)h_p(y) = 2^{-(n+p)/2}(n!p!)^{-1/2}H_n(x\sqrt{l/2})H_p(x\sqrt{l'/2}) \tag{4.6}$$

define a model of $(\lambda, l) \otimes (\lambda', l')$. We will use (2.13) and (4.5) to compute the basis functions $h_m^{(\lambda+\lambda'-a,\, l+l')} \equiv h_m^a(x,y)$ directly. It is easy to verify that the equations

$$J^- h_o^a = 0, \quad J^3 h_o^a = -(\lambda + \lambda')h_o^a \tag{4.7}$$

for $h_o^a(x,y)$ have unique solutions,

$$h_o^a(x,y) = c_a 2^{-a/2}(a!)^{-1/2}H_a[(x-y)\sqrt{\tfrac{1}{2}ll'/(l+l')}], \quad a = 0,1,2,\ldots, \tag{4.8}$$

where the c_a are constants. The remaining basis functions can be obtained from the recurrence relation

$$J^+ h_m^a = [(m+1)ll'/(l+l')]^{1/2}h_{m+1}^a, \quad m = 0,1,2,\cdots. \tag{4.9}$$

The solution is

$$h_m^a(x,y) = c_a(2ll')^{-m/2}(l+l')^m(m!)^{-1/2}H_a(u)H_m(v)$$

$$u = \sqrt{\frac{\frac{1}{2}ll'}{l+l'}}(x-y), \quad v = \frac{lx + l'y}{\sqrt{2(l+l')}}, \tag{4.10}$$

$$a, m = 0,1,2,\cdots.$$

To compute c_a we use the fact that

$$h_0^a = (a!)^{1/2}(l+l')^{-a/2}\sum_{k=0}^{a}\frac{l^{(a-k)/2}(l')^{k/2}(-1)^k}{\sqrt{k!\,(a-k)!}}h_{k,a-k}, \tag{4.11}$$

which follows from the explicit expression (2.16) for $K[l,k;l',a-k\,|\,a,0]$. Comparing the coefficient of x^a on both sides of this equation, we obtain

$$c_a = (-1)^a. \tag{4.12}$$

On the other hand, (2.14) yields the relation

$$h_m^a = \sum_{n,p} K[l,n;l',p\,|\,a,m]h_{n,p}. \tag{4.13}$$

Substitution of (2.16), (4.6), (4.10), and (4.12) into this relation yields the desired identity.

Another model of $(\lambda_1, l_1) \otimes (\lambda_2, l_2)$ is provided by the operators (4.3) and basis functions

$$h_{n,p}(x,\Theta) = 2^{-(n+p)/2}(n!p!)^{-1/2}\exp(-lx^2/4) \times H_n(x\sqrt{l_1/2})H_p(x\sqrt{l_2/2})e^{i(n+p-\lambda_1-\lambda_2)\Theta} = h_n(x,\Theta)h_p(x,\Theta), \tag{4.14}$$

where $l = l_1 + l_2$. Indeed,

$$J^+h_{n,p} = h_n(x,\Theta)e^{i\Theta}\left(\frac{-\partial}{\partial x} + \frac{l_2 x}{2}\right)h_p(x,\Theta) + h_p(x,\Theta)e^{i\Theta}\left(\frac{-\partial}{\partial x} + \frac{l_1 x}{2}\right)h_n(x,\Theta) = \sqrt{l_2(p+1)}\,h_{n,p+1} + \sqrt{l_1(n+1)}\,h_{n+1,p} \tag{4.15}$$

with similar formulas for the other operators (4.3). On the other hand, from (2.13) it is obvious that the basis functions h_m^a for this model are given by

$$h_m^a(x,\Theta) = c_a 2^{-m/2}(m!)^{-1/2}\exp(-lx^2/4) \times H_m(x\sqrt{l/2})e^{i(m-\lambda_1-\lambda_2+a)\Theta}, \tag{4.16}$$

where c_a is a constant. We can use the identity (3.11) with $l = l_1$, $l' = l_2$ to compute c_a. Indeed, comparing coefficients of x^0, we find

$$c_a = \begin{cases}(-2)^{-a/2}(a!)^{1/2}/(a/2)! & \text{if } a \text{ is even}\\ 0 & \text{if } a \text{ is odd.}\end{cases} \tag{4.17}$$

(Note that (4.18) is actually the special case $x = y$ of the first identity derived in this section. However, the method of proof is much simpler.)

For our next model we observe that the operators

$$K^+ = -e^{i\Theta}\frac{\partial}{\partial x}, \quad K^- = e^{-i\Theta}\frac{\partial}{\partial x}, \quad K^3 = -i\frac{\partial}{\partial \Theta}, \quad E = 0 \tag{4.19}$$

and basis functions

$$k_n(x,\Theta) = (-i)^n e^{i(n+s)\Theta + i\rho x}, \quad n = 0, \pm 1, \pm 2, \cdots, \tag{4.20}$$

define the representation $[\rho, s]$. Therefore, the operators (4.3) and basis functions

$$k_n \otimes h_j(x,\Theta) = (-i)^n 2^{-j/2}(j!)^{-1/2} \times \exp(+i\rho x - lx^2/4)H_j(x\sqrt{l/2}) \times \exp[i(n+j+s-\lambda)\Theta], \quad j, \pm n = 0,1,2,\ldots, \tag{4.21}$$

determine a model of $[\rho, s] \otimes (\lambda, l)$. From the explicit form of the operators (4.3), we can directly compute the natural basis functions $h_m^{(\lambda-s+a,\, l)}(x,\Theta) \equiv h_m^a(x,\Theta)$ corresponding to the Clebsch–Gordan series. The results are clearly

$$h_m^a(x,\Theta) = c_a 2^{-m/2}(m!)^{-1/2}\exp(-lx^2/4)H_m(x\sqrt{l/2}) \times e^{i(m-\lambda+s-a)\Theta}, \tag{4.22}$$

where c_a is a constant. We compute the constant by evaluating the expression

$$h_m^a(x,\Theta) = \sum_{j=0}^{\infty} E(-j-a, j; a, 0; \rho^2/l)k_{-j-a} \otimes h_j(x,\Theta) \tag{4.23}$$

at $x = 0$. the result is $c_a = i^a$, so the identity

$$h_m^a = \sum_{j,n} E(n,j;a,m;\rho^2/l)\,k_n \otimes h_j \tag{4.24}$$

becomes (after some simplification)

$$\exp(\rho^2 - 2i\rho x)H_m(x) = \sum_{j=0}^{\infty}(-2i\rho)^{m-j}L_j^{(m-j)}(2\rho^2)H_j(x). \tag{4.25}$$

A different group-theoretic interpretation of (4.25) is presented in Ref. 3, p. 106.

The operators

$$K^+ = \rho e^{i\Theta}, \quad K^- = \rho e^{-i\Theta}, \quad K^3 = -i\frac{\partial}{\partial\Theta}, \quad E = 0 \tag{4.26}$$

and basis functions

$$k_n(\Theta) = e^{i(n+s)\Theta}, \quad n = 0, \pm 1, \pm 2, \cdots, \tag{4.27}$$

define another model of $[\rho, s]$. It follows from this remark and expressions (4.3) and (4.4) that the operators

$$J^+ = e^{i\Theta}\left(\frac{-\partial}{\partial x} + \frac{lx}{2} + \rho\right), \quad J^- = e^{-i\Theta}\left(\frac{\partial}{\partial x} + \frac{lx}{2} + \rho\right)$$
$$E = l, \quad J^3 = -i\frac{\partial}{\partial\Theta} \tag{4.28}$$

and basis functions

$$k_n \otimes h_j(x, \Theta) = k_n(\Theta) h_j(x, \Theta) = e^{i(n+s)\Theta} 2^{-j/2}(j!)^{-1/2} \exp(-lx^2/4) \times H_j(x\sqrt{l/2})\, e^{i(j-\lambda)\Theta},$$
$$j, \pm n = 0, 1, 2, \cdots, \tag{4.29}$$

define a model of $[\rho, s] \otimes (\lambda, l)$. In particular

$$J^+[k_n h_j] = (\rho e^{i\Theta} k_n) h_j + k_n e^{i\Theta}\left(-\frac{\partial}{\partial x} + \frac{lx}{2}\right) h_j,$$
$$J^3[k_n h_j] = \left(-i\frac{\partial}{\partial\Theta} k_n\right) h_j + k_n\left(-i\frac{\partial}{\partial\Theta} h_j\right), \tag{4.30}$$

with similar interpretations of the remaining operators. We can again compute the basis functions $h_m^{(\lambda-s+a,l)}(x,\Theta) \equiv h_m^a(x,\Theta)$ directly from (4.28) and (2.7):

$$h_m^a(x, \Theta) = c_a 2^{-m/2}(m!)^{-1/2} \exp[-l(x + 2\rho/l)^2/4] \times H_m[(x - 2\rho/l)\sqrt{l/2}]\, e^{i(m-\lambda+s-a)\Theta}. \tag{4.31}$$

As usual, we compute c_a by evaluating (4.23) at $x = 0$. The result is $c_a = 1$, so our new identity becomes

$$\exp[-2x\rho - \rho^2] H_m(x + 2\rho) = \sum_{j=0}^{\infty} (2\rho)^{m-j} L_j^{(m-j)}(2\rho^2) H_j(x). \tag{4.32}$$

A different group-theoretic derivation of this formula is given in Ref. 3, p. 106.

We omit the routine computation of the identities for Hermite polynomials obtained by decomposing $(\lambda, l) \otimes (\lambda', -l')$.

As a concluding remark we note that the identity

$$2^{-n/2}(n!)^{-1/2} H_n[(2l)^{-1/2}(J^+ + J^-)] h_o = h_n,$$
$$n = 0, 1, 2, \ldots, \tag{4.33}$$

holds for the model of (λ, l) defined by (4.1), (4.2), since $J^+ + J^- = lx$. Therefore, (4.33) must hold for *all* models of (λ, l) as classified in Refs. 3–6. This identity is by no means obvious for the remaining models considered in this paper.

5. IDENTITIES FOR LAGUERRE FUNCTIONS

As shown in Ref. 3, p. 111, the operators

$$J^+ = e^{i\Theta}\left(\frac{\partial}{\partial x} - l\right), \quad J^- = e^{-i\Theta}\left(-x\frac{\partial}{\partial x} + \frac{i\partial}{\partial\Theta}\right),$$
$$J^3 = -i\frac{\partial}{\partial\Theta}, \quad E = l \tag{5.1}$$

and basis functions

$$h_n(x, \Theta) = (n!)^{1/2} l^{n/2} (lx)^{\lambda-n} L_n^{(\lambda-n)}(lx)\, e^{i(n-\lambda)\Theta},$$
$$n = 0, 1, 2, \cdots, \tag{5.2}$$

form a model of (λ, l). It follows that the operators (5.1) and basis functions

$$h_{n,p}(x, \Theta) = h_n \otimes h'_p = (n!p!)^{1/2} l_1^{\lambda_1 - n/2} l_2^{\lambda_2 - p/2} \times x^{\lambda_1+\lambda_2-n-p} L_n^{(\lambda_1-n)}(l_1 x) L_p^{(\lambda_2-p)}(l_2 x)\, e^{i(n+p-\lambda_1-\lambda_2)\Theta}. \tag{5.3}$$

define a model of $(\lambda_1, l_1) \otimes (\lambda_2, l_2)$, where $l_1 + l_2 = l$. Here,

$$J^+(h_n h'_p) = h_n e^{i}\left(\frac{\partial}{\partial x} - l_2\right) h'_p + h_p e^{i\Theta}\left(\frac{\partial}{\partial x} - l_1\right) h_n \tag{5.4}$$

with a similar interpretation for the other operators. The basis functions $h_m^{(\lambda_1+\lambda_2-a,l)}(x,\Theta) \equiv h_m^a(x,\Theta)$ for this representation are easily obtained from (5.1) and (5.2):

$$h_m^a(x, \Theta) = c_a(m!)^{1/2} l^{m/2} (lx)^{\lambda_1+\lambda_2-a-m} L_m^{(\lambda_1+\lambda_2-a-m)}(lx) \times e^{i(m-\lambda_1-\lambda_2+a)\Theta}, \quad m, a = 0, 1, 2, \cdots. \tag{5.5}$$

The value of c_a follows by equating coefficients of $x^{\lambda_1+\lambda_2-a}$ on both sides of expression (4.11) ($l = l_1$, $l' = l_2$):

$$c_a = \frac{\Gamma(\lambda_2 + 1)}{\Gamma(\lambda_2 - a + 1)} l_1^{\lambda_1+a/2} l_2^{\lambda_2-a/2} (l_1 + l_2)^{-\lambda_1-\lambda_2+a/2} \times {}_1F_1(-\lambda_1 - a; \lambda_2 - a + 1; -l_2/l_1). \tag{5.6}$$

Thus,

$$\sum_{n,p} K[l_1, n; l_2, p \,|\, a, m](n!p!)^{1/2} l_1^{\lambda_1-n/2} \times l_2^{\lambda_2-p/2} L_n^{(\lambda_1-n)}(l_1 x) L_p^{(\lambda_2-p)}(l_2 x) = c_a(m!)^{1/2}(l_1 + l_2)^{\lambda_1+\lambda_2-a-m/2} L_m^{(\lambda_1+\lambda_2-n-p)}[(l_1 + l_2)x]. \tag{5.7}$$

In the special case where λ_1 and λ_2 are integers, this identity reduces to (3.9).

We can construct another model related to Laguerre polynomials by observing that the operators

$$K^+ = e^{i\Theta}\frac{\partial}{\partial x}, \quad K^- = e^{-i\Theta}\left(-x\frac{\partial}{\partial x} + \frac{i\partial}{\partial\Theta}\right), \quad K^3 = -i\frac{\partial}{\partial\Theta},$$
$$E = 0 \tag{5.8}$$

and basis functions

$$k_n(x, \Theta) = x^{-(n+s)/2} J_{-n-s}(2\rho\sqrt{x})\, e^{i(n+s)\Theta},$$
$$n = 0, \pm 1, \pm 2, \ldots, \tag{5.9}$$

form a model of $[\rho, s]$. Hence, the operators (5.1) and basis functions

$$k_n \otimes h_j(x, \Theta) = (j!)^{1/2} l^{j/2} (lx)^{\lambda-j} x^{-(n+s)/2} \times J_{-n-s}(2\rho\sqrt{x}) L_j^{(\lambda-j)}(lx)\, e^{i(n+s+j-\lambda)\Theta},$$
$$j, \pm n = 0, \pm 1, \pm 2, \ldots, \tag{5.10}$$

determine a model of $[\rho, s] \otimes (\lambda, l)$. The basis functions $h_m^{(\lambda-s+a,l)} \equiv h_m^a$ can be computed directly from (5.1) and (5.2):

$$h_m^a(x, \Theta) = c_a(m!)^{1/2} l^{m/2} (lx)^{(\lambda-s+a-m)} L_m^{(\lambda-s+a-m)}(lx) \times e^{i(m-\lambda+s-a)\Theta}, \quad a, m = 0, 1, 2, \cdots. \tag{5.11}$$

We compute the constants c_a by comparing coefficients of $x^{\lambda-s+a}$ on both sides of (4.23). The result is

$$c_a = \frac{\exp(-\rho^2/2l)}{\Gamma(a-s+1)}\left(\frac{\rho}{l}\right)^{a-s} {}_1F_1\left(\frac{-\lambda; a-s+1; \rho^2}{l}\right)$$
$$= \exp\left(\frac{-\rho^2}{2l}\right)\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+a-s+1)}\left(\frac{\rho}{l}\right)^{a-s} L_\lambda^{(a-s)}\left(\frac{\rho^2}{l}\right), \tag{5.12}$$

so that the identity

$$k_n \otimes h_j = \sum_m E(n,j;a,m;\rho^2/l)\, h_m^{(\lambda-s+a,l)} \tag{5.13}$$

reads

$$\exp(\rho^2)(\rho^2 x)^{(n+s)/2} J_{-n-s}(2\rho\sqrt{x}) L_j^{(\lambda-j)}(x)$$
$$= \sum_{m=0}^{\infty} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+m-n-j-s+1)} (\rho^2)^{m-j} L_j^{(m-j)}(\rho^2)$$
$$\times L_\lambda^{(m-n-j-s)}(\rho^2) L_m^{(\lambda-s-n-j)}(x). \tag{5.14}$$

If $\lambda = j = 0,\ n+s = -a, l = 1$, this formula simplifies to the well-known expression

$$\exp(\rho^2)(\rho^2 x)^{-\alpha/2} J_\alpha(2\rho\sqrt{x}) = \sum_{m=0}^{\infty} \frac{\rho^{2m} L_m^{(\alpha)}(x)}{\Gamma(m+\alpha+1)}. \tag{5.15}$$

The special case of (5.14) with $j = 0$ was first derived by Erdelyi in 1937 (see Ref. 9, p. 141). However, the general formula with $j \neq 0$ may be new.

It is a routine computation to obtain models of the representation $(\lambda, l) \otimes (\lambda', -l')$, but this will be omitted.

6. DIFFERENCE OPERATOR MODELS

In this section, we construct Lie algebra models using difference operators. These models were classified in Ref. 4.

The operators

$$K^+ = e^{+i\Theta}(-L+1),$$
$$K^- = e^{-i\Theta}\left(-(x+1)R + x + 1 + \frac{i\partial}{\partial\Theta}\right), \tag{6.1}$$
$$K^3 = -i\frac{\partial}{\partial\Theta}, \qquad E = 0$$

and basis functions

$$k_n(x,\Theta) = \rho^{-n} L_x^{(-n-s)}(\rho^2)\, e^{i(n+s)\Theta}, \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{6.2}$$

define a model of $[\rho, s]$. Here,

$$Rf(x,\Theta) = f(x+1,\Theta), \qquad Lf(x,\Theta) = f(x-1,\Theta). \tag{6.3}$$

On the other hand, the operators

$$H^+ = -le^{+i\Theta}, \quad H^- = ie^{-i\Theta}\left(\frac{\partial}{\partial\Theta} + i\lambda\right), \quad H^3 = -i\frac{\partial}{\partial\Theta},$$
$$E = l \tag{6.4}$$

and functions

$$h_j(\Theta) = (-1)^j l^{j/2}(j!)^{-1/2} e^{i(j-\lambda)\Theta}, \qquad j = 0, 1, 2, \ldots, \tag{6.5}$$

define a model of (λ, l). Thus, the operators

$$J^+ = e^{+i\Theta}(-L-l+1),$$
$$J^- = e^{-i\Theta}\left(-(x+1)R + x - \lambda + 1 + \frac{i\partial}{\partial\Theta}\right), \tag{6.6}$$
$$J^3 = -i\frac{\partial}{\partial\Theta}, \qquad E = l$$

and functions

$$k_n \otimes h_j(x,\Theta) = (-\sqrt{l})^j (j!)^{-1/2} \rho^{-n} L_x^{(-n-s)}(\rho^2) e^{i(n+j+s-\lambda)\Theta}, \tag{6.7}$$

determine a model of $[\rho, s] \otimes (\lambda, l)$. We can compute the basis functions $h_m^{(\lambda-s+a,l)}(x,\Theta) \equiv h_m^a(x,\Theta)$ directly from (6.6), (2.7), and (2.22), with the result

$$h_m^a(x,\Theta) = c_a(-\sqrt{l})^m (m!)^{-1/2} \frac{\Gamma(x-s+a+1)}{\Gamma(x+1)}$$
$$\times {}_2F_1(-m, s-a; s-a-x; l^{-1}) e^{i(m-\lambda+s-a)\Theta}. \tag{6.8}$$

(Here, c_a could be a periodic function of x with period one. However, it is easy to check that c_a is actually a constant.) The constant c_a can be evaluated from expression (4.23). Indeed, comparing (4.23) and (5.14) with $j = 0,\ x = 0$, we find

$$c_a = \rho^a \exp(\rho^2 - \rho^2/2l)/\Gamma(a-s+1). \tag{6.9}$$

Thus, the identity (5.13) becomes

$$\exp[\rho^2(1-l)] L_x^{(\alpha)}(l\rho^2)$$
$$= j! \sum_{m=0}^{\infty} \binom{x+\alpha+m-j}{x} \frac{(-l\rho^2)^{m-j}}{m!}$$
$$\times {}_2F_1(-m, j-\alpha-m; j-a-m-x; l^{-1})$$
$$\times L_j^{(m-j)}(\rho^2). \tag{6.10}$$

For our last example we consider the operators

$$J^+ = e^{i\Theta}(-L-l-l'+1),$$
$$J^- = e^{-i\Theta}\left(-(x+1)R + x - \lambda + 1 + \frac{i\partial}{\partial\Theta}\right), \tag{6.11}$$
$$J^3 = -i\frac{\partial}{\partial\Theta}, \qquad E = l + l'.$$

The functions

$$h_{n,p} = h_n \otimes h_p'(x,\Theta) = \frac{(-\sqrt{l})^n}{\sqrt{n!}} \frac{(-\sqrt{l'})^p}{\sqrt{p!}} \frac{\Gamma(x+\lambda'+1)}{\Gamma(x+1)}$$
$$\times {}_2F_1(-p, -\lambda'; -\lambda'-x; l'^{-1})$$
$$\times \exp(n+p-\lambda-\lambda')i\Theta, \quad n, p = 0, 1, 2, \ldots, \tag{6.12}$$

and these operators define a model of $(\lambda, l) \otimes (\lambda', l')$. [In particular, the action of the operators (6.4) on the $\{h_n(\Theta)\}$ yields (λ, l).] Computing the basis functions $h_m^a \equiv h_m^{(\lambda+\lambda'-a,l+l')}(x,\Theta)$ directly, we find in analogy with (6.8):

$$h_m^a(x,\Theta) = c_a \frac{(-\sqrt{l+l'})^m}{\sqrt{m!}} \frac{\Gamma(x+\lambda'-a+1)}{\Gamma(x+1)}$$
$$\times {}_2F_1(-m, a-\lambda'; a-\lambda'-x; (l+l')^{-1})$$
$$\times e^{i(m+a-\lambda-\lambda')\Theta}. \tag{6.13}$$

Using (4.11) to evaluate the constant, we obtain

$$c_a = \binom{\lambda'}{a}\sqrt{a!}\,(l+l')^{-a/2}(l/l')^{a/2}. \tag{6.14}$$

The resulting identity is

$$\binom{\lambda'}{a}\frac{\Gamma(x+\lambda'-a+1)}{\Gamma(x+\lambda'+1)}(-1)^{a+m}(l-l')^m l'^{-a} \times {}_2F_1(-m;a-\lambda',a-\lambda'-x;(l+l')^{-1}) = \sum_{n=0}^{a+m} \times \frac{(-1)^n l^{m-n}{}_2F_1(-m,n-a-m;-n-m+1;-l'/l)}{(a+m-n)!\,\Gamma(n-m+1)} \times {}_2F_1(n-a-m,-\lambda';-\lambda'-x;l'^{-1}). \quad (6.15)$$

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Note on the Explicit Form of Invariant Operators for $O(n)$

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A complete set of invariants of $O(n)$ is constructed explicitly and a method of deriving the corresponding invariants of $O(p,q)$ is briefly remarked.

It is assumed that the invariant operators for a Lie group are clarified implicitly by the researches of Killing, Cartan, and Weyl. It, however, is important to know the explicit form for the invariants in applications to physics. The subject is discussed by some authors[1-3] and an interesting form for the invariants is given for some special groups. But it seems that the explicit form for the invariants is not so simple as the Casimir operator. In this note, we give a complete system of independent invariants suitable for uniquely labeling the irreducible inequivalent representations of $O(n)$. A further discussion will be given in the near future together with some simple applications.[4]

The infinitesimal generators D_{jk}, $j, k = 1, 2, \ldots, n$, of $O(n)$ are defined as the quantities which satisfy the commutation relations

$$[D_{jk}, D_{lm}] = i(\delta_{jl}D_{km} + \delta_{km}D_{jl} - \delta_{jm}D_{kl} - \delta_{kl}D_{jm}), \quad (1)$$

where D_{jk} is antisymmetric ($D_{jk} = -D_{kj}$) and Hermitian. As is well known, the orthogonal group $O(n)$ has $[n/2]$ invariant operators, where $[n/2]$ is equal to $n/2$ or $(n-1)/2$ corresponding to even n or odd n. One of these invariants is the well-known Casimir operator

$$F^{(n)} = \tfrac{1}{2}D_{jk}D_{jk}, \quad (2)$$

where the superscript n of F denotes the dimension number. Unless stated otherwise, similar notation and the summation convention from 1 to n will be used.

We can give the result for the other invariant operators $G_p^{(n)}$ as follows:

$$G_p^{(n)} = \sum_{i_1<i_2<\cdots<i_{n-2p-2}} (C^{(n)}_{i_1 i_2\cdots i_{n-2p-2}})^2, \quad (3)$$

$$G_p^{(n)} = C^{(n)} \quad \text{for an even } n \text{ and } p = (n-2)/2, \quad (4)$$

where p in (3) takes $1, 2, \ldots, (n-4)/2$ for an even n and $1, 2, \ldots, (n-3)/2$ for an odd n. It is straightforward to show that the $G_p^{(n)}$ in (3) and (4) are invariant. The sum on the right-hand side of (3) is taken over all satisfying the condition $i_1 < i_2 < \cdots < i_{n-2p-2}$. The C in (3) and (4) are given by virtue of D_{jk} as follows:

$$C^{(n)}_{i_1 i_2\cdots i_{n-2p-2}} = \frac{1}{2^{p+1}(p+1)!}\mathcal{E}_{i_1 i_2\cdots i_n} D_{i_{n-2p-1}i_{n-2p}} \times D_{i_{n-2p+1}i_{n-2p+2}}\cdots D_{i_{n-1}i_n},$$

$$\mathcal{E}_{i_1 i_2\cdots i_n} = \begin{cases} +1, & \text{for an even permutation } (i_1 i_2\cdots i_n) \text{ of } (12\cdots n),\\ -1, & \text{for an odd permutation } (i_1 i_2\cdots i_n) \text{ of } (12\cdots n)\\ 0, & \text{otherwise.}\end{cases} \quad (5)$$

Thus together with (2) and (3) [and (4) for an even n], we have given the $[n/2]$ invariant operators for $O(n)$, whose explicit expressions can be easily given.

It can be seen that these invariant operators are independent and suitable for labeling the irreducible representations of $O(n)$. In order to see the situation, let us give an outline of the proof according to Biedenharn[1] and Micu[2]: When an invariant is evaluated in terms of the highest weight L and only the highest-order terms [only the terms containing the generators $H_j \equiv D_{2j-1\,2j}(j = 1, 2, \ldots, [n/2])$] in the invariant are considered, it becomes an invariant of the group S (the group of reflections on hyperplanes perpendicular to the roots). That is, the invariants $F^{(n)}$ and $G_p^{(n)}$ become

$$F^{(n)} \to \overline{F}^{(n)} = \sum^{[n/2]} L_j L_j, \quad (6)$$

$$G_p^{(n)} \to \overline{G}_p^{(n)} = \sum_{i_1<i_2<\cdots<i_{p+1}} (L_{i_1}L_{i_2}\cdots L_{i_{p+1}})^2, \quad (7)$$

$$G_p^{(n)} \to \overline{G}_p^{(n)} = L_1 L_2 \cdots L_{n/2} \quad \text{for an even } n \text{ and } p = (n-2)/2. \quad (8)$$

These invariants of S have the properties: Their Jacobian does not vanish identically and factorizes into $N[= n(n-2)/8$ for an even n and $(n-1)(n+1)/8$ for an odd $n]$ linear forms which, when equated to zero, give the reflecting hyperplanes that generate the