If the representation is not irreducible or not a factor representation, we learn from the above argument that it decomposes into a direct sum of representations in $P_1\mathfrak{H}$, $(P_2 - P_1)\mathfrak{H}$, $(P_3 - P_2)\mathfrak{H}$ etc. such that $U(f)$ is p_n-continuous in $(P_n - P_{n-1})\mathfrak{H}$. For fixed subrepresentation one can apply the same argument to $V(g)$, and thus we find that *the representation is a direct sum of subrepresentations in each of which* $U(f)$ *and* $V(g)$ *are continuous with respect to some Hilbertian norm* (depending on the subrepresentation).

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Clebsch–Gordan Coefficients and Special Function Identities. II. The Rotation and Lorentz Groups in 3-Space

Willard Miller, Jr.
Mathematics Department, University of Minnesota, Minneapolis, Minnesota 55455
(Received 17 February 1971)

It is shown that the construction of concrete models of Clebsch–Gordan decompositions for tensor products of irreducible group representations leads to a wide variety of special function identities. In this paper the representation theory of the rotation and Lorentz groups in 3-space is used to give elegant derivations of identities involving Laguerre, Gegenbauer, hypergeometric, and generalized hypergeometric functions. Some of these identities may be new in this general form.

INTRODUCTION

In Ref. 1, which we refer to as I, a method was described whereby a knowledge of the Clebsch–Gordan decomposition for the tensor product of two representations of a group G, could be used to derive special function identities. The idea is easy to describe. Suppose G has a family of irreducible representations $\{D_u\}$ with Clebsch–Gordan series

$$D_u \otimes D_v \cong \sum_w \oplus D_w,$$

such that each irreducible representation D_w occurs at most once in the tensor product. If $\{j_n^{(u)}\}$ is a canonical basis for D_u, then there exists a relation of the form

$$\text{(a)}\quad \tilde{j}_h^{(w)} = \sum_{n,m} C(u,n;v,m\,|\,w,h)\, j_n^{(u)} \otimes j_m^{(v)},$$

where the constants $C(\cdot|\cdot)$ are Clebsch–Gordan coefficients. Suppose we have an explicit function-space model of the representation $D_u \otimes D_v$. Then the vectors $j_n^{(u)} \otimes j_m^{(v)}$ will be special functions and if the model is simple enough, the special functions $\tilde{j}_h^{(w)}$ can be computed directly. In this case, expression (a) becomes an identity relating the special functions $j_n^{(u)} \otimes j_m^{(v)}$ and $\tilde{j}_h^{(w)}$. This identity can be inverted since the coefficients $C(\cdot|\cdot)$ satisfy orthogonality relations.

The above method is useful for a given group if there is a procedure for constructing a variety of models of the group representations. In Refs. 2–4, a number of such models are cataloged for groups of common occurence in physics. Here we use these models to give elegant derivations of identities associated with the rotation and homogeneous Lorentz groups in 3-space. Some of these identities may be new in this general form; certainly their close relationship to one another and to group theory is new.

Most of the following explicit examples are associated with the Lorentz group G_3 but the analogous examples for $SO(3)$ are usually self-evident.

In physical applications, integral forms of these identities appear when one computes matrix elements corresponding to a quantum mechanical system with symmetry group $SO(3)$ or G_3.[5,6] However, the group theoretic method has validity independent of the computation of matrix elements, so the results of this paper are not presented in integral form. The reader can write most of the following identities in various integral forms by using well-known orthogonality relations for the Laguerre, Gegenbauer, and hypergeometric functions.

1. THE GROUPS $SU(2)$ AND G_3

The group $SU(2)$ consists of all 2×2 unitary unimodular matrices. In Euler angles, every $A \in SU(2)$ can be written as

$$A(\varphi_1,\theta,\varphi_2) = \begin{pmatrix} e^{-i(\varphi_1+\varphi_2)/2}\cos(\theta/2), & ie^{-i(\varphi_1-\varphi_2)/2}\sin(\theta/2) \\ ie^{i(\varphi_1-\varphi_2)/2}\sin(\theta/2), & e^{i(\varphi_1+\varphi_2)/2}\cos(\theta/2) \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \qquad |a|^2 + |b|^2 = 1. \quad (1.1)$$

If $ab \neq 0$ the Euler angles can be defined uniquely by

$$\cos(\theta/2) = |a|, \quad \sin(\theta/2) = |b|, \quad 0 \le \theta \le \pi,$$

$$-\tfrac{1}{2}(\varphi_1+\varphi_2)=\arg a,\quad \tfrac{1}{2}(\varphi_2-\varphi_1)+\tfrac{1}{2}\pi=\arg b,$$
$$-2\pi\le\varphi_1,\varphi_2<2\pi. \tag{1.2}$$

However, if $ab=0$ these angles are not unique. As a basis for the Lie algebra $su(2)$ we choose the matrices $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$, such that

$$A(0,\theta,0)=\exp\theta\mathcal{J}_1\quad A\left(\frac{\pi}{2},\theta,-\frac{\pi}{2}\right)=\exp\theta\mathcal{J}_2,$$
$$A(\varphi,0,0)=A(0,0,\varphi)=\exp\varphi\mathcal{J}_3. \tag{1.3}$$

These matrices satisfy the commutation relations

$$[\mathcal{J}_1,\mathcal{J}_2]=\mathcal{J}_3,\quad[\mathcal{J}_3,\mathcal{J}_1]=\mathcal{J}_2,\quad[\mathcal{J}_2,\mathcal{J}_3]=\mathcal{J}_1. \tag{1.4}$$

Another convenient basis is given by

$$\mathcal{J}^{\pm}=\mp\mathcal{J}_2+i\mathcal{J}_1,\qquad \mathcal{J}^3=i\mathcal{J}_3, \tag{1.5}$$

which belong to the complexification of $su(2)$. Here,

$$[\mathcal{J}^3,\mathcal{J}^{\pm}]=\pm\mathcal{J}^{\pm},\qquad[\mathcal{J}^+,\mathcal{J}^-]=2\mathcal{J}^3. \tag{1.6}$$

The irreducible unitary representations of $SU(2)$ are D_u, $2u=0,1,2,\dots$, each defined on a $(2u+1)$-dimensional Hilbert space $\mathcal{H}_u$ with ON basis $\{p_m: m=-u,-u+1,\dots,u-1,u\}$. The defining relations are

$$J^3p_m=mp_m,\qquad J^{\pm}p_m=[(u\mp m)(u\pm m+1)]^{1/2}p_{m\pm1},$$
$$m=-u,\dots,u, \tag{1.7}$$

where $J^{\pm}, J^3$ are the linear operators corresponding to $\mathcal{J}^{\pm}, \mathcal{J}^3$, respectively, in the Lie algebra representation induced by D_u. The matrix elements $U_{n,m}(A)$ of the unitary operators $\mathbf{U}(A)$ on $\mathcal{H}_u$ which determine this representation are

$$U_{n,m}(A)=\langle p_n, U(A)p_m\rangle=\left(\frac{(u+m)!\,(u-n)!}{(u+n)!\,(u-m)!}\right)^{1/2}$$
$$\times\ a^{u+n}\bar{a}^{u-m}\bar{b}^{m-n}\frac{1}{\Gamma(m-n+1)}$$
$$\times\ F(-u-n,m-u,m-n+1;-|b/a|^2)$$
$$=(i)^{n-m}\left(\frac{(u+m)!\,(u-n)!}{(u+n)!\,(u-m)!}\right)^{1/2}$$
$$\times\ e^{-i(n\varphi_1+m\varphi_2)}P_u^{-n,m}(\cos\theta), \tag{1.8}$$

where

$$P_u^{r,m}(x)=\left(\frac{1+x}{2}\right)^{(m-r)/2}\left(\frac{1-x}{2}\right)^{(m+r)/2}$$
$$\times\frac{1}{\Gamma(m+r+1)}F(u+m+1,-u+m;$$
$$m+r+1;\tfrac{1}{2}(1-x)) \tag{1.9}$$

and A is given by (1.1). Here, $\langle\cdot,\cdot\rangle$ is the inner product on $\mathcal{H}_u$, linear in the second argument and $F(a,b;c;z)$ is the hypergeometric function, see Ref. 7, Vol. 1.

The group G_3 consists of all 2×2 complex matrices of the form

$$A=\begin{pmatrix}a & b\\ \bar{b} & \bar{a}\end{pmatrix},\qquad a,b,\in\mathbb{C},\qquad \det A=|a|^2-|b|^2=1. \tag{1.10}$$

This is a real 3-parameter matrix group isomorphic to $SL(2,R)$.[2] Furthermore, G_3 is the twofold covering group of the homogeneous Lorentz group in 3-space.[8] We can choose real coordinates (μ,ρ,ν) for A so that

$$A(\mu,\rho,\nu)=\begin{pmatrix}e^{-i(\mu+\nu)/2}\cosh(\rho/2) & e^{i(\nu-\mu)/2}\sinh(\rho/2)\\ e^{i(\mu-\nu)/2}\sinh(\rho/2) & e^{i(\mu+\nu)/2}\cosh(\rho/2)\end{pmatrix}. \tag{1.11}$$

Here we require,

$$|a|=\cosh\rho,\qquad|b|=\sinh\rho,\qquad 0\le\rho<\infty,$$
$$\mu=-\arg a-\arg b,\qquad \nu=\arg b-\arg a. \tag{1.12}$$

The matrices $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$, such that

$$A(0,\rho,\pi)=\exp\rho\mathcal{J}_1,\qquad A(0,\rho,0)=\exp\rho\mathcal{J}_2,$$
$$A(\mu,0,0)=A(0,0,\mu)=\exp\mu\mathcal{J}_3, \tag{1.13}$$

form a basis for the Lie algebra $\mathcal{G}_3$ of G_3. The commutation relations are

$$[\mathcal{J}_1,\mathcal{J}_2]=-\mathcal{J}_3,\quad[\mathcal{J}_3,\mathcal{J}_1]=\mathcal{J}_2,\quad[\mathcal{J}_3,\mathcal{J}_2]=-\mathcal{J}_1. \tag{1.14}$$

A more convenient basis for many purposes is $\mathcal{J}^{\pm}=-\mathcal{J}_2\pm i\mathcal{J}_1$, $\mathcal{J}^3=i\mathcal{J}_3$ in the complexification of $\mathcal{G}_3$. Here the commutation relations are

$$[\mathcal{J}^+,\mathcal{J}^-]=2\mathcal{J}^3,\ [\mathcal{J}^3,\mathcal{J}^{\pm}]=\pm\mathcal{J}^{\pm}, \tag{1.15}$$

identical with (1.6).

We consider a class D_u^+ of irreducible unitary representations of G_3, defined for $u>0$ (discrete series). Here, D_u^+ can be realized on the Hilbert space $\mathcal{H}$ with ON basis $\{j_n: n=0,1,2,\dots\}$. The defining relations are

$$J^3j_n=(u+n)j_n,\qquad J^+j_n=[(2u+n)(n+1)]^{1/2}j_{n+1},$$
$$J^-j_n=-[n(2u+n-1)]^{1/2}\,j_{n-1},\qquad n=0,1,\dots, \tag{1.16}$$

where $J^{\pm}, J^3$ are the representation operators corresponding to $\mathcal{J}^{\pm}, \mathcal{J}^3$, respectively. (To be more precise, D_u^+ is a global representation of G_3 only for $2u$, an integer. For $2u$ not an integer, D_u^+ is a local representation of G_3 and a global irreducible representation of the simply connected covering group of G_3 (see Refs. 2, 8, and 9). The matrix elements of D_u^+ are

$$V_{n,m}(A)=\langle j_n, V(A)j_m\rangle$$
$$=\left(\frac{\Gamma(2u+n)m!}{\Gamma(2u+m)n!}\right)^{1/2}\cdot a^n\bar{a}^{-2u-m}\bar{b}^{m-n}$$
$$\times\frac{F(-n,2u+m;m-n+1;|b/a|^2)}{\Gamma(m-n+1)}$$
$$=\left(\frac{\Gamma(2u+n)m!}{\Gamma(2u+m)n!}\right)^{1/2}e^{-i[\mu(u+n)+\nu(u+m)]}\,\mathfrak{P}^{-u-n,u+m}_{-u}(\cosh\rho), \tag{1.17}$$

where the coordinates of A are given by (1.10), (1.11), and

$$\mathfrak{P}_\nu^{\mu,\xi}(z)=\frac{1}{\Gamma(\xi+\mu+1)}\left(\frac{z+1}{2}\right)^{(\xi-\mu)/2}\left(\frac{z-1}{2}\right)^{(\xi+\mu)/2}$$
$$\times F(\nu+\xi+1,\xi-\nu;\mu+\xi+1;\tfrac{1}{2}(1-z)). \tag{1.18}$$

As is well known, the Clebsch–Gordan series for $SU(2)$ is

$$D_u \otimes D_v \cong \sum_{w=|u-v|}^{u+v} \oplus D_w. \quad (1.19)$$

The vectors $\{p_{n,m} = p_n \otimes p'_m : n = -u, -u+1, \ldots, m = -v, -v+1, \ldots, v\}$ form a natural basis for the representation space $\mathcal{H}_u \otimes \mathcal{H}'_v$, while a canonical basis for the subspace transforming according to D_w can be denoted $\{p_k^w : k = -w+1, \ldots, w\}$. The Clebsch–Gordan (CG) coefficients relating these two bases are

$$C(u, n; v, m \,|\, w, k) = \langle p_{n,m}, p_k^w\rangle', \quad (1.20)$$

where $\langle\cdot,\cdot\rangle'$ is the inner product on $\mathcal{H}_u \otimes \mathcal{H}'_v$. If the basis vectors are chosen appropriately, the CG coefficients are defined by the generating function

$$\exp[\alpha(x_2 - x_3) + \beta(x_3 - x_1) + \gamma(x_1 - x_2)] = \sum_{j_1+j_2+j_3=0}^{\infty} \sum_{m_i=-j_i}^{j_i} (j_1 + j_2 + j_3 + 1)^{1/2} \frac{\alpha^{-j_1+j_2+j_3}\beta^{j_1-j_2+j_3}}{[(-j_1 + j_2 + j_3)!(j_1 - j_2 + j_3)!} \times \frac{\gamma^{j_1+j_2-j_3} x_1^{j_1+m_1} x_2^{j_2+m_2} x_3^{j_3+m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}}{(j_1 + j_2 - j_3)!\,(j_1 + m_1)!\,(j_1 - m_1)!\,(j_2 + m_2)!(j_2 - m_2)!(j_3 + m_3)!(j_3 - m_3)!]^{1/2}}, \quad (1.21)$$

where the 3-j coefficients are

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_3-m_3}}{\sqrt{2j_3+1}} C(j_1, m_1; j_2, m_2 \,|\, j_3 - m_3) \quad (1.22)$$

and the sum is taken over all j_i, m_i for which (1.21) makes sense. In particular, $C(u, n; v, m \,|\, w, k) = 0$ unless $k = n + m$ and $|u - v| \le w \le u + v$. The various symmetries and explicit formulas for the CG coefficients which abound in the literature can all be obtained from (1.21), see Refs. 10 and 11.

The Clebsch–Gordan series for the tensor product $D_u^+ \otimes D_v^+$ of G_3 representations is[2,6]

$$D_u^+ \otimes D_v^+ \cong \sum_{s=0}^{\infty} \oplus D_{u+v+s}^+. \quad (1.23)$$

The vectors $\{j_{n,m} = j_n \otimes j'_m : n, m = 0, 1, \cdots\}$ form a natural ON basis for the representation space $\mathcal{H} \otimes \mathcal{H}'$. A canonical basis for the subspace of $\mathcal{H} \otimes \mathcal{H}'$ transforming according to D_{u+v+s}^+ can be denoted $\{j_h^s : h = 0, 1, \cdots\}$. The CG coefficients are

$$E(u, n; v, m \,|\, s, h) = \langle j_{n,m}, j_h^s\rangle', \quad (1.24)$$

where $\langle\cdot,\cdot\rangle'$ is the inner product on $\mathcal{H} \otimes \mathcal{H}'$. With an appropriate choice of basis vectors, the CG coefficients are given by the generating function.

$$\left(\frac{(2u + 2v + 2s - 1)\Gamma(2u + 2v + s - 1)\Gamma(2v + s)}{s!\,\Gamma(2u)\Gamma(2v)}\right)^{1/2} \times (1 - by)^{-2u-s}(1 - bx)^{-2v-s}(y - x)^s = \sum_{h,n,m=0}^{\infty} \left(\frac{\Gamma(2u + 2v + 2s + h)}{h!}\right)^{1/2} \times E(u, n; v, m \,|\, s, h) y^n x^m b^h, \quad |bx| < 1, \ |by| < 1. \quad (1.25)$$

We can expand the left-hand side of (1.25) to obtain explicit expressions for the CG coefficients. In general they are rather complicated finite sums. However in the special cases $s = 0$ or $h = 0$, the sum contains only one term and the CG coefficient reduces to the square root of a quotient of gamma functions, as the reader can easily verify.

From the definitions (1.20), (1.24) it follows that the CG coefficients satisfy orthogonality relations. Indeed the coefficients $E(\cdot)$ are real and satisfy

$$\sum_{s,h=0} E(u, n_1; v, m_1 \,|\, s, h) E(u, n_2, v, m_2 \,|\, s, h) = \delta_{n_1 n_2}\delta_{m_1 m_2}, \quad (1.26)$$
$$\sum_{n,m=0}^{\infty} E(u, n; v, m \,|\, s_1, h_1) E(u, n; v, m \,|\, s_2, h_2) = \delta_{s_1 s_2}\delta_{h_1 h_2}.$$

The coefficients $C(\cdot)$ satisfy similar relations except that the sums are finite.

2. IDENTITIES FOR THE MATRIX ELEMENTS OF $SO(3)$ AND G_3

Just as in I, Sec. 2, we can use products of matrix elements of the representations D_u, D_u^+ to construct new models of these representations. Since the methods are identical with I we present only the results.

For fixed b and c, the functions

$$p_h^{(u+v+s)}(A, A') = \sum_{n,m=0}^{\infty} E(u, n; v, m \,|\, s, h) \times V_{b,n}^{(u)}(A) V_{c,m}^{(v)}(A'), \quad h = 0, 1, 2, \cdots, \quad (2.1)$$

form a canonical basis for a model of D_{u+v+s}^+ under the group action

$$[P(B)f](A, A') = f(AB, A'B), \quad A, A', B \in G_3 \quad (2.2)$$

on functions defined on $G_3 \times G_3$. Hence, $V_{b,n}^{(u)}(A)$ is the matrix element (1.17) corresponding to the representation D_u^+. Note that the sum on the right-hand side of (2.1) is finite since $E(u, n; v, m \,|\, s, h) = 0$ unless $n + m = s + h$.

Using the transformation properties of the basis $p_h^{(u+v+s)}$, we can also show

$$p_h^{(u+v+s)}(A, A') = \sum_{j=0}^{\infty} E(u, s + j - c; v, c \,|\, s, j) \times V_{j,h}^{(u+v+s)}(A') V_{b,s+j-c}^{(u)}(A(A')^{-1}). \quad (2.3)$$

Equating (2.1) and (2.3) we obtain a family of identities obeyed by the matrix elements. In particular, for $A = A'$ the identity reduces to the formula

$$\sum_{n,m} E(u, n; v, m \mid s, h) V^{(u)}_{b,n}(A) V^{(v)}_{c,m}(A) = E(u, b; v, c \mid s, b + c - s) V^{(u+v+s)}_{b+c-s,h}(A), \quad (2.4)$$

since $V^{(u)}_{n,m}(E) = \delta_{n,m}$ for E the identity matrix.

The construction of models of the representations D_u of $SU(2)$ is analogous to that given above, and formulas (2.1) and (2.3) can easily be modified for this case. Of special interest is the case where the basis contains only one element $p_0^{(0)}(A, A')$, i.e., this function transforms according to the identity representation D_0. Nonzero functions $p_0^{(0)}(A, A')$ can be constructed only if $u = v$, in which case the analogy of (2.1) is

$$p_0^{(0)}(A, A') = \sum_{n=-u}^{u} C(u, n; u, -n \mid 0, 0)\, U^{(u)}_{b,n}(A)\, U^{(u)}_{c,-n}(A') \quad (2.5)$$

with fixed b, c. The analogy of (2.3) is

$$p_0^{(0)}(A, A') = C(u, -c; u, c \mid 0, 0) U^{(u)}_{b,-c}(A(A')^{-1}). \quad (2.6)$$

Equating (2.5) and (2.6) we obtain a family of addition theorems for the matrix elements. The simplest case, $b = c = 0, u = l$, yields the well-known addition theorem

$$P_l[\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\varphi - \varphi')] = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi) \quad (2.7)$$

for the Legendre polynomials (see Ref. 12, p. 68).

3. DIFFERENTIAL OPERATOR MODELS

In this section we construct new models of the representations D_u^+ as classified in Ref. 2, Chap. 5, and use these models and the results of Sec. 1 to obtain special function identities.

The Type B operators

$$J^+ = e^{i\theta}\left(x \frac{\partial}{\partial x} - i \frac{\partial}{\partial \theta} - x\right), \qquad J^- = e^{-i\theta}\left(x \frac{\partial}{\partial x} + i \frac{\partial}{\partial \theta}\right), \quad J^3 = -i \frac{\partial}{\partial \theta} \quad (3.1)$$

and basis functions

$$j_n(x, \theta) = \left(\frac{n!}{\Gamma(n + 2u)}\right)^{1/2} \times x^u L_n^{(2u-1)}(x)\, e^{i(u+n)\theta}, \quad n = 0, 1, 2, \ldots, \quad (3.2)$$

form a model of D_u^+, i.e., they satisfy expressions (1.16). Here $L_n^{(\alpha)}(x)$ is a generalized Laguerre polynomial (see Ref. 7, Vol. 1).

It follows that the functions

$$j_{n,m}(x, \theta) = j_n^{(1)}(x, \theta)\, j_m^{(2)}(x, \theta) = \left(\frac{n!}{\Gamma(n + 2u)}\right)^{1/2} \times x^u L_n^{(2u-1)}(ax) e^{i(u+n)\theta} \left(\frac{m!}{\Gamma(m + 2v)}\right)^{1/2} \times x^v L_m^{(2v-1)}([1 - a]x)\, e^{i(v+m)\theta}, \quad n, m = 0, 1, 2, \ldots, \quad (3.3)$$

and the operators (3.1) define a model of $D_u^+ \otimes D_v^+$ where a, u, v are real constants such that $u > 0, v > 0$. Indeed,

$$\begin{aligned} J^+(j_n^{(1)} j_m^{(2)}) &= j_n^{(1)} e^{i\theta}\left(x \frac{\partial}{\partial x} - i \frac{\partial}{\partial \theta} - (1 - a)x\right) j_m^{(2)} + j_m^{(2)} e^{i\theta}\left(x \frac{\partial}{\partial x} - i \frac{\partial}{\partial \theta} - ax\right) j_n^{(1)} = [(2u + n) \\ &\quad \times (n + 1)]^{1/2} j_{n+1}^{(1)} j_m^{(2)} + [(2v + m)(m + 1)]^{1/2} j_n^{(1)} j_{m+1}^{(2)} \\ &= e^{i\theta}\left(x \frac{\partial}{\partial x} - i \frac{\partial}{\partial \theta} - x\right) j_n^{(1)} j_m^{(2)}, \end{aligned} \quad (3.4)$$

with similar interpretations of J^- and J^3.

We now compute the basis vectors j_h^s, $s, h = 0, 1, 2, \ldots$, corresponding to the Clebsch–Gordan series (1.23). From (1.24) we have

$$j_h^s(x, \theta) = \sum_{n,m=0}^{\infty} E(u, n; v, m \mid s, h)\, j_n^{(1)}(x, \theta)\, j_m^{(2)}(x, \theta). \quad (3.5)$$

[Recall that $E(\cdot) = 0$ unless $n + m = s + h$.] On the other hand, we can compute the j_h^s directly for this model by using the fact that they satisfy (1.16) with $n = h$, $u = u + v + s$. Indeed, from (3.2),

$$j_h^s(x, \theta) = c_s \left(\frac{h!}{\Gamma(h + 2u + 2v + 2s)}\right)^{1/2} \times x^{u+v+s} L_h^{(2u+2v+2s-1)}(x)\, e^{i(u+v+s+h)\theta}, \quad (3.6)$$

where c_s is a constant. To determine c_s, we equate (3.5) and (3.6) in the case $h = 0$. In this special case, (3.5) simplifies to

$$j_0^s = \left(\frac{s!\,\Gamma(2u + 2v + s - 1)\,\Gamma(2u + s)\Gamma(2v + s)}{\Gamma(2u + 2v + 2s - 1)}\right)^{1/2} \times \sum_{n=0}^{\infty} (-1)^n [(s - n)!\, n!\, \Gamma(2u + n)\Gamma(2v + s - n)]^{-1/2} \times j_n^{(1)} j_{s-n}^{(2)}. \quad (3.7)$$

Substituting (3.3) and (3.6) into this expression, comparing coefficients of x^{u+v+s} on both sides of the resulting equation we find

$$c_s = \frac{(a - 1)^s}{\Gamma(2u)} \left(\frac{\Gamma(2u + s)\Gamma(2u + 2v + s - 1)}{s!\,\Gamma(2v + s)} \times (2u + 2v + 2s - 1)\right)^{1/2} F\left(1 - s - 2v, -s; 2u; \frac{a}{a - 1}\right), \quad s = 0, 1, 2, \ldots, \quad (3.8)$$

where $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function (see Ref. 7, Vol. 1) Note that c_s is a polynomial of order s in a. The final identity is obtained by substituting (3.3), (3.6), and (3.8) into (3.5).

For $a = 0$ this identity simplifies to

$$x^s L_h^{(2u+2v+2s-1)}(x) (-1)^s \times \left(\frac{h!\,\Gamma(2u + s)\Gamma(2u + 2v + s - 1)(2u + 2v + 2s - 1)}{x!\,\Gamma(2v + s)\Gamma(2u + 2v + 2s + h)}\right)^{1/2}$$

$$= \sum_{n,m} E(u,n;v,m\,|\,s,h)\left(\frac{m!\,\Gamma(2u+n)}{n!\,\Gamma(2v+m)}\right)^{1/2} L_m^{(2v-1)}(x). \tag{3.9}$$

A second model of D_u^+ is defined by the operators

$$J^{\pm} = e^{\pm i\theta}\left((z^2-1)^{1/2}\frac{\partial}{\partial z} \pm \frac{iz}{(z^2-1)^{1/2}}\frac{\partial}{\partial\theta} \mp \frac{r}{(z^2-1)^{1/2}}\right),$$
$$J^3 = -i\frac{\partial}{\partial\theta}, \tag{3.10}$$

and basis functions

$$j_n^{(z,\theta)} = [\Gamma(2u+n)n!]^{-1/2}\mathfrak{P}_{-u}^{-r,-u-n}(z)e^{i(u+n)\theta},$$
$$n = 0,1,2,\ldots,\quad z = \cosh\rho, \tag{3.11}$$

where r is an arbitrary constant. It follows that the functions

$$j_{n,m}^{(z,\theta)} = j_n(z,\theta)j_m(z,\theta) = [\Gamma(2u+n)n!]^{-1/2}$$
$$\times \mathfrak{P}_{-u}^{-r_1,-u-n}(z)\,e^{i(u+n)\theta}[\Gamma(2v+m)m!]^{-1/2}$$
$$\times \mathfrak{P}_{-v}^{-r_2,-v-m}(z)\,e^{i(v+m)\theta},$$
$$r_1 + r_2 = r,\quad n,m = 0,1,2,\cdots, \tag{3.12}$$

and the operators (3.10) define a realization of $D_u^+ \otimes D_v^+$. Indeed, writing $J^+ = \tilde{J}^+ - e^{i\theta}r(z^2-1)^{-1/2}$, we have

$$J^+(j_nj_m) = j_n(\tilde{J}^+ - e^{i\theta}r_2(z^2-1)^{-1/2})j_m$$
$$+ j_m(\tilde{J}^+ - e^{i\theta}r_1(z^2-1)^{-1/2})j_n$$

with a similar interpretation of J^- and J^3. From (3.11) we see that the basis functions j_h^s corresponding to the Clebsch-Gordan series (1.23) must be

$$j_h^s = c_s[\Gamma(2u+2v+2s+h)h!]^{-1/2}$$
$$\times \mathfrak{P}_{-u-v-s}^{-r,-u-v-s-h}(z)\,e^{i(u+v+s+h)\theta}. \tag{3.13}$$

To compute the constant c_s we substitute (3.12) and (3.13), $h = 0$, into (3.7). Canceling the common factor

$$\left(\frac{z-1}{2}\right)^{-(u+v+s+r)/2}\left(\frac{z+1}{2}\right)^{-(u+v+s-r)/2}$$

on both sides of the equation and setting $z = 1$, we obtain

$$c_s = \frac{\Gamma(-u-v-s-r+1)}{\Gamma(2u)\Gamma(-r_1-u+1)\Gamma(-r_2-v-s-1)}$$
$$\times\left(\frac{\Gamma(2u+s)\Gamma(2u+2v+s-1)}{s!\,\Gamma(2v+s)}\right.$$
$$\left.\times(2u+2v+2s-1)\right)^{1/2}$$
$$\times\,{}_3F_2(-s,-2v-s+1,u+r_1;$$
$$2u,-r_2-v-s-1;1). \tag{3.14}$$

Our final identity is obtained by substituting (3.12), (3.13), and (3.14) into (3.5). In the very special case $r_1 = r_2 = s = 0$ this identity reduces to

$$\frac{\Gamma(-u-v+1)}{\Gamma(1-u)\Gamma(-v-1)}\left(\frac{\Gamma(2u+2v)}{\Gamma(2u)\Gamma(2v)\Gamma(2u+2v+h)h!}\right)^{1/2}$$
$$\times\mathfrak{P}_{-u-v-s}^{u+v+h}(z) = \sum_{n=0}^{h} E(u,n;v,h-n\,|\,0,h)$$
$$\times\frac{\mathfrak{P}_{-u}^{u+n}(z)\,\mathfrak{P}_{-v}^{v+s-n}(z)}{[\Gamma(2u+n)\Gamma(2v+s-n)n!(s-n)!]^{1/2}}, \tag{3.15}$$

where $\mathfrak{P}_v^{\mu}(z)$ is a Legendre function of the first kind (see Ref. 7, Vol. 1). [In this special case $s = 0$, the coefficients $E(\cdot)$ are easy to evaluate explicitly.]

For our next model of D_u^+ we choose operators

$$J^{\pm} = e^{\pm i\theta}\left((x^2-1)\frac{\partial}{\partial x} \mp ix\frac{\partial}{\partial\theta}\right),$$
$$J^3 = -\frac{\partial}{\partial\theta}, \tag{3.16}$$

and basis functions

$$j_n(x,\theta) = \left(\frac{n!}{\Gamma(2u+n)}\right)^{1/2}(x^2-1)^{u/2}C_n^u(x)e^{i(u+n)\theta},$$
$$n = 0,1,2,\ldots, \tag{3.17}$$

where $C_n^u(x)$ is a Gegenbauer polynomial (see Ref. 7, Vol. 2). It follows that a model of $D_u^+ \otimes D_v^+$ is determined by the operators (3.16) and basis functions

$$j_{n,m}(x,\theta) = j_n(x,\theta)j_m(x,\theta) = \left(\frac{n!\,m!}{\Gamma(2u+n)\Gamma(2v+m)}\right)^{1/2}$$
$$\times(x^2-1)^{(u+v)/2}C_n^u(x)C_m^v(x)\,e^{i(u+v+n+m)\theta},$$
$$n,m = 0,1,2,\cdots. \tag{3.18}$$

The basis functions j_h^s transforming according to D_{u+v+s}^+ can be obtained directly from (3.17):

$$j_h^s(x,\theta) = c_s\left(\frac{h!}{\Gamma(2u+2v+2s+h)}\right)^{1/2}(x^2-1)^{(u+v+s)/2}$$
$$\times C^{u+v+s}(x)e^{i(u+v+s+h)\theta},\quad s,h = 0,1,2,\cdots. \tag{3.19}$$

To determine the constants c_s we substitute (3.18) and (3.19) into (3.7) and divide through by the common factor $(x^2-1)^{(u+v)/2}$. If s is odd, the right-hand side of the resulting expression is odd and the left-hand side is even. Thus

$$c_s = 0,\quad s \text{ odd}. \tag{3.20}$$

If s is even, we compare coefficients of x^s on both sides of the equation to obtain

$$c_s = \left(\frac{\Gamma(2u+s)(2u+2v+2s-1)}{s!\,\Gamma(2v+s)}\right)^{1/2}\frac{2^s\Gamma(v+s)}{\Gamma(2u)\Gamma(v)}$$
$$\times\,{}_3F_2(u,-s,-2v-s+1;2u,-v-s+1;1),$$
$$s \text{ even}. \tag{3.21}$$

Substituting (3.18)–(3.21) into (3.5), we obtain our general identity. In the special case $s = 0$, this formula reduces to

$$\left(\frac{h!(2u+2v-1)}{\Gamma(2u+2v+h)\Gamma(2u)\Gamma(2v)}\right)^{1/2} C_h^{u+v}(x)$$
$$= \sum_{n=0}^{h} E(u,n;v,h-n\,|\,0,h)$$
$$\times\left(\frac{n!(h-n)!}{\Gamma(2u+n)\Gamma(2v+h-n)}\right)^{1/2} C_n^u(x)C_{h-n}^v(x), \tag{3.22}$$

where the coefficients $E(\cdot)$ can be simply evaluated.

Our next model of D_u^+ is defined by operators

$$J^+ = e^{i\theta}\left(x(1-x)\frac{\partial}{\partial x} - i\frac{\partial}{\partial\theta} - qx\right),$$
$$J^- = e^{i\theta}\left(x\frac{\partial}{\partial x} + i\frac{\partial}{\partial\theta}\right), \qquad (3.23)$$
$$J^3 = -i\frac{\partial}{\partial\theta},$$

and basis functions

$$j_n(x,\theta) = \left(\frac{\Gamma(2u+n)}{n!}\right)^{1/2} x^u F(-n, u+q; 2u; x) e^{i(u+n)\theta},$$
$$n = 0,1,2,\ldots, \qquad (3.24)$$

where q is a constant and $F(\alpha,\beta;\gamma,x)$ is a hypergeometric function (see Ref. 7, Vol. 1). It follows easily that the operators (3.23) and basis functions

$$j_{n,m}(x,\theta) = \left(\frac{\Gamma(2u+n)\Gamma(2v+m)}{n!m!}\right)^{1/2} x^{u+v}$$
$$\times F(-n, u+q_1; 2u; x)$$
$$\times F(-m, v+q_2; 2v; x)\, e^{i(u+v+n+m)\theta},$$
$$n, m = 0,1,2,\ldots, \qquad (3.25)$$

define a model of $D_u^+ \otimes D_v^+$ where $q = q_1 + q_2$. From (3.24) we see that the basis vectors j_h^s transforming according to D_{u+v+s}^+ are given by

$$j_h^s(x,\theta) = c_s\left(\frac{\Gamma(2u+2v+2s+h)}{h!}\right)^{1/2} x^{u+v+s}$$
$$\times F(-h, u+v+s+q; 2u+2v+2s; x)$$
$$\times e^{i(u+v+s+h)\theta}, \quad s,h = 0,1,2,\cdots. \qquad (3.26)$$

To compute the constants c_s we substitute (3.25) and (3.26) into (3.7) and equate coefficients of x^{u+v+s} on both sides of the resulting expression. We find

$$c_s = \left(\frac{\Gamma(2u+2v+s-1)\Gamma(2u+s)}{s!\,(2u+2v+2s-1)\Gamma(2v+s)}\right)^{1/2}$$
$$\times \frac{\Gamma(2v)\Gamma(v+q_2+s)}{\Gamma(2u+2v+2s-1)\Gamma(v+q_2)}\, {}_3F_2(-s, u+q_1,$$
$$-2v-s+1; 2u, -v-q_2-s+1; 1). \qquad (3.27)$$

Substituting (3.25)–(3.27) into (3.5), we obtain our general identity. In the special case $s = 0$, it reduces to

$$\left(\frac{\Gamma(2u)\Gamma(2v)\Gamma(2u+2v+h)}{\Gamma(2u+2v)\,h!}\right)^{1/2}$$
$$\times F(-h, u+v+q; 2u+2v; x)$$
$$= \sum_{n=0}^{h} E(u,n,;v,h-n|0,h)$$
$$\times\left(\frac{\Gamma(2u+n)\Gamma(2v+h-n)}{n!(h-n)!}\right)^{1/2}$$
$$\times F(-n, u+q_1; 2u; x)$$
$$\times F(-h+n, v+q_2; 2v; x). \qquad (3.28)$$

The reader can discover other interesting special cases of this general identity by varying q_1 and q_2, e.g., set $q_1 = -u$.

4. A DIFFERENCE OPERATOR MODEL

As shown in Refs. 3 or 13, the operators

$$J^+ = e^{i\theta}\left((x-1)L - x - i\frac{\partial}{\partial\theta} + q\right),$$
$$J^- = e^{-i\theta}\left(-(x+r)E + x + i\frac{\partial}{\partial\theta} + r + q - 1\right), \qquad (4.1)$$
$$J^3 = -i\frac{\partial}{\partial\theta}$$

and basis functions

$$j_n(x,\theta) = \left(\frac{\Gamma(2u+n)}{n!}\right)^{1/2}$$
$$\times \frac{\Gamma(u-q+1)\Gamma(x+r+q-u-n-1)}{\Gamma(x+r-n)}$$
$$\times {}_3F_2(-n, u-q-r+1, u-q+1; 2u,$$
$$x+r-n; 1)\, e^{i(u+n)\theta}, \quad n = 0,1,2,\cdots \qquad (4.2)$$

form a model of D_u^+, where r, q are constants and $Ef(x,\theta) = f(x+1,\theta)$, $Lf(x,\theta) = f(x-1,\theta)$. Furthermore, the operators

$$J^+ = e^{i\theta}\left(-i\frac{\partial}{\partial\theta} + v\right), \qquad J^- = e^{-i\theta}\ i\frac{\partial}{\partial\theta} + v,$$
$$J^3 = -i\frac{\partial}{\partial\theta}, \qquad (4.3)$$

and basis functions

$$j_m'(\theta) = \sqrt{\frac{\Gamma(2v+m)}{m!}}\ e^{i(v+m)\theta}, \quad m = 0,1,2,\ldots, \qquad (4.4)$$

form a model of D_v^+. Thus the operators (4.1) and basis functions

$$j_{n,m}(x,\theta) = j_n(x,\theta)j_m'(\theta) = \left(\frac{\Gamma(2u+n)\Gamma(2v+m)}{n!m!}\right)^{1/2}$$
$$\times \frac{\Gamma(u-q+v+1)\Gamma(x+r+q-u-v-n-1)}{\Gamma(x+r-n)}$$
$$\times {}_3F_2(-n, u+v-q-r+1, u+v-q+1;$$
$$2u, x+r-n; 1)\, e^{i(u+v+n+m)\theta}, \quad n,m = 0,1,2,\cdots, \qquad (4.5)$$

define a model of $D_u^+ \otimes D_v^+$. The basis functions $j_h^s(x,\theta)$ transforming according to D_{u+v+s}^+ can be obtained immediately from (4.2) with u replaced by $u+v+s$ and $n = h$:

$$j_h^s(x,\theta) = c_s\left(\frac{\Gamma(2u+2v+2s+h)}{h!}\right)^{1/2}$$
$$\times \frac{\Gamma(u+v+s-q+1)}{\Gamma(x+r-h)}$$
$$\times \Gamma(x+r+q-u-v-s-h-1)$$
$$\times {}_3F_2\,(-h, u+v+s-q-r+1, u+v+s$$
$$-q+1;\ 2u+2v+2s, x+r-h; 1)$$
$$\times e^{i(u+v+s+h)\theta}. \qquad (4.6)$$

To compute the constants c_s, we substitute (4.5) and (4.6) into (3.7) and set $x = -r + 1$. We can then sum the right-hand side to obtain

$$c_s = \left(\frac{\Gamma(2u + 2v + s - 1)\Gamma(2u + s)\Gamma(2v + s)}{s!\,(2u + 2v + 2s - 1)}\right)^{1/2}$$

$$\times \frac{(-1)^s}{\Gamma(2u + 2v + 2s - 1)}$$

$$\times {}_3F_2(-s, u + v - q - r + 1, u + v - q + 1; \times 2u, u + v - q + 1; 1). \qquad (4.7)$$

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Spectrum Generating Algebras and Symmetries in Mechanics. I

Robert Hermann*

Department of Mathematics, Rutgers University, New Brunswick, New Jersey

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Certain differential-geometric and Lie group theoretic facts that are useful in the systematic study and search for spectrum generating algebras are presented.

1. INTRODUCTION

Dothan has pointed out[1] several ideas and possible directions of research in connection with the "spectrum generating Lie algebras" of quantum mechanics. Typically, such ideas also have analogs in classical mechanics. Since the problems in classical mechanics often have a geometric foundation, one finds interconnections between geometry and Lie group theory, quantum mechanics and elementary particle physics. The aim of this paper is to survey more extensively some of these links than was possible in Dothan's paper.

2. POISSON BRACKETS STRUCTURES AND CANONICAL TRANSFORMATIONS ON MANIFOLD

We adopt Ref. 2 as a basic reference for the ideas and notations of differential geometry on manifolds. Let M be a manifold of even dimension, with a closed two-differential form ω of maximal rank on M. A diffeomorphism $\phi: M \to M$ is a *canonical transformation* if ϕ preserves the form ω, i.e., $\phi^*(\omega) = \omega$. A vector field $X \in V(M)$ defines an *infinitesimal canonical* transformation if

$$X(\omega) = 0 \qquad (2.1)$$

$[X(\omega)$ denotes the Lie derivative[2] of the form ω by the vector field $X]$. The set of vector fields X satisfying (2.1) forms a Lie algebra [under Jacobi bracket $(X, Y) \to [X, Y]]$ of vector fields, that we denote by $V(\omega)$. It may be thought of as the "Lie algebra" of the group of canonical transformations.

Let $F(M)$ denote the C^∞, real-valued functions on M. The form ω defines a Lie algebra structure $(f_1, f_2) \to \{f_1, f_2\}$ called the *Poisson bracket*. To define it, for $f \in F(M)$, let X_f be the vector field such that

$$df = X_f \lrcorner\, \omega. \qquad (2.2)$$

Set

$$\{f_1, f_2\} = -X_{f_1}(f_2) \qquad (2.3)$$

for $f_1, f_2 \in F(M)$.

Then, one can prove the following results.

The bracket $\{\ ,\ \}$ defined by 2.3 makes $F(M)$ into a Lie algebra. (2.4)

The mapping $f \to X_f$ is a Lie algebra homomorphism of $F(M)$ into $V(\omega)$. (2.5)

The kernel of this homomorphism consists of the constant functions on M.

To recover the classical expression for Poisson bracket to be found in all mechanics books, suppose (p_i, q_i), $1 \le i, j \le m$ is a coordinate system for M such that

$$\omega = dp_i \wedge dq_i.$$

Then, for $f \in F(M)$,

$$X_f = -\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} + \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}. \qquad (2.6)$$

Given $h \in F(M)$, the integral curves of the vector field X_h are the solutions of Hamilton's equations, with h the Hamiltonian.[2] Thus, if h is the function that represents the total energy of the mechanical system, a basic problem is to study these integral curves, i.e., to study the one-parameter group of canonical transformations generated by X_h. Now, in Ref. 2 certain general insights of the "Lie theory" of ordinary differential equations have been explained. In particular, they apply to the problem of finding the integral curves of X_h.

Definition: A function $f \in F(M)$ is a *symmetry* of h if

$$\{f, h\} = 0. \qquad (2.7)$$

If f satisfies (2.7), then it follows from (2.5) that $[X_f, X_h] = 0$, hence that the one-parameter group generated by X_f and X_h commute. The aim of the theory of "spectrum generating algebras", stated in rather vague terms, is to study Lie subalgebras of $F(M)$, whose elements f satisfy commutation relations