cations, edited by E. M. Loebl (Academic, New York, 1968).
4 H. Weyl, *The Classical Groups* (Princeton U.P., Princeton, N.J., 1946); H. Weyl, *Gruppentheorie und Quantenmechanik* (Hirzel-Verlag, Leibzig, 1931).
5 L. C. Biedenharn, J. Math. Phys. **4**, 436 (1963); G. E. Baird and L. C. Biedenharn, J. Math. Phys. **5**, 1723, 1730 (1964); **6**, 1847 (1965).
6 G. E. Baird and L. C. Biedenharn, J. Math. Phys. **4**, 1449 (1963).
7 V. Bargmann and M. Moshinsky, Nucl. Phys. **23**, 177 (1961); M. Moshinsky, J. Math. Phys. **4**, 1128 (1963).
8 J. Nagel and M. Moshinsky, J. Math. Phys. **6**, 682 (1965).
9 I. M. Gel'fand and M. L. Zetlin, Dokl. Akad. Nauk SSSR **71**, 825 (1950).
10 M. Moshinsky, J. Math. Phys. **7**, 691 (1966).
11 M. Ciftan and L. C. Biedenharn, J. Math. Phys. **10**, 221 (1969).
12 M. Ciftan, J. Math. Phys. **10**, 1635 (1969).
13 P. A. M. Dirac, *Quantum Mechanics* (Oxford U.P., London, 1935).
14 P. O. Löwdin, Rev. Mod. Phys. **39**, 254 (1967); P. O. Löwdin and O. Goscinsky, Research Report No. 243, Quantum Chemistry Group (Uppsala, Sweden).
15 W. A. Goddard III, Phys. Rev. **157**, 81 (1967); **169**, 120 (1968); J. Chem. Phys. **48**, 1008, 5337 (1968).
16 L. C. Biedenharn, *Boulder Lectures in Theoretical Physics* Vol. V, edited by W. E. Brittin, B. W. Downs, and J. Downs (Interscience, New York, 1963).
17 P. Jordan, Z. Physik **94** (1925).
18 J. Schwinger (unpublished 1952), reprinted in *Selected Papers in Quantum Theory of Angular Momentum*, edited by L. C. Biedenharn and H. van Dam (Academic, New York, 1965).
19 S. S. Schweber, *Relativistic Quantum Field Theory* (Harper and Row, New York, 1962).
20 In a different context H. Han and L. C. Biedenharn [Phys. Rev. Letters **24**, 118 (1970)] give a formula for the diagonal matrix elements of the operators $\sum_{i<j} U(ij)$ and $\sum_{i<j<k} U(ijk)$ in $U(3)$ multiplets.
21 D. E. Rutherford, *Substitutional Analysis* (Edinburgh U.P., Edinburgh, 1948); see also Ref. 13.
22 G. de B. Robinson, *Representation Theory of the Symmetric Group* (U. of Toronto Press, Toronto, 1961); H. Boerner, *Darstellungen von Gruppen* (Springer-Verlag, Berlin, 1967); B. G. Wybourne, *Symmetry Principles and Atomic Spectroscopy* (Interscience, New York, 1970); M. Hamermesh, *Group Theory* (Addison-Wesley, Reading, Mass., 1964).

Lie Theory and the Lauricella Functions F_D

Willard Miller, Jr.
School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455
(Received 2 March 1972)

It is shown that the Lauricella functions F_D in n variables transform as basis vectors corresponding to irreducible representations of the Lie algebra $sl(n+3,\mathbb{C})$. Group representation theory can then be applied to derive addition theorems, transformation formulas, and generating functions for the F_D. It is clear from this analysis that the use of $SL(m,\mathbb{C})$ symmetry in atomic and elementary particle physics will lead inevitably to the remarkable functions F_D.

INTRODUCTION

In a recent paper,[1] Ciftan has shown that the Appell function F_1 arises naturally from a study of the representation theory of the special linear groups. The author proved in Ref. 2 that this was due to the fact that $SL(5,\mathbb{C})$ was the dynamical symmetry group of F_1. Here we generalize this observation by demonstrating that $SL(n+3,\mathbb{C})$ is the dynamical symmetry group of the Lauricella functions F_D in n variables (Recall that F_1 is an F_D with $n=2$). We further show that exploitation of the $SL(n+3,\mathbb{C})$ symmetry yields elegant and simple derivations of addition theorems, transformation formulas, and generating functions for the F_D. It follows from this analysis that the implementation of $SL(m,\mathbb{C})$ symmetry in atomic and particle physics will necessarily lead to the functions F_D.

The methods employed in this paper are rather straightforward generalizations of those employed in Refs. 2 and 3.

1. THE DYNAMICAL SYMMETRY GROUP

The Lauricella function F_D is defined by the series

$$F_D(a;b_1,\dots,b_n;c;x_1,\dots,x_n) = \sum_{m_1\cdots m_j=0}^{\infty} \frac{(a,m_1+\cdots+m_n)(b_1,m_1)\cdots(b_n,m_n)}{(c,m_1+\cdots+m_n)m_1!\cdots m_n!} \times x_1^{m_1}\cdots x_n^{m_n}, \tag{1.1}$$

convergent for $|x_1|<1,\dots,|x_n|<1$.[4,5]

Here,

$$(a,n)=a(a+1)\cdots(a+n-1)=(a)_n, \tag{1.2}$$

and it is assumed that $c\neq 0,-1,-2,\cdots$. We define the following partial differential operators acting on a space of functions of $2n+2$ complex variables, $s,u_1,\dots,u_n,t,x_1,\dots,x_n$:

$$E_\alpha = s\Big(\sum_{j=1}^n x_j\partial_{x_j} + s\partial_s\Big), \quad E_{\alpha\beta_k\gamma} = su_kt\partial_{x_k},$$

$$E_{\beta_k} = u_k(x_k\partial_{x_k} + u_k\partial_{u_k}), \quad E_{-\gamma} = t^{-1}\Big(\sum_{j=1}^n x_j\partial_{x_j} + t\partial_t - 1\Big),$$

$$E_{\alpha\gamma} = st\Big(\sum_{j=1}^n (1-x_j)\partial_{x_j} - s\partial_s\Big),$$

$$E_\gamma = t\Big(\sum_{j=1}^n (1-x_j)\partial_{x_j} + t\partial_t - s\partial_s - \sum_{j=1}^n u_j\partial_{u_j}\Big),$$

$$E_{-\alpha} = s^{-1}\Big(\sum_{j=1}^n x_j(1-x_j)\partial_{x_j} + t\partial_t - s\partial_s - \sum_{j=1}^n x_ju_j\partial_{u_j}\Big),$$

$$E_{-\beta_k} = u_k^{-1}\Big(x_k(1-x_k)\partial_{x_k} + x_k\sum_{j\neq k}(1-x_j)\partial_{x_j} + t\partial_t - x_ks\partial_s - \sum_{h=1}^n u_h\partial_{u_h}\Big),$$

$$E_{\beta_k\gamma} = u_kt\,((x_k-1)\partial_{x_k} + u_k\partial_{u_k}),$$

$$E_{-\alpha,-\gamma} = s^{-1}t^{-1}\Big(\sum_{j=1}^n x_j(1-x_j)\partial_{x_j} - \sum_{j=1}^n x_ju_j\partial_{u_j} + t\partial_t - 1\Big),$$

$$E_{-\alpha,-\beta_k,-\gamma} = s^{-1}u_k^{-1}t^{-1}\Big(\sum_{j=1}^n x_j(x_j-1)\partial_{x_j} - t\partial_t + x_ks\partial_s + \sum_{l\neq k} x_lu_l\partial_{u_l} - x_k + 1\Big),$$

$$E_{-\beta_k,-\gamma} = u_k^{-1}t^{-1}\Big(x_k(x_k-1)\partial_{x_k} + \sum_{j\neq k}(x_k-1)x_j\partial_{x_j} + x_ks\partial_s - t\partial_t + 1\Big),$$

$$E_{\beta_k,-\beta_p} = u_ku_p^{-1}((x_k-x_p)\partial_{x_k} + u_k\partial_{u_k}),$$

$$J_\alpha = s\partial_s - \tfrac{1}{2}t\partial_t, \qquad J_{\beta_k} = u_k\partial_{u_k} - \tfrac{1}{2}t\partial_t + \tfrac{1}{2}\sum_{j\neq k}u_j\partial_{u_j},$$

$$J_\gamma = t\partial_t - \tfrac{1}{2}\left(s\partial_s + \sum_{j=1}^{n} u_j\partial_{u_j} + 1\right), \quad k,p = 1,2,\ldots,n. \tag{1.3}$$

We define basis functions $f_{\alpha,\beta_1,\ldots,\beta_n,\gamma}(s,u_1,\ldots,u_n,t,x_1,\ldots,x_n)$ in this space by

$$f_{\alpha,\beta_1,\ldots,\beta_n,\gamma}(s,u_1,\ldots,u_n,t,x_1,\ldots x_n) = f_{\alpha\beta_j\gamma}(s,u_j,t,x_j) = [\Gamma(\gamma-\alpha)\Gamma(\alpha)/\Gamma(\gamma)]\, F_D(\alpha;\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n) \times s^\alpha u_1^{\beta_1}\cdots u_n^{\beta_n} t^\gamma, \tag{1.4}$$

where $\gamma \neq 0,-1,-2,\cdots$ and $\Gamma(z)$ is the gamma function.[6] The action of the above operators on the basis functions is

$$\begin{aligned}
&E_\alpha f_{\alpha,\beta_j,\gamma} = (\gamma-\alpha-1)f_{\alpha+1,\beta_j,\gamma},\\
&E_{\alpha\beta_k\gamma} f_{\alpha,\beta_j,\gamma} = \beta_k f_{\alpha+1,\hat\beta_k,\gamma+1},\\
&E_{\beta_k} f_{\alpha,\beta_j,\gamma} = \beta_k f_{\alpha,\hat\beta_k,\gamma},\\
&E_{-\gamma} f_{\alpha,\beta_j,\gamma} = (\gamma-\alpha-1)f_{\alpha,\beta_j,\gamma-1},\\
&E_{\alpha\gamma} f_{\alpha,\beta_j,\gamma} = \left(\sum_{j=1}^{n}\beta_j - \gamma\right) f_{\alpha+1,\beta_j,\gamma+1},\\
&E_\gamma f_{\alpha,\beta_j,\gamma} = \left(\gamma - \sum_{j=1}^{n}\beta_j\right) f_{\alpha,\beta_j,\gamma+1},\\
&E_{-\alpha} f_{\alpha,\beta_j,\gamma} = (\alpha-1) f_{\alpha-1,\beta_j,\gamma},\\
&E_{-\beta_k} f_{\alpha,\beta_j,\gamma} = \left(\gamma - \sum_{j=1}^{n}\beta_j\right) f_{\alpha,\tilde\beta_k,\gamma},\\
&E_{\beta_k\gamma} f_{\alpha,\beta_j,\gamma} = \beta_k f_{\alpha,\hat\beta_k,\gamma+1},\\
&E_{-\alpha,-\gamma} f_{\alpha,\beta_j,\gamma} = (\alpha-1) f_{\alpha-1,\beta_j,\gamma-1},\\
&E_{-\alpha,-\beta_k,-\gamma} f_{\alpha,\beta_j,\gamma} = (1-\alpha) f_{\alpha-1,\tilde\beta_k,\gamma-1},\\
&E_{-\beta_k,-\gamma} f_{\alpha,\beta_j,\gamma} = (\alpha-\gamma+1) f_{\alpha,\beta_k,\gamma-1},\\
&E_{\beta_k,-\beta_p} f_{\alpha,\beta_j,\gamma} = \beta_k f_{\alpha,\beta_1,\ldots,\beta_k+1,\ldots,\beta_p-1,\ldots,\beta_n,\gamma},\\
&J_\alpha f_{\alpha,\beta_j,\gamma} = (\alpha - \tfrac{1}{2}\gamma) f_{\alpha,\beta_j,\gamma},\\
&J_{\beta_k} f_{\alpha,\beta_j,\gamma} = \left(\beta_k - \tfrac{1}{2}\gamma + \tfrac{1}{2}\sum_{l\neq k}\beta_l\right) f_{\alpha,\beta_j,\gamma},\\
&J_\alpha f_{\alpha,\beta_j,\gamma} = \left[\gamma - \tfrac{1}{2}\left(\alpha + \sum_{l=1}^{n}\beta_l + 1\right)\right] f_{\alpha,\beta_j,\gamma},\\
&\qquad k,p = 1,2,\ldots,n.
\end{aligned} \tag{1.5}$$

(Here the E operators and the J operators are independent of the parameters α, β_j, γ. The subscripts merely indicate the action of these operators.) The symbols $\hat\beta_k$ and $\tilde\beta_k$ are defined by

$$\begin{aligned}
\hat\beta_k &= \beta_1,\ldots,\beta_{k-1},\beta_k+1,\beta_{k+1},\ldots,\beta_n,\\
\tilde\beta_k &= \beta_1,\ldots,\beta_{k-1},\beta_k-1,\beta_{k+1},\ldots,\beta_n.
\end{aligned} \tag{1.6}$$

Relations (1.5) can be verified by routine computation. Furthermore, it is straightforward to show that the operators (1.3) form a basis for a simple Lie algebra of dimension $(n+3)^2-1$, i.e., a basis for $sl(n+3,\mathbb{C})$.

To determine the group action of $SL(n+3,\mathbb{C})$ induced by the operators (1.3), we note that each of the triplets

$$\begin{aligned}
&\{J^+,J^-,J^3\} \equiv \{E_\alpha, E_{-\alpha}, J_\alpha\},\\
&\{E_{\beta_k}, E_{-\beta_k}, J_{\beta_k}\}, \quad \{E_\gamma, E_{-\gamma}, J_\gamma\},\\
&\{E_{\alpha\beta_k\gamma}, E_{-\alpha,-\beta_k,-\gamma}, J_\alpha + J_{\beta_k} + J_\gamma\},\\
&\{E_{\alpha\gamma}, E_{-\alpha,-\gamma}, J_\alpha + J_\gamma\}, \quad \{E_{\beta_k\gamma}, E_{-\beta_k,-\gamma}, J_{\beta_k} + J_\gamma\},\\
&\{E_{\beta_l,-\beta_p}, E_{-\beta_l,\beta_p}, J_{\beta_l} - J_{\beta_p}\}, \quad k = 1,\ldots,n,\ 1\le l<p\le n,
\end{aligned} \tag{1.7}$$

satisfies the commutation relations

$$[J^3,J^\pm] = \pm J^\pm, \quad [J^+,J^-] = 2J^3, \tag{1.8}$$

and forms a basis for a subalgebra of $sl(n+3,\mathbb{C})$ isomorphic to $sl(2,\mathbb{C})$. Furthermore, each triplet generates a local Lie subgroup of $SL(n+3,\mathbb{C})$ isomorphic to $SL(2,\mathbb{C})$ and the subgroups so obtained suffice to generate the full group action of $SL(n+3,\mathbb{C})$.

We pass from the Lie algebra action generated by $\{J^+,J^-,J^3\}$ to the group action via the relation

$$\mathbf{T}(A) = \exp[-(b/d)J^+]\,\exp(-cdJ^-)\,\exp(\tau J^3), \quad e^{\tau/2} = d^{-1}, \tag{1.9}$$

where

$$A = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in SL(2,\mathbb{C}), \quad ad-bc=1; \tag{1.10}$$

see Ref. 7. We find that the triplet $\{E_\alpha, E_{-\alpha}, J_\alpha\}$ generates the group action

$$\mathbf{T}_1(A)f(s,u_1,\ldots,u_n,t,x_1,\ldots,x_n) = f\left(\frac{as+c}{d+bs}, \frac{u_j(as+c)}{as+c(1-x_j)}, \frac{ts}{as+c}, \frac{x_js}{(d+bs)(as-cx_j+c)}\right) \tag{1.11}$$

and the triplet $\{E_{\beta_k}, E_{-\beta_k}, J_{\beta_k}\}$

$$\mathbf{T}_{2,k}(A)f(s,u_j,u_k,t,x_j,x_k) = f\left(\frac{s(au_k+c)}{au_k+c(1-x_k)}, \frac{u_j}{u_k}(au_k+c), \frac{au_k+c}{d+bu_k}, \frac{u_kt}{au_k+c}, \frac{au_kx_j+c(x_j-x_k)}{au_k+c(1-x_k)}, \frac{x_ku_k}{(d+bu_k)(au_k-cx_k+c)}\right). \tag{1.12}$$

In (1.11) the index j runs from 1 to n, but in (1.12) j runs from 1 to n excluding k. The triplet $\{E_\gamma, E_{-\gamma}, J_\gamma\}$ generates

$$\mathbf{T}_3(A)f(s,u_j,t,x_j) = \left(a+\frac{c}{t}\right)^{-1} f\left(s(d+bt), u_j(d+bt), \frac{at+c}{d+bt}, [dx_j - bt(1-x_j)]\left(a+\frac{c}{t}\right)\right), \tag{1.13}$$

the triplet $\{E_{\alpha\beta_k\gamma}, E_{-\alpha,-\beta_k,-\gamma}, J_\alpha + J_{\beta_k} + J_\gamma\}$ generates

$$\mathbf{T}_{4,k}(A)f(s,u_j,u_k,t,x_j,x_k) = \left(a+\frac{c(1-x_k)}{u_kt_s}\right)^{-1} f\left(as-\frac{cx_k}{u_kt}, u_j\left(\frac{asu_kt-cx_k}{asu_kt+c(x_j-x_k)}\right),\right.$$

$$au_k - \frac{cx_k}{st}, t\left(\frac{asu_kt + c(1-x_k)}{asu_kt - cx_k}\right), x_j\left(\frac{asu_kt + c(1-x_k)}{asu_kt + c(x_j - x_k)}\right),$$
$$(x_kd - bsu_kt)(a + c - cx_k)\Bigg), \quad (1.14)$$

the triplet $\{E_{\alpha\gamma}, E_{-\alpha,-\gamma}, J_\alpha + J_\gamma\}$ generates

$$\mathbf{T}_5(A)f(s,u_j,t,x_j) = \left(a - \frac{c}{st}\right)^{-1} f\left(\frac{s}{d-bst}, \frac{u_jst}{ast - cx_j}, at - \frac{c}{s}, \frac{(dx_j - bst)(ast - c)}{(ast - cx_j)(d - bst)}\right), \quad (1.15)$$

the triplet $\{E_{\beta_k\gamma}, E_{-\beta_k,-\gamma}, J_{\beta_k} + J_\gamma\}$ generates

$$\mathbf{T}_{6,k}(A)f(s,u_j,u_k,t,x_j,x_k) = \left(a + \frac{c}{ut}\right)^{-1} f\left(\frac{su_kt}{au_kt + cx_k}, u_j, \frac{u_k}{d + bu_kt}, at + \frac{c}{u_k}, \frac{x_j(au_kt + c)}{au_kt + cx_k}, \frac{(dx_k + bu_kt)(au_kt + c)}{(d + bu_kt)(au_kt + cx_k)}\right), \quad (1.16)$$

and the triplet $\{E_{\beta_k,-\beta_p}, E_{-\beta_k,\beta_p}, J_{\beta_k} - J_{\beta_p}\}$ generates

$$\mathbf{T}_{7,k,p}(A)f(s,u_j,u_p,t,x_j,x_k,x_p) = f\left(s, u_j, \frac{u_ku_p}{du_p + bu_k}, \frac{u_pu_k}{au_k + cu_p}, t, x_j, \frac{dx_ku_p + bx_pu_k}{du_p + bu_k}, \frac{ax_pu_k + cx_ku_p}{au_k + cu_p}\right),$$
$$1 \le k < p \le n. \quad (1.17)$$

Let

$$C_k = E_\alpha E_{\beta_k} - E_{\alpha\beta_k\gamma}E_{-\gamma}, \quad 1 \le k \le n. \quad (1.18)$$

It is straightforward to check that the solution f of the simultaneous equations

$$J_\alpha f = (\alpha - \tfrac{1}{2}\gamma)f, \quad J_{\beta_k} f = \left(\beta_k - \tfrac{1}{2}\gamma + \tfrac{1}{2}\sum_{l \ne k} \beta_l\right)f,$$
$$J_\gamma f = \left[\gamma - \tfrac{1}{2}\left(\alpha + \sum_{l=1}^n \beta_l + 1\right)\right]f,$$
$$C_kf = 0, \; k = 1, \dots, n, \quad (1.19)$$

analytic in a neighborhood of $x_1 = x_2 = \cdots = x_n = 0$ is

$$f = F_D(\alpha; \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n)s^\alpha u_1^{\beta_1} \cdots u_n^{\beta_n} t^\alpha, \quad (1.20)$$

unique to within a multiplicative constant. In fact the first $n + 2$ equations imply

$$f = F(x_1, \dots, x_n)s^\alpha u_1^{\beta_1} \cdots u_n^{\beta_n} t^\gamma$$

and the last n imply

$$\left[\left(\sum_{j=1}^n x_j\partial_{x_j} + \alpha\right)(x_k\partial_{x_k} + \beta_k) - \partial_{x_k}\left(\sum_{j=1}^n x_j\partial_{x_j} + \gamma - 1\right)\right] F = 0, \quad k = 1, \dots, n \quad (1.21)$$

which are the partial differential equations for F_D.[4] The operators C_k do not commute with all the elements of $sl(n + 3, \mathbb{C})$, but each such element maps a solution f of $C_kf = 0$, $k = 1, \dots, n$, into another solution. It follows that the operators $\mathbf{T}_j(A)$ also map solutions into solutions. Furthermore, if $f(s, u_j, t, x_j)$ is a solution of $C_kf = 0$, $k = 1, \dots, n$, which has a Laurent expansion

$$f = \sum_{\alpha,\beta_j,\gamma} g_{\alpha\beta_j\gamma}(x_j)s^\alpha u_1^{\beta_1} \cdots u_n^{\beta_n} t^\gamma, \quad (1.22)$$

and if f is analytic at $x_1 = x_2 = \cdots = x_n = 0$, then it follows from the above remarks that

$$g_{\alpha\beta_j\gamma} = k(\alpha\beta_j\gamma)F_D(\alpha; \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n), \quad (1.23)$$

where $k(\alpha\beta_j\gamma)$ is a constant.

Let $\alpha^0, \beta_j^0, \gamma^0$, $1 \le j \le n$, be fixed complex numbers, not integers, and let $\alpha = \alpha^0 + h$, $\beta_j = \beta_j^0 + n_j$, $\gamma = \gamma^0 + m$, where b, n_j, m run over all integers. The basis vectors $\{f_{\alpha\beta_j\gamma}\}$, (1.4), and operators (1.3) define an infinite-dimensional irreducible representation $P(\alpha^0, \beta_j^0, \gamma^0)$ of $sl(n + 3, \mathbb{C})$. Using operators (1.10)–(1.17), we can extend this Lie algebra representation to a local group representation of $SL(n + 3, \mathbb{C})$.

In order to compute the matrix elements of this group representation with respect to the basis functions $\{f_{\alpha\beta_j\gamma}\}$, it is useful to consider the following simple realization of $\rho(\alpha^0, \beta_j^0, \gamma^0)$ in terms of differential operators in $n + 2$ complex variables: $s, u_1, \dots, u_n, t$. The basis functions in this new model are

$$f_{\alpha\beta_j\gamma}(s, u_1, \dots, u_n, t) = s^\alpha u_1^{\beta_1} \cdots u_n^{\beta_n} t^\gamma \quad (1.24)$$

and the Lie derivatives are

$$E_\alpha = s(t\partial_t - s\partial_s - 1), \quad E_{\alpha\beta_k\gamma} = su_k^2 t\partial_{u_k},$$
$$E_{\beta_k} = u_k^2\partial_{u_k}, \quad E_{-\gamma} = t^{-1}(t\partial_t - s\partial_s - 1),$$
$$E_{\alpha\gamma} = s\left(\sum_{j=1}^n u_j\partial_{u_j} - t\partial_t\right), \quad E_\gamma = t\left(t\partial_t - \sum_{j=1}^n u_j\partial_{u_j}\right),$$
$$E_{-\alpha} = s^{-1}(s\partial_s - 1), \quad E_{-\beta_k} = u_k^{-1}\left(t\partial_t - \sum_{j=1}^n u_j\partial_{u_j}\right),$$
$$E_{\beta_k\gamma} = u_k^2 t\partial_{u_k}, \quad E_{-\alpha,-\gamma} = s^{-1}t^{-1}(s\partial_s - 1),$$
$$E_{-\alpha,-\beta_k,-\gamma} = s^{-1}u_k^{-1}t^{-1}(1 - s\partial_s),$$
$$E_{-\beta_k,-\gamma} = u_k^{-1}t^{-1}(s\partial_s - t\partial_t + 1),$$
$$E_{\beta_k,-\beta_p} = u_k^2u_p^{-1}\partial_{u_k}, \quad J_\alpha = s\partial_s - \tfrac{1}{2}t\partial_t,$$
$$J_{\beta_k} = u_k\partial_{u_k} - \tfrac{1}{2}t\partial_t + \tfrac{1}{2}\sum_{l \ne k} u_l\partial_{u_k},$$
$$J_\gamma = t\delta_t - \tfrac{1}{2}\left(s\partial_s + \sum_{l=1}^n u_l\partial_u + 1\right). \quad (1.25)$$

As is simple to verify, these operators and basis functions satisfy relations (1.5), so that they determine a model of $\rho(\alpha^0, \beta_j^0, \gamma^0)$. We extend this model to encompass the group action by computing the operators $\mathbf{T}_j(A)$ analogous to (1.11)–(1.17):

$$\mathbf{T}_1(A)f(s, u_j, t) = (d - bs)^{-1}\left(a - \frac{c}{s}\right)^{-1} f\left(\frac{as - c}{d - bs}, u_j, t(d - bs)\right),$$

$$\mathbf{T}_{2,k}(A)f(s,u_j,u_k,t) = f\left(s, u_j\left(a + \frac{c}{u_k}\right), \frac{au_k + c}{d + bu_k}, \frac{tu_k}{au_k + c}\right),$$

$$\mathbf{T}_3(A)f(s,u_j,t) = \left(a + \frac{c}{t}\right)^{-1} f\left(\frac{st}{at+c}, u_j(d+bt), \frac{at+c}{d+bt}\right),$$

$$\mathbf{T}_{4,k}(A)f(s,u_j,u_k,t) = \left(a + \frac{c}{su_k t}\right)^{-1} f\left(as + \frac{c}{u_k t}, u_j, \frac{u_k}{d + bu_k st}, t\right),$$

$$\mathbf{T}_5(A)f(s,u_j,t) = \left(a - \frac{c}{st}\right)^{-1} f\left(as - \frac{c}{t}, u_j(d - bst), \frac{t}{d - bst}\right),$$

$$\mathbf{T}_{6,k}(A)f(s,u_j,u_k,t) = \left(a + \frac{c}{u_k t}\right)^{-1} f\left(\frac{su_k t}{c + au_k t}, u_j, \frac{u_k}{d + bu_k t}, at + \frac{c}{u_k}\right),$$

$$\mathbf{T}_{7,k,p}(A)f(s,u_j,u_k,u_p,t) = \left(d - \frac{bu_k}{u_p}\right)^{-1}\left(a - c\frac{u_p}{u_k}\right)^{-1} \times f(s, u_j, au_k - cu_p, du_p - bu_k, t). \quad (1.26)$$

The matrix elements corresponding to a representation $\mathbf{T}(A)$ of $SL(2,\mathbb{C})$ induced by a triplet $\{J^+, J^-, J^3\}$ acting on a basis f_m according to the rule

$$J^{\pm} f_m = (-\omega \pm m) f_{m\pm 1}, \quad J^3 f_m = m f_m, \quad \omega \in \mathbb{C}, \quad (1.27)$$

have been computed many times before.[7] The result is

$$T_{n'n}(A) = a^{\omega + m_0 + n'} d^{\omega - m_0 - n} c^{n-n'} \frac{\Gamma(\omega + m_0 + n + 1)}{\Gamma(\omega + m_0 + n' + 1)} \times \frac{{}_2F_1(-\omega - m_0 - n', -\omega + m_0 + n; n - n' + 1; bc/ad)}{\Gamma(n - n' + 1)}, \quad (1.28)$$

where

$$\mathbf{T}(A)f_{m_0+n} = \sum_{n'=-\infty}^{\infty} T_{n'n}(A) f_{m_0+n'}, \quad n = 0, \pm 1, \pm 2, \ldots, \quad (1.29)$$

and A is in a sufficiently small neighborhood of the identity element. From this result it is easy to compute the matrix elements of each of the operators (1.11)–(1.17).

For more complicated group elements, however, the model (1.26) is very convenient. Consider the $(2n + 3)$-dimensional complex Lie algebra G with basis $\{J^+, J^-, J^3, E_j^+, E_j^-, j = 1, \ldots, n\}$ and commutation relations

$$\begin{aligned}
&[J^3, J^{\pm}] = \pm J^{\pm}, [J^+, J^-] = 2J^3,\\
&[J^+, E_j^-] = -E_j^+, [J^-, E_j^+] = -E_j^-,\\
&[J^+, E_j^+] = [J^-, E_j^-] = 0,\\
&[J^3, E_j^{\pm}] = \pm\tfrac{1}{2}E_j^{\pm}, [E_j^+, E_k^+] = 0,\\
&[E_j^+, E_k^-] = [E_j^-, E_k^-] = 0, \quad j,k = 1, \ldots, n.
\end{aligned} \quad (1.30)$$

This is the Lie algebra of the group G of $(n+2) \times (n+2)$ matrices

$$\{A, \mathbf{g}^{(1)}, \ldots, \mathbf{g}^{(n)}\} = \left(\begin{array}{c|c} A & \mathbf{g}^{(1)}, \ldots, \mathbf{g}^{(n)} \\ \hline 0 & E_n \end{array}\right), \quad A \in SL(2,\mathbb{C}),\ g_l^{(k)} \in \mathbb{C}, \quad (1.31)$$

and multiplication law

$$\{A, \mathbf{g}^{(j)}\}\{A', \mathbf{g}'^{(j)}\} = \{AA', A\mathbf{g}'^{(j)} + \mathbf{g}^{(j)}\}. \quad (1.32)$$

Here $\mathbf{g}^{(j)} = \{g_1^{(j)}, g_2^{(j)}\}$ is a column 2-vector and E_n is the $n \times n$ identity matrix. The Lie algebra $\mathcal{G}$ is related to G by the expression

$$\{A, \mathbf{g}^{(j)}\} = \exp(g_1^{(1)}E_1^+ + g_2^{(1)}E_1^-) \cdots \exp(g_1^{(n)}E_n^+ + g_2^{(n)}E_n^-) \times \exp[-(b/d)J^+] \exp(-cdJ^-) \exp(\tau J^3),$$
$$e^{\tau/2} = d^{-1}. \quad (1.33)$$

The Lie algebra $\mathcal{G}$ can be embedded as a subalgebra of $sl(n+3,\mathbb{C})$ in many distinct ways, but for purposes of illustration we consider only the example $\{J^+, J^-, J^3, E_j^+, E_j^-\} \equiv \{E_\alpha, E_{-\alpha}, J_\alpha, E_{\alpha\beta_j\gamma}, E_{\beta_j\gamma}\}$. Using this embedding, we compute the action of G in our $(n+2)$-variable model:

$$\begin{aligned}
\mathbf{T}(A, \mathbf{g}^{(j)})f(s,u_j,t) &= \exp(g_1^{(1)}E_{\alpha\beta,\gamma} + g_2^{(1)}E_{\beta,\gamma}) \times \cdots \exp(g_1^{(n)}E_{\alpha\beta_n\gamma} + g_2^{(n)}E_{\beta_n\gamma})\mathbf{T}_1(A)f\\
&= (d - bs)^{-1}\left(a - \frac{c}{s}\right)^{-1} f\left(\frac{as - c}{d - bs}, \frac{u_j}{1 - g_1^{(j)}u_j st - g_2^{(j)}u_j t}, t(d - bs)\right).
\end{aligned} \quad (1.34)$$

Applying $\mathbf{T}(A, \mathbf{g}^{(j)})$ to the basis vectors $f_{hn_jm}(s,u_j,t) = s^{\alpha^0+h} u_1^{\beta_1^0+n_1} \cdots u_n^{\beta_n^0+n_n} t^{\gamma^0+m}$, we obtain

$$\mathbf{T}\{A, \mathbf{g}^{(j)}\} f_{h,n_jm} = \sum_{h'n_k', m=-\infty}^{\infty} T(A, \mathbf{g}^{(j)})_{hn_jm}^{h'n_k'm'} f_{h'n_k'm'}, \quad (1.35)$$

or

$$T(A, \mathbf{g}^{(j)})_{hn_jm}^{h'n_g'm'} = 0$$

unless $n_j' \geq n_j$, $j = 1, \ldots, n$, and $m' - m = \sum_{j=1}^{n}(n_j' - n_j)$; (1.36)

$$\mathbf{T}(A, \mathbf{g}^{(j)})_{hn_jm}^{h+k, n_k+l_k, m+\Sigma l_j} = \binom{-\beta_1^0 - n_1}{l_1} \cdots \binom{-\beta_n^0 - n_n}{l_n} \times \frac{\Gamma(1 - k - \alpha^0 - h)}{\Gamma(1 - \alpha^0 - h)} [-g_2^{(1)}]^{l_1} \cdots [-g_2^{(n)}]^{l_n} a^{\alpha-1} d^{\gamma-\alpha-1}$$
$$\times \frac{(a/c)^k}{\Gamma(1-k)} F_D\left(1 - k - \alpha; -l_1, \ldots, -l_n, \alpha - \gamma + 1; 1 - k; \frac{-g_1^{(1)}c}{g_2^{(1)}a}, \ldots, \frac{-g_1^{(n)}c}{g_2^{(n)}a}, \frac{bc}{ad}\right), \quad l_j \geq 0.$$

The group property (1.32) leads immediately to the addition theorem

$$T\{AB, A\mathbf{h}^{(j)} + \mathbf{g}^{(j)}\}_{hn_jm}^{h'n_k'm'} = \sum_{HN_sM} T\{A, \mathbf{g}^{(j)}\}_{HN_sM}^{h'n_k'm'} \times T\{B, \mathbf{h}^{(j)}\}_{hn_jm}^{HN_sM} \quad (1.37)$$

for the F_D.

Equation (1.35) with matrix elements (1.36) is also valid for the $(2n+2)$-variable model. In this case the basis functions are given by (1.4) and the operator $\mathbf{T}\{A, \mathbf{g}^{(j)}\}$ by

$$\mathbf{T}\{A, \mathbf{g}^{(j)}\} f(s,u_j,t,x_j) = f\Big(\frac{as+c}{d+bs}, \frac{u_j(as+c)}{as(1-g_2^{(j)}u_jt)+c(1-x_j-g_1^{(j)}su_jt)}, \frac{ts}{as+c}, \frac{s(x_j+g_1^{(j)}su_jt-g_2^{(j)}u_jt)}{(d+bs)[as(1-g_2^{(j)}u_jt)+c(1-x_j-g_1^{(j)}su_jt)]}\Big). \quad (1.38)$$

A variety of addition theorems for the F_D can be obtained from (1.35) and (1.38) by specialization of the group parameters. Since this is routine, we give no examples.

2. TRANSFORMATION FORMULAS AND GENERATING FUNCTIONS

We next show that the transformation formulas for the F_D are consequences of the $SL(n+3,\mathbb{C})$ symmetry. Let

$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2,\mathbb{C}). \quad (2.1)$$

Expressions (1.4) and (1.11) imply

$$\mathbf{T}_1(I) f_{\alpha\beta_j\gamma} = (-1)^{\alpha+\gamma}[\Gamma(\gamma-\alpha)\Gamma(\alpha)/\Gamma(\gamma)] \times F_D(\alpha;\beta_j;\gamma,x_j/(x_j-1))(1-x_1)^{-\beta_1}\cdots \times (1-x_n)^{-\beta_n} s^{-\alpha+\gamma} u_1^{\beta_1}\cdots u_n^{\beta_n} t^{\gamma}. \quad (2.2)$$

However, $\mathbf{T}_1(I)f_{\alpha\beta_j\gamma}$ is a simultaneous eigenfunction of $J_\alpha, J_{\beta_1},\dots,J_{\beta_n},J_\gamma$, analytic at $x_1=\cdots=x_n=0$. Thus,

$$\mathbf{T}_1(I) f_{\alpha\beta_j\gamma} = kF_D(\gamma-\alpha;\beta_j;\gamma;x_j)s^{-\alpha+\gamma}u_1^{\beta_1}\cdots u_n^{\beta_n}t^{\gamma}. \quad (2.3)$$

Setting $x_1=\cdots=x_n=0$ in (2.2) and (2.3), we can evaluate the constant k and obtain the transformation formula

$$(1-x_1)^{-\beta_1}\cdots(1-x_n)^{-\beta_n}F_D(\alpha;\beta_j;\gamma;x_j/(x_j-1)) = F_D(\gamma-\alpha;\beta_j;\gamma;x_j).^4 \quad (2.4)$$

Similarly, $\mathbf{T}_{2,k}(I)f_{\alpha\beta_j\gamma}$ yields the formulas

$$(1-x_k)^{-\alpha}F_D(\alpha;\beta_j,\beta_k;\gamma;(x_k-x_j)/(x_k-1),x_k/(x_k-1)) = F_D\Big(\alpha;\beta_j,\gamma-\sum_{l=1}^n\beta_l;\gamma;x_j,x_k\Big), \quad k=1,\dots,n. \quad (2.5)$$

The remaining transformation formulas for the F_D can be obtained by composition from (2.4) and (2.5).

Computing $\mathbf{T}_3(I)f_{\alpha\beta_j\gamma}$, we find that

$$F_D(\alpha;\beta_j;\alpha+\sum_j\beta_j-\gamma+1;1-x_j) \quad (2.6)$$

is a solution of Eqs. (1.21), analytic at $x_1=\cdots=x_n=1$. Computing $\mathbf{T}_5(I)f_{\alpha\beta_j\gamma}$, we see that

$$x_1^{-\beta_1}\cdots x_n^{-\beta_n}F_D(\sum_l\beta_l-\gamma+1;\beta_j;\sum_l\beta_l-\alpha+1;x_j^{-1}) \quad (2.7)$$

is another solution of (1.21). Similarly, $\mathbf{T}_{6,k}(I)f_{\alpha\beta_j\gamma}$ yields the solution

$$x_k^{-\alpha}F_D(\alpha;\beta_j,\alpha-\gamma+1;\alpha-\beta_k+1;x_j/x_k,1/x_k). \quad (2.8)$$

For A close to the identity in $SL(2,\mathbb{C})$ the expressions $\mathbf{T}_{j,k}(A)f_{\alpha\beta_j\gamma}$ can be expanded by use of the matrix elements (1.28). However, for A far from the identity, say $A=I$, these expansions are no longer valid. For example,

$$(\exp cE_{\alpha\gamma})f(s,u_j,t,x_j) = f\Big(\frac{s}{1+cst},u_j,t,\frac{x_j+cst}{1+cst}\Big). \quad (2.9)$$

For $|c|$ small we find

$$(\exp cE_{\alpha\gamma})f_{\alpha\beta_j\gamma} = \sum_{h=0}^\infty \binom{\sum_l\beta_l-\gamma}{h} f_{\alpha+h,\beta_j,\gamma+h}c^h,$$

i.e.,

$$(1+c)^{-\alpha}F_D\Big(\alpha;\beta_j;\gamma;\frac{x_j+c}{1+c}\Big) = \sum_{h=0}^\infty\binom{\sum_l\beta_l-\gamma}{h} \times \frac{(\alpha)_h}{(\gamma)_h}F_D(\alpha+h;\beta_j;\gamma+h;x_j)c^h, \quad |c|<1. \quad (2.10)$$

If $c=1$ and $|\tau|<1$, where $\tau=s^{-1}t^{-1}$, then $(\exp E_{\alpha\gamma})f_{\alpha\beta_j\gamma}$ is not analytic at $x_1=\cdots=x_n=\tau=0$. However, we can apply $\exp E_{\alpha\beta}$ to the solution (2.6) and use (1.22), (1.23) to obtain

$$(1+\tau)^{-\alpha}F_D(\alpha;\beta_j;\alpha+\sum\beta_l-\gamma+1;\tau(1-x_j)/(1+\tau)) = \sum_{h=0}^\infty C_hF_D(-h;\beta_j;\gamma-\alpha-h;x_j)\tau^h. \quad (2.11)$$

To evaluate the constants C_h, we set $x_1=\cdots x_n=0$:

$$(1+\tau)^{-\alpha}F_D(\alpha,\beta_j;\alpha+\sum_l\beta_l-\gamma+1;\tau/(1+\tau)) = (1+\tau)^{-\alpha}\,{}_2F_1(\alpha,\sum_l\beta_l;\alpha+\sum_l\beta_l-\gamma+1;\tau/(1+\tau)) = \sum_{h=0}^\infty C_h\tau^h.$$

Thus,

$$C_h = \binom{-\alpha}{h}\,{}_2F_1(-h,\sum_l\beta_l;\alpha+\sum_l\beta_l-\gamma+1;1) = \binom{-\alpha}{h}\frac{(\alpha-\gamma+1)_h}{(\alpha+\sum\beta_l-\gamma+1)_h} \quad (2.12)$$

from Ref. 7, p. 211, and Vandermonde's theorem.

Expanding $\mathbf{T}_1(A)f_{\alpha\beta_j\gamma}$ as a power series in $\tau=s^{-1}$, we obtain

$$a^{\alpha-\gamma}b^{-\alpha}\Big(1+\frac{c\tau}{a}\Big)^{\alpha+\Sigma\beta_l-\gamma}\Big(1+\frac{d\tau}{b}\Big)^{-\alpha}\Big(1+\frac{c\tau(1-x_1)}{a}\Big)^{-\beta_1} \times\cdots\Big(1+\frac{c\tau(1-x_n)}{a}\Big)^{-\beta_n} \times F_D\Big(\alpha;\beta_j;\gamma;\frac{x_j\tau}{(b+dt)[a+c\tau(1-x_j)]}\Big) = \sum_{h=0}^\infty k_hF_D(-h;\beta_j;\gamma;x_j)\tau^h. \quad (2.13)$$

Setting $x_1=\cdots=x_n=0$ and using identity (5.124), Ref. 7, p. 206, we find

$$k_h = \Big(\frac{a}{b}\Big)^\alpha a^{-\gamma-h}c^h\binom{-\gamma}{h}\,{}_2F_1(-h,\alpha;\gamma;-1/bc), \quad ad-bc=1. \quad (2.14)$$

If $a=d=b=1$ and $c=0$, the identity becomes

$$(1+\tau)^{-\alpha}F_D\left(\alpha;\beta_j;\gamma;\frac{x_j\tau}{1+\tau}\right)=\sum_{h=0}^{\infty}\binom{-\alpha}{h}F_D(-h;\beta_j;\gamma;x_j)\tau^h,\qquad |\tau|<1,\qquad(2.15)$$

and, if $a=c=1, b=-w^{-1}$, it reduces to

$$(1+\tau)^{\alpha+\Sigma\beta_l-\gamma}[1+(1-w)\tau]^{-\alpha}\prod_{l=1}^{n}[1+(1-x_l)\tau]^{-\beta_l}\times F_D\left(\alpha;\beta_j;\gamma;\frac{-x_j\tau w}{[1+(1-w)\tau][1+(1-x_j)\tau]}\right)$$
$$=\sum_{h=0}^{\infty}\binom{-\gamma}{h}\,{}_2F_1(-h,\alpha;\gamma;w)F_D(-h;\beta_j;\gamma;x_j)\tau^k,$$
$$|\tau|<\min(1,|1-x_j|^{-1},|1-w|^{-1}).\qquad(2.16)$$

More generally, we can derive generating functions for the F_D through the characterization of a solution f of $C_jf=0,\ j=1,\dots,n$, by the requirement that f is a simultaneous eigenfunction of $n+2$ independent operators constructed from $sl(n+3,\mathbb{C})$. Such a characterization of $f_{\alpha\beta_j\gamma}$ is given by (1.19).

As an example we compute the solution f of the simultaneous equations

$$E_\alpha f=f,\quad J_{\beta_k}f=\left(\beta_k+\tfrac{1}{2}\sum_{j\neq k}\beta_j-\tfrac{1}{2}\gamma\right)f,$$
$$(J_\gamma+\tfrac{1}{2}J_\alpha)f=\left(\tfrac{3}{4}\gamma-\tfrac{1}{2}\sum_l\beta_1-\tfrac{1}{2}\right)f,$$
$$C_kf=0,\ k=1,\dots,n,\qquad(2.17)$$

which is analytic at $x_1=\cdots=x_n=0$. The first $n+2$ equations have the general solution

$$f=h(x_1/s,\dots,x_n/s)\exp(-s^{-1})u_1^{\beta_1}\cdots u_n^{\beta_n}t^\gamma,$$

where h is an arbitrary function. Substitution of this expression into $C_kf=0,\ 1\le k\le n$, yields

$$h(x_1,\dots,x_n)=\Phi(\beta_1,\dots,\beta_n;\gamma;x_1,\dots,x_n)=\sum_{m_1\cdots m_n=0}^{\infty}\frac{(\beta_1)_{m_1}\cdots(\beta_n)_{m_n}}{(\gamma)_{m_1+\dots+m_n}}\frac{x_1^{m_1}\cdots x_n^{m_n}}{m_1!\dots m_n!}$$
$$=\lim_{\alpha\to\infty}F_D\left(\alpha;\beta_j;\gamma;\frac{x_j}{\alpha}\right),\qquad(2.18)$$

unique up to a constant multiple. Expanding $\mathbf{T}_1(A)f$ as a power series in $\tau=s^{-1}$, we obtain

$$\exp\left[-\left(\frac{d\tau+b}{a+c\tau}\right)\right](a+c\tau)^{\Sigma\beta_l-\gamma}\prod_{l=1}^{n}[a+c\tau(1-x_l)]^{-\beta_l}\times\Phi\left(\beta_j;\gamma;\frac{x_j\tau}{(a+c\tau)[a+c\tau(1-x_j)]}\right)$$
$$=\sum_{k=0}^{\infty}r_kF_D(-k;\beta_j;\gamma;x_j)\tau^k,\qquad ad-bc=1.\qquad(2.19)$$

Setting $x_1=\cdots=x_n=0$ and using the generating function for Laguerre polynomials [(5.101), Ref. 7, p. 190], we find

$$r_k=a^{-\gamma}e^{-b/a}\left(\frac{c}{a}\right)^kL_k^{(\gamma-1)}\left(\frac{1}{ac}\right),\qquad(2.20)$$

where $L_n^{(\alpha)}(x)$ is a generalized Laguerre polynomial. If $b=c=0,\ a=d=1$, the identity simplifies to

$$\exp(-\tau)\Phi(\beta_j;\gamma;x_j\tau)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}F_D(-k;\beta_j;\gamma;x_j)\tau^k.\qquad(2.21)$$

If $a=c=d^{-1}=w^{-1/2},\ b=0$, we find

$$\exp\left(\frac{-w\tau}{1+\tau}\right)(1+\tau)^{\Sigma\beta_l-\gamma}[1+\tau(1-x_1)]^{-\beta_1}\cdots\times[1+\tau(1-x_n)]^{-\beta_n}\Phi\left(\beta_j;\gamma;\frac{x_jw\tau}{(1+\tau)[1+\tau(1-x_j)]}\right)$$
$$=\sum_{k=0}^{\infty}L_k^{(\gamma-1)}(w)F_D(-k;\beta_j;\gamma;x_j)\tau^k,$$
$$|\tau|<\min(1,|x_j-1|^{-1}).\qquad(2.22)$$

If $b=-c=1,\ a=d=0$, then $\mathbf{T}_1(A)f$ becomes

$$e^s(1-x_1)^{-\beta_1}\dots(1-x_n)^{-\beta_n}s^\gamma\,\Phi\left(\beta_j;\gamma\,\frac{x_js}{1-x_j}\right)\times u_1^{\beta_1}\dots u_n^{\beta_n}t^\gamma.$$

Expanding this function in powers of s, we obtain

$$e^s(1-x_1)^{-\beta_1}\dots(1-x_n)^{-\beta_n}\Phi\left(\beta_j;\gamma;\frac{x_js}{1-x_j}\right)=\sum_{k=0}^{\infty}\frac{s^k}{k!}F_D(\gamma+k;\beta_j;\gamma;x_j).\qquad(2.23)$$

Although the derivation of these generating functions is completely routine, an exhaustive classification of such generating functions awaits the classification of all algebraically irreducible representations of $sl(n+3,\mathbb{C})$.

The various confluent forms of the functions F_D have symmetry algebras corresponding to contractions of the algebra $gl(n+3,\mathbb{C})\cong sl(n+3,\mathbb{C})\oplus(\mathcal{E})$. For example, consider the confluent function

$$\psi(\alpha;\beta_1,\dots,\beta_{n-1};\gamma;x_1,\dots,x_n)=\sum_{m_1,\dots,m_m=0}^{\infty}\frac{(\gamma)_{m_1+\dots+m_n}}{(\gamma)_{m_1+\dots+m_n}}(\beta_1)_{m_1}\cdots(\beta_{n-1})_{m_{n-1}}\frac{x_1^{m_1}\dots x_n^{m_n}}{m_1!\dots m_n!}$$
$$=\lim_{\beta_n\to\infty}F_D\left(\alpha;\beta_1,\dots,\beta_{n-1},\beta_n;\gamma;x_1,\dots,x_{n-1},\frac{x_n}{\beta_n}\right).\qquad(2.24)$$

To obtain the symmetry algebra, we introduce new operators

$$E'_{\alpha\beta_n\gamma}=\frac{1}{\beta_n}E_{\alpha\beta_n\gamma},\quad E'_{\alpha\gamma}=\frac{1}{\beta_n}E_{\alpha\gamma},\quad E'_\gamma=\frac{1}{\beta_n}E_\gamma$$

$$E'_{-\beta_k}=\frac{1}{\beta_n}E_{-\beta_k},\quad k\neq n,\quad E'_{\beta_n\gamma}=\frac{1}{\beta_n}E_{\beta_n\gamma},$$

$$E'_{\beta_n-\beta_k}=\frac{1}{\beta_n}E_{\beta_n,-\beta_k},$$

$J'_{\beta_n}=(1/\beta_n)J_{\beta_n}$ and $E'_\xi=E_\xi$ for all other elements of $sl(n+3,\mathbb{C})$.

Formally letting $\beta_n\to\infty$, we obtain a contracted Lie algebra not isomorphic to $sl(n+3,\mathbb{C})$. The operators which raise and lower u_n are now redundant. Dropping these operators, we are left with an $(n+2)^2$-dimensional non-semi-simple Lie algebra, the symmetry algebra of ψ. This algebra can be used to derive identities for the ψ functions in a manner analogous to that for F_D.

Vilenkin's method of integral transforms and the $(n+2)$-variable model (1.26) can be used to derive Mellin–Barnes integral identities for F_D and its confluent forms. The procedure is completely analogous to that given in Refs. 8, 2, and 3, but the details are complicated.

[1] M. Ciftan, J. Math. Phys. **10**, 1635 (1969).
[2] W. Miller, "Lie Theory and the Appell Functions F_1" (submitted to SIAM J. Math. Anal.).
[3] W. Miller, SIAM J. Math. Anal. 3, 31 (1972).
[4] P. Appell and J. Kampe De Feriet, *Functions hypergeometriques et hyperspheriques* (Gauthiers-Villars, Paris, 1926).
[5] L. J. Slater, *Generalized Hypergeometric Functions* (Cambridge U. P., London, 1966).
[6] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. 1.
[7] W. Miller, *Lie Theory and Special Functions* (Academic, New York, 1968), p. 21.
[8] N. Y. Vilenkin, *Special Functions and the Theory of Group Representations*, transl. (Amer. Math. Soc., Providence, R. I., 1968).

Averages of the Components of Random Unit Vectors

J. F. McDonald

Department of Mathematics, University of Windsor, Windsor, Ontario, Canada

(Received 15 December 1971; Revised Manuscript Received 18 April 1972)

It is shown that a parametrization of the orthogonal and unitary groups due to Hurwitz can be used to evaluate averages of components of random unit vectors for those two spaces. Explicit results are given for moments which are general enough to include most cases of interest in applications.

1. INTRODUCTION

Several years ago, Ullah developed a method for evaluating averages of components of random unit vectors.[1] The technique is applicable for an N-dimensional orthogonal, unitary, or symplectic space.[2] However, the method as given is restricted to the even moments of a single vector for the unitary and symplectic spaces and is restricted to moments involving at most two orthogonal unit vectors for the orthogonal space. Unfortunately, it does not seem possible to extend the method to averages which involve a larger number of vectors.

A possible alternative to Ullah's method is the explicit parametrization of the group of transformations involved. The advantage to this approach is that in principle there is no restriction on the number of vectors involved.

It would appear that the major obstacle is the parametrization itself. That is, one must parametrize the group in such a way that the calculation is tractable. In particular, one must be able to express any element of the rotation matrix explicitly in terms of the parameters, and one must be able to determine the corresponding volume element in the parameter space.

Fortunately, such a parametrization for the orthogonal and unitary groups is known. These parametrizations are due to Hurwitz.[3]

We shall show that these parametrizations are indeed satisfactory for the explicit evaluation of averages which involve any number of vectors.

2. THE GENERAL ROTATION MATRIX

The general rotation matrix for an N-dimension orthogonal or unitary space can be built up out of successive two-dimensional rotations as follows. Let the $N \times N$ matrices $\alpha_r{}^{(s)}$ be defined as

$$\begin{aligned} [\alpha_r{}^{(s)}]_{ij} &= \delta_{ij}, \qquad i \neq r, r+1, \\ &= a_{N-1-r}^{(s)}\ \delta_{jr} + b_{N-1-r}^{(s)}\ \delta_{jr+1}, \qquad i = r \\ &= c_{N-1-r}^{(s)}\ \delta_{jr} + d_{N-1-r}^{(s)}\ \delta_{jr+1}, \qquad i = r+1, \end{aligned} \tag{1}$$

where $s = 1, 2, \ldots, N-1$ and $r = N-s, N-s-1, \ldots, N-1$.

The matrix $\alpha_r^{(s)}$ is a rotation in the corresponding two-dimensional subspace. For the orthogonal space $\alpha_r^{(s)} \tilde{\alpha}_r^{(s)} = 1$, and for the unitary space $\alpha_r^{(s)} \alpha_r^{(s)+} = 1$. These matrices will be parametrized below for the orthogonal and unitary spaces.

Next we define the matrices $E^{(s)}$ as

$$E^{(s)} = \widetilde{\lceil \mathrm{L} \rceil}_{r=1}\ \alpha_{N-r}^{(s)}, \tag{2}$$

where $\widetilde{\lceil \mathrm{L} \rceil}$ means that successive factors are to the left. It follows easily from mathematical induction that

$$E^{(s)} = \begin{bmatrix} I_{N-s-1} & 0 \\ 0 & T^{(s)} \end{bmatrix}, \tag{3}$$

where I_{N-s-1} is the $(N-s-1) \times (N-s-1)$ unit matrix, and $T^{(s)}$ is the $(s+1) \times (s+1)$ matrix with elements

$$T_{11}{}^{(s)} = a_{s-1}^{(s)}, \tag{4}$$

$$T_{1j}^{(s)} = a_{s-j}^{(s)} \prod_{r=s-j+1}^{s-1} b_r^{(s)}, \qquad j \geqslant 2, \tag{5}$$

$$T_{ii}^{(s)} = a_{s-i}^{(s)}\, d_{s-i+1}^{(s)}, \qquad i \geqslant 2, \tag{6}$$

$$T_{ij}^{(s)} = a_{s-j}^{(s)} d_{s-i+1}^{(s)} \prod_{r=s-j+1}^{s-i} b_r{}^{(s)}, \quad j > i \geqslant 2, \tag{7}$$

$$T_{i\,i-1}^{(s)} = c_{s-i+1}^{(s)}, \qquad i \geqslant 2, \tag{8}$$

$$T_{ij}^{(s)} = 0, \qquad i > j+1. \tag{9}$$

In the above equations $a_{-1} \equiv 1$.

Finally, the general rotation matrix A is given by

$$A = \widetilde{\lceil \mathrm{R} \rceil}_{s=1}^{N-1}\ E^{(s)}, \tag{10}$$

where $\widetilde{\lceil \mathrm{R} \rceil}$ means that successive factors are to the right. It follows easily from the definition of $T^{(s)}$ that