Lie theory and separation of variables. 3. The equation $f_{tt}-f_{ss} = \gamma^2 f$

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Kalnins has related the 11 coordinate systems in which variables separate in the equation $f_{tt}-f_{ss}=\gamma^2 f$ to 11 symmetric quadratic operators L in the enveloping algebra of the Lie algebra of the pseudo-Euclidean group in the plane $E(1,1)$. There are, up to equivalence, only 12 such operators and one of them, L_E, is not associated with a separation of variables. Corresponding to each faithful unitary irreducible representation of $E(1,1)$ we compute the spectral resolution and matrix elements in an L basis for seven cases of interest and also give overlap functions between different bases: Of the remaining five operators three are related to Mathieu functions and two are related to exponential solutions corresponding to Cartesian type coordinates. We then use these results to derive addition and expansion theorems for special solutions of $f_{tt}-f_{ss}=\gamma^2 f$ obtained via separation of variables, e.g., products of Bessel, Macdonald and Bessel, Airy and parabolic cylinder functions. The exceptional operator L_E is also treated in detail.

INTRODUCTION

In Refs. 1 and 2, Winternitz and coworkers introduced a group theoretical method for the description of separation of variables in the principal partial differential equations of mathematical physics. We apply their idea in this paper to study several coordinate systems in which separation of variables is possible in the equation

$$(*)\quad (\partial_s^2-\partial_t^2)f(s,t)=-\gamma^2 f(s,t),\quad \gamma>0.$$

The symmetry group of (*) is $E(1,1)$ the pseudo-Euclidean group in the plane. Its Lie algebra $e(1,1)$ is three-dimensional with basis P_1, P_2, M and commutation relations

$$[M,P_1]=P_2,\ [M,P_2]=P_1,\ [P_1,P_2]=0.$$

A two-variable model of $e(1,1)$ is

$$(**)\quad P_1=\partial_s,\quad P_2=\partial_t,\quad M=-s\partial_t-t\partial_s$$

in which case (*) becomes

$$(P_1^2-P_2^2)f=-\gamma^2 f.$$

According to the prescription in Refs. 1 and 2 one should characterize solutions f of (*) by requiring in addition that f is an eigenfunction of an operator L, $Lf=\lambda f$, where L belongs to the factor space $T=S/S\cap C$. Here, C is the center of the universal enveloping algebra U of $e(1,1)$ and S is the space of all symmetric second order elements in U. In our case, $S\cap C=\{\alpha(P_1^2-P_2^2)\}$, α any constant. $E(1,1)$ acts on T via the adjoint representation and we do not distinguish between operators L on the same orbit.

From the examples presented in Refs. 1 and 2 one might expect that each system of equations

$$(P_1^2-P_2^2)f=-\gamma^2 f,\quad Lf=\lambda f$$

where P_1, P_2, M are given by (**), is related to a coordinate system in which (*) separates, that all separable coordinate systems can be so obtained, and that there is a one-to-one relationship between orbits and separable coordinate systems. However, in Ref. 3 Kalnins has shown that this is not quite true. In fact, there are 12 orbits and 11 coordinate systems in which (*) separates. One orbit (with representative L_E in this paper) does not correspond to a separable coordinate system. Of the separable coordinate systems two, Cartesian and spherical polar, have well-known group theoretical interpretations (Ref. 4, Chap. V), three lead to various types of Mathieu equations, and two correspond to other Cartesian-type coordinates. The remaining four systems are related to parabolic cylinder, Bessel, Macdonald, and Airy functions, respectively, and correspond to operators L_D, L_B, L_K, L_A in T.

A study of parabolic coordinates with respect to the spectral resolution of L_D was carried out in Ref. 5. Here we undertake an analogous study of L_B, L_K, L_A and L_E. In Secs. 1 and 2 we compute the spectral resolutions of the self-adjoint operators L_G, $G=B,K,A,E$, corresponding to each of the irreducible faithful unitary representations of $E(1,1)$. In particular, we compute the matrix elements of the unitary group representation operators in an L_G-basis and we calculate the overlap functions relating two different bases.

In Sec. 3 we show how to construct models of the irreducible representations of $E(1,1)$ in which the Lie algebra operators take the form (**) and the Hilbert space vectors f satisfy (*). These models allow us to apply the results of Sec. 1 to obtain properties of those special solutions of (*) which can be obtained through separation of variables. (Of special interest here is L_E which does not lead to separation of variables.)

Finally, in Sec. 4 we study the spectral resolution of L_K corresponding to nonunitary representations of the complex Euclidean group $CE(2)$ and obtain a series of identities for products of modified Bessel and Macdonald functions.

1. THE REPRESENTATIONS OF $E(1,1)$

The pseudo-Euclidean group $E(1,1)$ is the group of all real matrices

$$A(\theta,a,b,)=\begin{pmatrix}\cosh\theta & \sinh\theta & a\\ \sinh\theta & \cosh\theta & b\\ 0 & 0 & 1\end{pmatrix},\quad -\infty<\theta,a,b<\infty.$$

It acts on the pseudo-Euclidean plane via the transformation $\mathbf{z}\to A\mathbf{z}$ where

$$\mathbf{z}=\begin{pmatrix}t\\ s\\ 1\end{pmatrix}$$

and preserves the form $(t_1-t_2)^2-(s_1-s_2)^2$.

The irreducible faithful unitary representations of $E(1,1)$ are well-known to be indexed by a parameter $\gamma>0$. Each such representation can be defined by operators $\mathbf{T}(\theta,a,b)$,

$$\mathbf{T}(\theta,a,b)f(x)=\exp[i\gamma(a\cosh x+b\sinh x)]f(x+\theta) \qquad (1.2)$$

acting on the Hilbert space $L_2(R)$ of Lebesgue square integrable functions $f(x)$ on the real line. The inner product is

$$\langle f,g\rangle=\int_{-\infty}^{\infty}f(x)\overline{g(x)}\,dx,\quad f,g\in L_2(R).$$

The Lie algebra $e(1,1)$ of $E(1,1)$ contains a basis $\{P_1,P_2,M\}$ with commutation relations

$$[M,P_1]=P_2,\quad [M,P_2]=P_1,\quad [P_1,P_2]=0 \qquad (1.3)$$

and related to the group via the exponential mapping

$$A(\theta,a,b)=\exp(aP_1+bP_2)\exp(\theta M).$$

The corresponding operators in $L_2(R)$ induced by the group action (1.2) are easily shown to be

$$P_1=i\gamma\cosh x,\quad P_2=i\gamma\sinh x,\quad M=\partial_x. \qquad (1.4)$$

Vilenkin (Ref. 4, Chap. V) has studied the unitary representation of $E(1,1)$ in terms of the spectral resolution for the operator

$$L_M=M^2$$

(or M) on $L_2(R)$. In particular, he has determined the matrix elements of the group operators (1.2) with respect to this resolution. In Ref. 5 the representations of $E(1,1)$ were examined with respect to the spectral resolution of the operator

$$L_D=MP_1+P_1M. \qquad (1.5)$$

It was shown that L_D has a one parameter family of self-adjoint extensions $L_{D,\alpha}$, $0\leq\alpha<2$. Each $L_{D,\alpha}$ has discrete spectrum $-2\gamma(\alpha+2n)$, $n=0,\pm1,\pm2,\cdots$ and normalized eigenfunctions

$$f_n^{D,\alpha}(x)=\sqrt{2\pi}\,\exp(x/2)(1+ie^x)^{\alpha+2n-1/2}(1-ie^x)^{-\alpha-2n-1/2}. \qquad (1.6)$$

(In every example treated in this paper the L-operator is initially defined on the subspace of $L_2(R)$ consisting of C^∞-functions with compact support. One then searches for all self-adjoint extensions of this symmetric operator.)

This case was in sharp contrast to that of L_M where there was a single self-adjoint extension with continuous spectrum covering the negative real axis with generalized eigenfunctions

$$f_\lambda^M(x)=\frac{\exp(i\lambda x)}{\sqrt{2\pi i}},\quad -\infty<\lambda<\infty, \qquad (1.7)$$

$$Mf_\lambda^M=i\lambda f_\lambda^M,\quad \langle f_\lambda^M,f_\mu^M\rangle=\delta(\lambda-\mu).$$

The spectral resolution was obtained via the Fourier transform. The relationship between these two bases was computed in Ref. 5.

In this paper we study the spectral resolutions in $L_2(R)$ of self-adjoint extensions of the symmetric operators found in Ref. 3:

$$\begin{aligned}L_B&=M^2-(P_1+P_2)^2,\\ L_K&=M^2+(P_1+P_2)^2,\\ L_E&=M(P_1-P_2)+(P_1-P_2)M,\\ L_A&=M(P_1-P_2)+(P_1-P_2)M+(P_1+P_2)^2.\end{aligned} \qquad (1.8)$$

For each resolution we compute the matrix elements of the unitary operators (1.2). In addition we determine the unitary transformations which allow us to pass from one spectral resolution to another.

A. The Bessel function or B basis

It follows from (1.4) that

$$L_B=D_x^2+\gamma^2\exp(2x)=v^2D_v^2+vD_v+\gamma^2v^2,$$
$$v=\exp(x),\quad D_x=\frac{d}{dx},\quad D_v=\frac{d}{dv}. \qquad (1.9)$$

This operator is symmetric on $L_2(R)$ with deficiency indices (1,1). Thus there is a one-parameter family $L_{B,\alpha'}$, $0\leq\alpha'<2\pi$, of self-adjoint extensions of L_B. The domain of each extension is

$$\mathcal{D}_\alpha=\{f\in\mathcal{D}_{L^*_B}:\lim_{v\to+\infty}v[h_{\alpha'}(v)D_vf(v)-f(v)D_vh_{\alpha'}(v)]=0\}$$

where $\mathcal{D}_{L^*_B}$ is the domain of the adjoint of L_B in $L_2(R)$ and

$$h_{\alpha'}(v)=J_\beta(\gamma v)+\exp(i\alpha')J_{\bar\beta}(\gamma v),\quad \beta=\exp(i\pi/4),$$

where $J_\nu(z)$ is a Bessel function. (All special functions in this paper are defined as in Ref. 6.)

Each $L_{B,\alpha'}$ has discrete spectrum and an orthonormal basis of eigenfunctions

$$f_n^{B,\alpha}(v)=\sqrt{2(\alpha+2n)}\,J_{\alpha+2n}(\gamma v),$$
$$v=\exp(x),\quad n=0,1,2,\cdots, \qquad (1.10)$$

where $0<\alpha\leq2$ and the fixed parameters α,α' are related by

$$\tan\left(\frac{\pi\alpha}{2}-\frac{\pi}{2\sqrt2}\right)=\left(\frac{1+\exp(\pi/\sqrt2)}{1-\exp(\pi/\sqrt2)}\right)\tan\frac{\alpha'}{2}.$$

(Our computations of spectral resolutions for first and second order ordinary differential operators, while certainly nontrivial, are straightforward,[7] so we omit the details.)

The relationship between different bases is easily computed:

$$f_m^{B,\alpha_1}(v)=\sum_{n=0}^{\infty}\langle f_m^{B,\alpha_1},f_n^{B,\alpha_2}\rangle f_n^{B,\alpha_2}(v),$$

$$\langle f_m^{B,\alpha_1},f_n^{B,\alpha_2}\rangle=2\sqrt{(\alpha_1+2m)(\alpha_2+2n)}\int_0^\infty J_{\alpha_1+2m}(v)J_{\alpha_2+2n}(v)\frac{dv}{v}$$

$$=\frac{\sqrt{(\alpha_1+2m)(\alpha_2+2n)}\sin\pi[(\alpha_1-\alpha_2)/2+m-n]}{\pi[(\alpha_1-\alpha_2)/2+m-n][(\alpha_1+\alpha_2)/2+m+n]}. \qquad (1.11)$$

The matrix elements of the unitary operator $\mathbf{T}(0,a,a)$, $a>0$ are

$$T_{mn}^{B,\alpha}(0,a,a)=\langle\exp a(P_1+P_2)f_n^{B,\alpha},f_m^{B,\alpha}\rangle$$

$$= 2\sqrt{(\alpha+2n)(\alpha+2m)}\int_0^\infty e^{iav}J_{\alpha+2n}(v)J_{\alpha+2m}(v)\frac{dv}{v}$$

$$= \frac{2\sqrt{(\alpha+2n)(\alpha+2m)}}{\Gamma(\alpha+2n+1)\Gamma(\alpha+2m)}(4a^2)^{\alpha+m+n}e^{-i\pi(\alpha+m+n)}$$

$$\times {}_4F_3\left(\begin{matrix}\alpha+n+m, \alpha+n+m+\frac{1}{2}, \alpha+n+m+\frac{1}{2}, \\ \alpha+n+m+1 \\ \alpha+2m+1, \alpha+2n+1, 2\alpha+2m+2n+1\end{matrix}\middle|\frac{1}{4a^2}\right) \qquad (1.12)$$

where $\Gamma(z)$ is the gamma function and ${}_pF_q$ is a generalized hypergeometric function.

Further,

$$T^{B,\alpha}_{mn}(0,-a,-a) = \overline{T^{B,\alpha}_{nm}}(0,a,a).$$

The integral in (1.12) is evaluated with the help of Lebesgue's dominated convergence theorem and the device of expanding $J_{\alpha+2n}(v)J_{\alpha+2m}(v)$ into a power series in v and integrating term by term. There is a similar unenlightening expression for the matrix elements $T^{B,\alpha}_{mn}(0,a,-a)$ which we omit.

The matrix elements of the operator $\mathbf{T}(\theta,0,0,)$ are

$$T^{B,\alpha}_{mn}(\theta,0,0) = \langle \exp(\theta M)f^{B,\alpha}_n, f^{B,\alpha}_m\rangle$$

$$= 2\sqrt{(\alpha+2n)(\alpha+2m)}\int_0^\infty J_{\alpha+2n}(e^\theta v)J_{\alpha+2m}(v)\frac{dv}{v}$$

$$= e^{-(\alpha+2m)\theta}\frac{\sqrt{(\alpha+2n)(\alpha+2m)}\,\Gamma(\alpha+n+m)}{\Gamma(\alpha+2m+1)\Gamma(1+n-m)}$$

$$\times {}_2F_1\left(\begin{matrix}\alpha+n+m, m-n \\ \alpha+2m+1\end{matrix}\middle| e^{-2\theta}\right)$$

for $\theta \geq 0$. (1.13)

[This is a Weber-Schafheithin integral (Ref. 6, Vol. II.)] Furthermore,

$$T^{B,\alpha}_{mn}(-\theta,0,0) = \overline{T^{B,\alpha}_{nm}}(\theta,0,0).$$

Note that the matrix elements (1.13) vanish if $m \geq n+1$.

B. The Macdonald function or K basis

From (1.4) it follows that

$$L_K = D_x^2 - \gamma^2 e^{2x} = v^2D_v^2 + vD_v - \gamma^2v^2. \qquad (1.14)$$

This operator is symmetric on $L_2(R)$ and has deficiency indices (0,0). Thus L_K has a unique self-adjoint extension (which we also call L_K) and a complete set of orthonormal eigenfunctions of L_K, f^K_x, which form a basis for the representation space. The spectral resolution of L_K can be obtained from the known form of the Lebedev integral transform (Ref. 6, Vol. II). The spectrum of L_K is continuous and an orthonormal basis of eigenfunctions is ($\mathrm{sh}z = \sinh z$, $\mathrm{ch}z = \cosh z$)

$$f^K_z(v) = \frac{1}{\pi}\sqrt{2z\,\mathrm{sh}\pi z}\,K_{iz}(\gamma v), \quad 0 < z < \infty. \qquad (1.15)$$

These basis functions satisfy the delta function normalization

$$\langle f^K_x, f^K_y\rangle = \delta(x-y).$$

The matrix elements of the unitary operator $\mathbf{T}(0,a,a)$, $a>0$, are in this basis

$$T^K_{xy}(0,a,a) = \langle \exp a(P_1+P_2)f^K_y, f^K_x\rangle$$

$$= \frac{\pi^2}{2}\sqrt{xy\,\mathrm{sh}\pi x\,\mathrm{sh}\pi y}\int_0^\infty \exp(iav)K_{ix}(\gamma v)K_{iy}(\gamma v)\frac{dv}{v}$$

$$= \delta(x-y)$$

$$+\frac{1}{4\pi^2}\sqrt{xy\,\mathrm{sh}\pi x\,\mathrm{sh}\pi y}\left[\tfrac{1}{4}a(\mathrm{ch}\pi x - \mathrm{ch}\pi y)\right.$$

$$\times {}_4F_3\left(\frac{1+ix+iy}{2}, \frac{1+ix-iy}{2}, \frac{1+iy-ix}{2},\right.$$

$$\left.\frac{1-ix-iy}{2}; \tfrac{1}{2}, 1, \tfrac{3}{2}; -\tfrac{1}{4}a^2\right)$$

$$+a^2\frac{(\mathrm{ch}\pi x + \mathrm{ch}\pi y)}{y^2-x^2}\,{}_4F_3\left(1+\frac{i(x+y)}{2},\right.$$

$$1+\frac{i(x-y)}{2}, 1+\frac{i(x-y)}{2},$$

$$\left.\left.1-\frac{i(x+y)}{2}; \tfrac{3}{2}, \tfrac{3}{2}, 2; -\tfrac{1}{4}a^2\right)\right]. \qquad (1.16)$$

This integral is evaluated by expanding the exponential in a power series in v and integrating term by term. We omit the evaluation of the matrix elements of the operator $\mathbf{T}(0,a,-a)$.

The matrix elements of the operator $\mathbf{T}(\theta,0,0)$ are

$$T^K_{xy}(\theta,0,0) = \langle \exp(-\theta M)f^K_y, f^K_x\rangle$$

$$= \frac{2}{\pi^2}\sqrt{xy\,\mathrm{sh}\pi x\,\mathrm{sh}\pi y}\int_0^\infty K_{ix}(v)K_{iy}(e^\theta v)\frac{dv}{v}$$

$$= \cos(\theta y)\delta(x-y)$$

$$+\frac{1}{4\pi^2}\sqrt{xy\,\mathrm{sh}\pi x\,\mathrm{sh}\pi y}$$

$$\times\left[e^{iy\theta}\frac{\Gamma(1-iy)}{\Gamma[(ix-iy-1)/2]\Gamma[(-ix-iy-1)/2]}\right.$$

$$\times {}_2F_1\left(\frac{ix-iy-1}{2}, \frac{-ix-iy-1}{2}; -iy; \exp(-2\theta)\right)$$

$$+\exp(-iy\theta)\frac{\Gamma(1+iy)}{\Gamma[(ix+iy-1)/2]\Gamma[(-ix+iy-1)/2]}$$

$$\left.\times {}_2F_1\left(\frac{ix+iy-1}{2}, \frac{-ix+iy-1}{2}; iy; \exp(-2\theta)\right)\right] \qquad (1.17)$$

for $\theta > 0$.

For $\theta < 0$ the matrix elements can be obtained from the relation

$$T^K_{xy}(-\theta,0,0)=\overline{T^K_{yx}}(\theta,0,0).$$

C. The exponential or E basis

The basis defining operator in the realization (1.4) has the form

$$L_E=i\gamma(2e^{-x}D_x-e^{-x})=i\gamma(2D_v-v^{-1}).$$

Solutions of the eigenfunction equation $L_EF(v)=\lambda F(v)$ then are

$$F(v)=Cv^{1/2}\exp\left(-\frac{i\lambda}{2\gamma}v\right). \qquad (1.18)$$

These eigenfunctions do not form a complete set on the Hilbert space $\mathcal{H}^+$ on which the representation (1.2) is defined, i.e., the space of functions $f(v)$ for $0\le v<\infty$, $v=e^x$. The correct group[3] in which to realize this basis is $E'=E(1,1)\oplus\{R,I\}$ where R is the reflection operator in the pseudo-Euclidean plane and I is the identity operator. R acts on the generators of $E(1,1)$ according to

$$R:M\to M,\quad R:P_i\to -P_i\quad (i=1,2) \qquad (1.19)$$

The Hilbert space H on which the irreducible representation labelled by γ is realized is now the direct sum of two Hilbert spaces $H=\mathcal{H}^+\oplus\mathcal{H}^-$ with $\mathcal{H}^-$ the space of functions $f(v)$ for $-\infty<v\le 0$ which are square integrable with respect to the measure dv/v and transform under the group E' according to (1.2) with $v=e^x$ (remember $R:e^x\to -e^x$). In fact, we can write symbolically $\mathcal{H}^-=R\mathcal{H}^+$. Accordingly, each $f(v)\in H$ $(-\infty<v<\infty)$ satisfies the integrability condition

$$\int_{-\infty}^{\infty}|f(v)|^2\frac{dv}{v}<\infty \qquad (1.20)$$

with the group action given quite generally by (1.2) with $e^x=v$. The operator L_E is then essentially self adjoint and the eigenfunctions correspond to a form of the exponential solutions of the momentum operator. The spectrum of L_E is the real axis and a complete set of orthonomal eigenfunctions is

$$f^E_\lambda(v)=\frac{1}{2}\left(\frac{-v}{2\pi}\right)^{1/2}\exp\left(-\frac{i\lambda}{2\gamma}v\right) \qquad (1.21)$$

where

$$\langle f^E_\lambda,f^E_{\lambda'}\rangle=\int_{-\infty}^{\infty}f^E_\lambda(v)\overline{f^E_{\lambda'}(v)}\frac{dv}{v}$$

$$=\delta(\lambda-\lambda').$$

In (1.21) we make the consistent convention that the square root $(-v)^{1/2}$ for v positive be taken as $+|v|^{1/2}$. The matrix elements in the E basis can be easily calculated.

The matrix element for the unitary operator $\mathbf{T}(0,a,a)$ is

$$T^E_{\lambda\lambda'}(0,a,a)=\frac{-1}{4\pi\gamma}\int_{-\infty}^{\infty}\exp\left(\frac{i}{2\gamma}(\lambda'-\lambda)+i\gamma a\right)v\,dv$$

$$=\delta(\lambda-\lambda'-2\gamma^2a). \qquad (1.22)$$

For the operator $T(0,a,-a)$ we have the result

$$T^E_{\lambda\lambda'}(0,a,-a)=\frac{-1}{4\pi\gamma}\int_{-\infty}^{\infty}\exp\left(\frac{i}{2\gamma}(\lambda'-\lambda)v+\frac{i\gamma a}{v}\right)dv$$

$$=\delta\;=\delta(\lambda-\lambda')+\gamma\sqrt{2}a/(\lambda'-\lambda)\,\mathrm{sign}(\lambda'-\lambda)$$

$$\times J_1(2a\sqrt{\lambda'-\lambda}). \qquad (1.23)$$

This matrix element can be evaluated by expanding $\exp(i\gamma a/v)$ in a power series then integrating term by term in the sense of generalized functions.[8] Alternatively, contour integration of the regular part of the matrix element will give the same result.

The matrix element of the operator $\mathbf{T}(\theta,0,0)$ is

$$T^E_{\lambda\lambda'}(\theta,0,0)=\frac{-1}{4\pi\gamma}\int_{-\infty}^{\infty}\exp\left(\frac{i}{2\gamma}(\lambda'-e^\theta\lambda)v\right)dv$$

$$=\delta(e^\theta\lambda-\lambda'). \qquad (1.24)$$

D. The Airy function or A basis

In the realization (1.4) L_A has the form

$$L_A=i\gamma(2e^{-x}D_x-e^{-x})-\gamma^2e^{2x} \qquad (1.25)$$

$$=i\gamma(2D_v-v^{-1})-\gamma^2v^2.$$

The solutions of the eigenfunction equation $L_AF(v)=\lambda F(v)$ are

$$F(v)=v^{1/2}\exp\left(\frac{i}{6}\gamma v^3-\frac{i}{2}\frac{\lambda}{\gamma}v\right). \qquad (1.26)$$

As with the E basis these eigenfunctions do not form a complete set on the space $\mathcal{H}^+$ of functions $f(v)$ with $0\le v<\infty$. This space is extended in exactly the same way as for the E basis. A complete set of orthonormal eigenfunctions on $H=\mathcal{H}^+\oplus\mathcal{H}^-$ is then

$$f^A_\lambda=\frac{1}{2}\left(\frac{-v}{\pi\gamma}\right)^{1/2}\exp\left(\frac{i}{6}\gamma v^3-\frac{i}{2}\frac{\lambda}{\gamma}v\right) \qquad (1.27)$$

with

$$\langle f^A_\lambda,f^A_{\lambda'}\rangle=\delta(\lambda-\lambda').$$

The matrix elements in this basis can be easily calculated. For the translations $\mathbf{T}(0,a,\pm a)$ the results are the same as for the E basis, viz., (1.22) and (1.23). For the matrix element of the operator $\mathbf{T}(\theta,0,0)$ we have a new result:

$$T^A_{\lambda\lambda'}(\theta,0,0)=\frac{-1}{4\pi\gamma}\int_{-\infty}^{\infty}\exp\left(\frac{i}{6}\gamma(e^{3\theta}-1)v^3+\frac{i}{2\gamma}(\lambda'-\lambda)v\right)dv$$

$$=\frac{-1}{4\gamma}\left(\frac{2}{\gamma}\right)^{1/3}(e^{3\theta}-1)^{-1/3}\,\mathrm{Ai}\left(\frac{\lambda'-\lambda}{[2\gamma^2(e^{3\theta}-1)]^{1/3}}\right) \qquad (1.28)$$

for $\theta>0$ and where $\mathrm{Ai}(z)$ is an Airy function. The matrix element for $\theta<0$ can be obtained by using the result

$$T^A_{\lambda\lambda'}(-\theta,0,0)=\overline{T^A_{\lambda'\lambda}}(\theta,0,0).$$

2. OVERLAP FUNCTIONS

In this section we compute functions of the form

$$U^{G,H}_{n,m}=\langle f^G_n,f^H_m\rangle=\int_{-\infty}^{\infty}f^G_n(x)\overline{f^H_m}(x)dx \qquad (2.1)$$

which allow us to pass from the $\{f^G\}$ basis to the $\{f^H\}$ basis via the expression

$$f_n^G(x)=\sum_m U_{n,m}^{G,H} f_m^H(x). \tag{2.2}$$

(For $H=M,K,A,E$ the sum should be replaced by an integral.) Note that

$$U_{n,m}^{G,H}=\overline{U}_{m,n}^{H,G}, \tag{2.3a}$$

$$U_{n,m}^{G,G}=\delta_{n,m}, \quad n,m \text{ discrete},$$

$$=\delta(n-m), \quad n,m \text{ continuous}, \tag{2.3b}$$

$$U_{n,m}^{G,H}=\sum_p U_{n,p}^{G,J} U_{p,m}^{J,H}. \tag{2.3c}$$

In the following we compute the various $U_{n,m}^{G,H}$ by substituting the explicit expressions for $f_n^G(x)$, $f_m^H(x)$ from Sec. 1 into (2.1) and evaluating the integral. In case both L_G and L_H have continuous spectrum then expressions (2.1), (2.2) must be interpreted in the sense of generalized functions.

First we relate all bases to the standard M basis:

$$U_{n,\lambda}^{G,M}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_n^G(x)\exp(-i\lambda x)dx.$$

The results are

$$U_{n,\lambda}^{B,M}=\langle f_n^{B,\alpha}, f_\lambda^M\rangle=\frac{\sqrt{\alpha+2n}}{\pi}\int_0^\infty J_{\alpha+2n}(\gamma v)v^{-1-i\lambda}dv \tag{2.4}$$

$$=\left(\frac{\alpha+2n}{\pi}\right)^{1/2}\frac{(\gamma/2)^{i\lambda}}{2}\frac{\Gamma(\alpha+n-i\lambda/2)}{\Gamma[1+n+(\alpha+i\lambda)/2]},$$

$$U_{x,\lambda}^{K,M}=\langle f_x^K, f_\lambda^M\rangle$$

$$=\frac{1}{\pi^{3/2}}\sqrt{x\operatorname{sh}\pi x}\int_0^\infty v^{-i\lambda-1}K_{ix}(\gamma v)\,dv$$

$$=\frac{i}{2}\left(\frac{x}{\pi\operatorname{sh}\pi x}\right)^{1/2}\left(\frac{(2\gamma)^{+ix}}{\Gamma(1+ix)}\delta(x-\lambda)-\frac{(2\gamma)^{-ix}}{\Gamma(1-ix)}\delta(x+\lambda)\right)$$

$$+\frac{1}{4\pi}\left(\frac{x\operatorname{sh}\pi x}{\pi}\right)\left(\frac{\gamma}{2}\right)^{i\lambda}\Gamma\left(\frac{ix-i\lambda}{2}\right)\Gamma\left(\frac{ix+i\lambda}{2}\right), \tag{2.5}$$

$$U_{K,\lambda}^{E,M}=\frac{i}{2\pi(2\gamma)^{1/2}}\int_0^\infty v^{-i\lambda-1/2}\exp\left(-\frac{iK}{2\gamma}v\right)dv$$

$$=-\frac{\Gamma(\frac{1}{2}-i\lambda)}{2\pi\sqrt{2\gamma}}e^{\epsilon(i\pi/4+\pi\lambda/2)}\left|\frac{K}{2\gamma}\right|^{-1/2+i\lambda} \tag{2.6}$$

where $\epsilon=+1$ if $K<0$ and -1 if $K>0$. We have

$$U_{K,\lambda}^{A,M}=\frac{i}{2\pi(2\gamma)^{1/2}}\int_0^\infty v^{-i\lambda-1/2}\exp\left(\frac{i\gamma}{6}v^3-\frac{iK}{2\gamma}v\right)dv$$

$$=\frac{i}{2\pi(2\lambda)^{1/2}}\left(\frac{\gamma}{6i}\right)^{(2i\lambda-1)/6}\sum_{n=0}^\infty\frac{\Gamma[(n-i\lambda)/3+\frac{1}{6}]}{n!}$$

$$\times\left(\frac{e^{-i\pi/6}K}{2\gamma}\right)^n\left(\frac{6}{\gamma}\right)^{n/3}. \tag{2.7}$$

This expression can also be written as a sum of three ${}_1F_2(a;b,c;z)$ hypergeometric functions.

It should be mentioned here that the overlap coefficients which we have given relating the E and A bases to the M basis are not the complete coefficients with respect to the group E' and hence are not unitary. The coefficients we have calculated only relate these bases on the Hilbert space $\mathcal{H}^+$. A similar calculation for the space $\mathcal{H}^-$ can be made but we do not do this here. For the unitary irreducible representations of E' in a B, K, or M basis we have basis functions labelled by an additional discrete label corresponding to the eigenvalues ± 1 of the reflection operator. This is because R commutes with the operators L_B, L_K, and L_M. For the A and E bases however R does not commute with L_A or L_E. Hence no such labels are required. For the purposes of this paper we have not introduced this discrete label, it being understood whenever we give an overlap function.

We now give a number of further overlap functions of interest:

$$U_{n,x}^{B,K}=\langle f_n^{B,\alpha}, f_x^K\rangle$$

$$=\frac{1}{2\pi}\sqrt{(\alpha+2n)x\operatorname{sh}\pi x}$$

$$\times\frac{\Gamma[n+(\alpha+ix)/2]\Gamma[n+(\alpha-ix)/2]}{\Gamma(1+\alpha+2n)}$$

$$\times{}_2F_1\left(n+\frac{\alpha+ix}{2},\ n+\frac{\alpha-ix}{2};\ 1+\alpha+2n;-1\right), \tag{2.8}$$

$$U_{\lambda,\lambda'}^{E,A}=\langle f_\lambda^E, f_{\lambda'}^A\rangle$$

$$=-(2\gamma^2)^{-1/3}\operatorname{Ai}((\lambda-\lambda')(2\gamma^2)^{-1/3}),$$

$$U_{x,\lambda}^{K,E}=-\frac{i\sqrt{x\operatorname{sh}\pi x}}{\pi}\frac{(2\gamma)^{2ix-1/2}}{(2\gamma^2+i\lambda)^{ix+1/2}}\Gamma(\tfrac{1}{2}+ix)\Gamma(\tfrac{1}{2}-ix)$$

$$\times{}_2F_1\left(\tfrac{1}{2}+ix,\tfrac{1}{2}+ix;1;\frac{2\gamma^2+i\lambda}{2\gamma^2-i\lambda}\right). \tag{2.9}$$

3. A TWO-VARIABLE MODEL FOR $E(1,1)$

As mentioned in Sec. 1, $E(1,1)$ acts as a transformation group in the pseudo-Euclidean plane. We choose this action in the s-t plane so that the Lie derivatives corresponding to the Lie algebra basis $\{P_1,P_2,M\}$ are

$$P_1=\partial_s,\quad P_2=\partial_t,\quad M=-t\partial_s-s\partial_t. \tag{3.1}$$

We now construct models of the irreducible representations of $E(1,1)$ where the Lie algebra acts via the operators (3.1) rather than (1.4). In particular, we construct the two-variable analogs of the basis functions $\{f_n^G\}$.

In the one-variable model we have $(P_2^2-P_1^2)f_n^G=\gamma^2 f_n^G$ for each basis function f, so we would expect the same equation to hold in the two-variable model, i.e.,

$$(\partial_t^2-\partial_s^2)F_n^G(s,t)=\gamma^2F_n^G(s,t),$$

where $F_n^G(s,t)$ is the two-variable function corresponding to $f_n^G(x)$. In the following we will define a mapping $f(x)\to F(s,t)$ from $L_2(R)$ to functions on the pseudo-Euclidean plane such that $(\partial_t^2-\partial_s^2)F=\gamma^2F$ and such the eigenfunctions $f_n^G(x)$ of L_G map to eigenfunctions $F_n^G(s,t)$ of the corresponding operator L_G constructed from (3.1). Because of the close relationship between separation of variables and operators L_G we can find simple expressions for the $\{F_n^G(s,t)\}$ in terms of the coordinates associated with L_G. (The single exception to this statement is the case $G=E$ where there is no associated coordinate system in which the variables separate.)

To make our construction precise we introduce the functions

$$h_{s,t}(x)=\exp[i\gamma(s\cosh x+t\sinh x)], \quad s,\ t\in \mathbf{C}. \tag{3.2}$$

which belong to $L_2(R)$ for $\operatorname{Im}\gamma(s\pm t)>0$. Given $f(x)\in L_2(R)$, we define a function $F(s,t)$ by

$$F(s,t)=\langle f,\overline{h_{s,t}}\rangle=\int_{-\infty}^{\infty}f(x)h_{s,t}(x)dx. \tag{3.3}$$

In particular, corresponding to a basis $\{f_n^G\}$ for $L_2(R)$ we obtain functions

$$F_n^G(s,t)=\langle f_n^G,\overline{h_{s,t}}\rangle. \tag{3.4}$$

The action (1.2) of $E(1,1)$ on $L_2(R)$ induces an action on the $F(s,t)$ which satisfies the homomorphism property:

$$\begin{aligned}[\mathrm{T}(\theta,a,b)F](s,t)&\equiv\langle \mathrm{T}(\theta,a,b)f,\overline{h_{s,t}}\rangle\\ &=\langle f,\mathrm{T}(\theta,a,b)^{-1}\overline{h_{s,t}}\rangle\\ &=F((s+a)\cosh\theta-(t+b)\sinh\theta,\\ &\qquad(t+b)\cosh\theta-(s+a)\sinh\theta).\end{aligned} \tag{3.5}$$

It is easy to check that the Lie derivatives corresponding to the group action (3.5) coincide with (3.1). Thus the operators (1.4) acting on f correspond to the operators (3.1) acting on F.

On the other hand, for f a basis vector f_n^G we have

$$\begin{aligned}[\mathrm{T}(\theta,a,b)F_n^G](s,t)&=\langle \mathrm{T}(\theta,a,b)f_n^G,\overline{h}_{s,t}\rangle\\ &=\sum_m T_{mn}^G(\theta,a,b)F_m^G(s,t)\end{aligned} \tag{3.6}$$

where the T_{mn}^G are the G-basis matrix elements. It follows from (3.5) and (3.6) that the $\{F_n^G\}$ transform under $E(1,1)$ exactly as the basis vectors $\{f_n^G\}$. In particular,

$$(P_2^2-P_1^2)F_n^G=\gamma^2F_n^G,\quad L_GF_n^G=\lambda_nF_n^G \tag{3.7}$$

[where P_1,P_2,L_G are expressed in terms of the operators (3.1)], provided $L_Gf_n^G=\lambda_nf_n^G$. Relations (3.6) also hold even for $\operatorname{Im}\gamma(s\pm t)=0$ if the $\{f_n^G\}$ belong to $L_1(R)$.

If $h_{s,t}\in L_2(R)$ it follows immediately that

$$h_{s,t}(x)=\sum_n \bar{f}_n^G(x)F_n^G(s,t) \tag{3.8}$$

where the right-hand side converges in $L_2(R)$ and also pointwise. (As usual, if L_G has continuous spectrum we replace the sum by an integral.) We can consider (3.8) as the expansion of a plane wave in a $\{F_n^G\}$ basis of solutions of the Helmholtz equation.

It follows directly from the definition of $h_{s,t}(x)$ that

$$\langle h_{s_1},t_1,h_{\bar{s}_2,\bar{t}_2}\rangle=2K_0(i\gamma[(s_1-s_2)^2-(t_1-t_2)^2]^{1/2}). \tag{3.9}$$

On the other hand, use of (3.8) yields

$$\langle h_{s_1,t_1},h_{\bar{s}_2,\bar{t}_2}\rangle=\sum_n F_n^G(s_1,t_1)\bar{F}_n^G(\bar{s}_2,\bar{t}_2). \tag{3.10}$$

Comparison of (3.9) and (3.10) yields another generating function for the $\{F_n^G\}$.

The overlap functions computed in Sec. 2 carry over immediately to the two variable model. Indeed, expression (2.2) relating the bases $\{f_n^G\}$ and $\{f_m^H\}$ yields

$$F_n^G(s,t)=\sum_m U_{n,m}^{G,H}F_m^H(s,t) \tag{3.11}$$

with the same overlap functions $U_{n,m}^{G,H}$.

It follows from these remarks that the functions $\{F_n^G\}$ will necessarily satisfy the identities (3.5)—(3.11) where the matrix elements $T_{mn}^G(\theta,a,b)$ and overlap functions $U_{n,m}^{G,H}$ have already been computed from the one-variable model. Moreover, due to the relationship between the operators L_G and separation of variables for the Helmholtz equation we can find simple expressions for the function $\{F_n^G\}$ in terms of the coordinate system related to L_G. Indeed, evaluating the integral (3.4) in each case, we find

$$\begin{aligned}&F_\lambda^M[\rho,\theta]=\sqrt{2}/\pi e^{-i\lambda\theta}K_{i\lambda}(i\gamma\rho),\\ &s=\rho\cosh\theta,\quad t=\rho\sinh\theta,\end{aligned} \tag{3.12}$$

$$\begin{aligned}F_n^{D,\alpha}[\xi,\eta]&=2\exp[3i\pi(\alpha+2n-1)/2]D_{-\alpha-2n-1/2}(\sqrt{-2\gamma}\,\xi)\\ &\quad\times D_{\alpha+2n-1/2}(\sqrt{-2\gamma}\,\eta),\\ &s=i\xi\eta,\quad t=(\eta^2-\xi^2)/2,\end{aligned} \tag{3.13}$$

$$\begin{aligned}&F_n^{B,\alpha}[u,v]=2\sqrt{2(\alpha+2n)}\,J_{\alpha+2n}(\gamma u)K_{\alpha+2n}(-i\gamma v),\\ &s=\frac{u^2+u^2v^2+v^2}{2uv},\quad t=\frac{u^2-u^2v^2+v^2}{2uv},\quad \left|\frac{u}{v}\right|<1,\end{aligned} \tag{3.14}$$

$$\begin{aligned}&F_x^K[u,v]=\frac{2}{\pi}\sqrt{x\,\mathrm{sh}\,\pi x}\;K_{ix}(\gamma u)K_{ix}(-i\gamma v),\\ &s=\frac{u^2-u^2v^2-v^2}{2uv},\quad t=\frac{u^2+u^2v^2-v^2}{2uv},\quad \left|\frac{u}{v}\right|>1,\end{aligned} \tag{3.15}$$

$$\begin{aligned}F_\lambda^E[s,t]&=\int_{-\infty}^{\infty}\exp\left(\frac{i\gamma(s+t)}{2}v+\frac{i\gamma(s-t)}{2v}\right)f_\lambda^E(v)\frac{dv}{v}\\ &=\frac{i}{2\sqrt{2\gamma^2(s+t)-2\lambda}}\exp\sqrt{\gamma(t^2-s^2)-(\lambda/\gamma)(t-s)},\\ &\qquad s+t>\lambda/\gamma^2,\quad t>s,\\ &=\frac{i}{2\sqrt{2\gamma^2(s+t)-2\lambda}}\cos\sqrt{\gamma(s^2-t^2)-(\lambda/\gamma)(s-t)},\\ &\qquad s+t>\lambda/\gamma^2,\quad s>t.\end{aligned} \tag{3.16}$$

Similar expressions can be given for the other ranges of s and t:

$$F_\lambda^A[x_1,x_2]=\int_0^{\infty}\exp\left(\frac{i\gamma(s+t)}{2}v+\frac{i\gamma(s-t)}{2v}\right)f_\lambda^A(v)\frac{dv}{v}$$

$$=A\phi_1(y_1)\phi_1(y_2)+B(\phi_1(y_1)\phi_2(y_2)$$

$$+\phi_1(y_2)\phi_2(y_1))+C\phi_2(y_1)\phi_2(y_2), \qquad (3.17)$$

where

$$\phi_1(y)={}_0F_1(\tfrac{2}{3},\tfrac{1}{9}y_1^3), \quad \phi_2(y)={}_0F_1(\tfrac{4}{3},\tfrac{1}{9}y_1^3).$$

The coefficients are given by

$$A=\frac{\Gamma(\frac{1}{6})}{6\sqrt{\pi\gamma}}\left(\frac{6}{\gamma}\right)^{1/6}, \quad B=-\frac{2i\gamma^{5/2}}{3}\left(\frac{6}{\gamma}\right)^{2/3},$$

$$C=\frac{16i\gamma^{11/2}}{3}\left(\frac{6}{\gamma}\right)^{5/6}\Gamma\left(\frac{5}{6}\right),$$

s and t are given by the relations

$$2s=-\tfrac{1}{2}(x_1-x_2)^2+(x_1+x_2),$$

$$2t=-\tfrac{1}{2}(x_1-x_2)^2-(x_1+x_2),$$

and

$$y_i=\frac{1}{4\gamma^2}\left(x_i\frac{\lambda}{4\gamma^2}\right) \quad (i=1,2).$$

The expression we have given for the A basis functions in the two parameter model can also be written as a sum of products of Airy functions. One comment should be made here concerning the $F_\lambda^E[s,t]$ functions. These functions indicate that for the two variable model the E basis functions do not afford a separation of variables. This is in agreement with an earlier result.[3]

4. REPRESENTATIONS OF $CE(2)$

For the purpose of relating Lie group theory to special functions it is imperative to study group representations which have no Hilbert space structure, in particular representations defined on spaces of analytic functions. Some example of these were given in Refs. 5 and 9. For such representations one can always assume that the group is complex and we shall do so here by allowing the parameters θ, a, b in (1.1) to take arbitrary complex values. Thus, we shall consider representations of the complex Euclidean group $CE(2)$.

The Lie algebra $ce(2)$ of $CE(2)$ consists of all complex linear combinations of the generators M, P_1, P_2 with commutation relations

$$[M,P_1]=P_2, \quad [M,P_2]=P_1, \quad [P_1,P_2]=0. \qquad (4.1)$$

We consider a model of this algebra in which the generators are given by

$$M=z\frac{d}{dz}, \quad P^+=\rho z, \quad P^-=\rho z^{-1}, \qquad (4.2)$$

$$P^\pm=P_1\pm P_2$$

acting on the space $\mathcal{J}^\nu$ of functions $f(z)$ analytic in the domain $|z|>0$ with periodicity $f(e^{2\pi i}z)=e^{2\pi i\nu}f(z)$. Here $\nu\in\mathbb{C}$ is not an integer. The eigenfunctions of the operator $L_K=M^2+(P_1+P_2)^2=M^2+(P^+)^2$ on this space are easily seen to be

$$f_n^K(z)=J_{\nu+n}(\rho z), \quad n=0, \pm 1, \cdots,$$

$$L_K f_n^K=(\nu+n)^2 f_n^K \qquad (4.3)$$

to within a constant factor. Moreover, as shown in Ref. 10, p. 204, every $f\in\mathcal{J}^\nu$ can be expanded as an infinite series in the eigenfunctions f_n^K

$$f(z)=\sum_{n=-\infty}^{\infty} c_n J_{\nu+n}(\rho z), \qquad (4.4)$$

where the coefficients c_n are given by

$$c_n=\frac{1}{2\pi i}\frac{\pi(\nu+n)}{\sin\pi(\nu+n)}\int f(z)J_{-\nu-n}(\rho z)\frac{dz}{z} \qquad (4.5)$$

and the pointwise convergence in (4.4) is uniform on compact subsets of the annulus. The path of integration in (4.5) can be chosen as a circle centered at the origin with radius $r>0$.

It follows from (4.2) that the action of $CE(2)$ on $\mathcal{J}^\nu$ is given by operators $\mathbf{T}(\theta,a,b)$,

$$[\mathbf{T}(\theta,a,b)f](z)=\exp\frac{\rho}{2}[(a+b)z+(a-b)/z]f(e^\theta z), \qquad (4.6)$$

and that $\mathcal{J}^\nu$ is invariant under this action. Thus, we can use expressions (4.4) and (4.5) to compute the matrix elements of the operators $M, P^\pm$ and $\mathrm{T}(\theta,a,b)$ in the $\{f_n^K\}$ basis. It is straightforward to show

$$P^+f_n^K=\sum_{m=0}^{\infty}(-1)^m(\nu+n+2m+1)f_{n+2m+1}^K,$$

$$P^-f_n^K=\frac{\rho^2}{2(\nu+n)}(f_{n-1}^K+f_{n+1}^K), \qquad (4.7)$$

$$Mf_n^K=\frac{(\nu+n)}{2}f_n^K-\sum_{m=0}^{\infty}(-1)^m(\nu+n+2m+2)f_{n+2m+2}^K,$$

$$T(0,a,a)_{m,n}=\begin{cases}\dfrac{(2a)^l\Gamma(\nu+m+1)}{\Gamma(\nu+n+1)l!}\,{}_4F_3\left(\begin{matrix}-\frac{l}{2},\ \frac{1-l}{2},\ 1-\frac{l}{2},\ \frac{1}{2}-\frac{l}{2}\\ 1-l,\ -\nu-m+1,\ \nu+n+1\end{matrix}\,\middle|\,\dfrac{-4}{a^2}\right), & \text{if } m-n=l\geq 0,\\ 0 & \text{if } m-n<0,\end{cases} \qquad (4.8)$$

$$T(0,a,-a)_{m,n}=\frac{(\rho^2a/2)^{n-m}\Gamma(\nu+m+1)}{\Gamma(\nu+n+1)\Gamma(n-m+1)}\times{}_0F_3\left(-\nu-m+1,\ \nu+n+1,\ n-m+1;\ -\frac{\rho^4a^2}{4}\right),$$

$$T(\theta,0,0)_{m,n}=\begin{cases}\dfrac{(-1)^l e^{(\nu+n)\theta}\Gamma(\nu+m+1)}{l!\,\Gamma(\nu+n+1)}\,{}_2F_1\left(\begin{matrix}-l,\ \nu+m-l\\ \nu+n+1\end{matrix}\,\middle|\,e^{2\theta}\right), & \text{if } m-n=2l,\ l=0,1,2,\ldots,\\ 0 & \text{otherwise.}\end{cases}$$

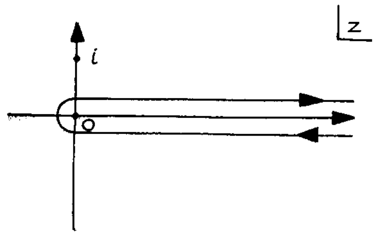

FIG. 1. Contour of Integration.

Note that in our model $P_1^2 - P_2^2 = P^+P^- = \rho^2$.

Next we construct a model of this representation in terms of functions $F_n^K(s,t)$ in the complex $s-t$ plane. Here,

$$P_1 = \partial_s, \quad P_2 = \partial_t, \quad M = -t\partial_s - s\partial_t, \tag{4.9}$$

so the basis functions $F_n^K(s,t)$ analogous to $f_n^K(z)$ must satisfy the equations

$$(\partial_t^2 - \partial_s^2)F_n^K(s,t) = -\rho^2 F_n^K(s,t), \quad L_K F_n^K(s,t) = (\nu+n)F_n^K(s,t). \tag{4.10}$$

In analogy with a similar construction in Ref. 5 and (3.3), we set

$$F_n^K(s,t) = \int_C \exp\frac{\rho}{2}[z(s+t) + z^{-1}(s-t)]f_n^K(z)\frac{dz}{z}, \quad \mathrm{Re}(s+t) < 0, \tag{4.11}$$

where C is the contour in the complex z plane (see Fig. 1). By differentiating under the integral sign in (4.11) and integration by parts it is easy to show that the generators (4.2) acting on $f_n^K(z)$ correspond to the generators (4.9) acting on $F_n^K(s,t)$. Thus Eqs. (4.10) must hold. This suggests that the $F_n^K(s,t)$ are simply expressable in terms of the u-v coordinates,

$$s = \frac{u^2 + u^2v^2 + v^2}{2uv}, \quad t = \frac{u^2 - u^2v^2 + v^2}{2uv}.$$

Indeed, a direct computation yields

$$F_n^K[u,v] = 4i\exp i(\pi/2)(n-\nu)\sin\pi\nu\sqrt{2(\nu+n)}\, I_{\nu+n}(-\rho u)K_{\nu+n}(-\rho v), \quad |u/v| < 1. \tag{4.12}$$

From (4.9) it follows that the action of $CE(2)$ on the basis $F_n^K(s,t)$ is given by

$$[\mathrm{T}(\theta,a,b)F_n^K](s,t) = F_n^K((s+a)\cosh\theta - (t+b)\sinh\theta, (t+b)\cosh\theta - (s+a)\sinh\theta). \tag{4.13}$$

On the other hand, by construction we have

$$[\mathrm{T}(\theta,a,b)F_n^K](s,t) = \sum_{m=-\infty}^{\infty} T(\theta,a,b)_{mn}F_m^K(s,t), \quad \mathrm{Re}[(s+t+a+b)e^{-\theta}] < 0, \tag{4.14}$$

where the matrix elements $T(\theta,a,b)_{mn}$ are given by (4.8). Comparison of expressions (4.12)–(4.14) yields addition theorems for the basis (4.12) whose direct derivation is not at all obvious. Other choices of the contour C in (4.11) will yield different bases satisfying (4.13) and (4.14).

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