

Four-body problem in d -dimensional space: ground state, (quasi)-exact-solvability. IV

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Abstract

Due to its great importance for applications, we generalize and extend the approach of our previous papers to study aspects of the quantum and classical dynamics of a 4-body system with equal masses in d -dimensional space with interaction depending only on mutual (relative) distances. The study is restricted to solutions in the space of relative motion which are functions of mutual (relative) distances only. The ground state (and some other states) in the quantum case and some trajectories in the classical case are of this type. We construct the quantum Hamiltonian for which these states are eigenstates. For $d \geq 3$, this describes a 6-dimensional quantum particle moving in a curved space with special d -independent metric in a certain d -dependent singular potential, while for $d = 1$ it corresponds to a 3-dimensional particle and coincides with the A_3 (4-body) rational Calogero model; the case $d = 2$ is exceptional and is discussed separately. The kinetic energy of the system has a hidden $sl(7, \mathbf{R})$ Lie (Poisson) algebra structure, but for the special case $d = 1$ it becomes degenerate with hidden algebra $sl(4, R)$. We find an exactly-solvable 4-body S_4 -permutationally invariant, generalized harmonic oscillator-type potential as well as a quasi-exactly-solvable 4-body sextic polynomial type potential with singular terms. The tetrahedron whose vertices correspond to the positions of the particles provides pure geometrical variables, volume variables, that lead to exactly solvable models. Their generalization to the n -body system as well as the case of non equal masses is briefly discussed.

INTRODUCTION

Consider 4 classical particles in d -dimensional space with potential depending on mutual relative distances alone. After separation of the center-of-mass motion, and assuming zero total (relative) angular momentum, the trajectories are defined by evolution of the relative (mutual) distances. It is an old question to find equations for trajectories which depend on relative distances only; in the 3-body case this problem can be traced back to J-L Lagrange (1772). In general, this problem was solved for the 3-body case in [1, 2]. The vector positions of 4 particles in a 3-dimensional space form a tetrahedron; the corresponding edges are nothing but the 6 relative distances between the particles. Thus, we can formulate the problem in terms of the evolution of such a geometrical object. We call it the *tetrahedron of interaction*.

The aim of the present paper is to construct the 4-body Hamiltonian which depends on the 6 relative distances and describes the motion of the tetrahedron of interaction in d -dimensional space. Our strategy is to study the quantum problem first for $d \geq 3$. Then, using geometrical variables obtained from the tetrahedron, we impose constraints on the edges (relative distances) and faces to degenerate the Hamiltonian to the planar $d = 2$ and 1-dimensional $d = 1$ cases. The corresponding classical Hamiltonian is obtained through the *de-quantization* procedure [3], of replacement of the quantum momentum by the classical one with preservation of positivity of kinetic energy. In [3], we studied the n -body system for $d \geq n - 1$ while in the present paper we will introduce new geometrical variables which allow to analyze the case $d < n - 1$.

The quantum Hamiltonian for 4 particles, in a d -dimensional Euclidean space, with a translation-invariant potential depending on relative (mutual) distances between particles only, is of the form

$$\mathcal{H} = - \sum_{i=1}^4 \frac{1}{2m_i} \Delta_i^{(d)} + V(r_{ij}) , \quad (1)$$

where $\Delta_i^{(d)}$ is the d -dimensional Laplacian,

$$\Delta_i^{(d)} = \frac{\partial^2}{\partial \mathbf{r}_i \partial \mathbf{r}_i} , \quad (2)$$

associated with the i th body with coordinate vector $\mathbf{r}_i \equiv \mathbf{r}_i^{(d)} = (x_{i,1}, x_{i,2}, x_{i,3} \dots, x_{i,d})$, and

$$r_{ij} = |\mathbf{r}_i - \mathbf{r}_j| , \quad i, j = 1, 2, 3, 4 , \quad (3)$$

the (relative) distance between particles i and j , $r_{ij} = r_{ji}$. For simplicity, unless otherwise stated, all masses in (1) are assumed to be equal: $m_i = m = 1$. The eigenvalue problem for \mathcal{H} is defined on the configuration space \mathbb{R}^{4d} .

The number of relative distances r_{ij} is equal to the number of edges of the tetrahedron which is formed by taking the particles' positions as vertices. We call this tetrahedron the *tetrahedron of interaction*, see for illustration Fig.1.

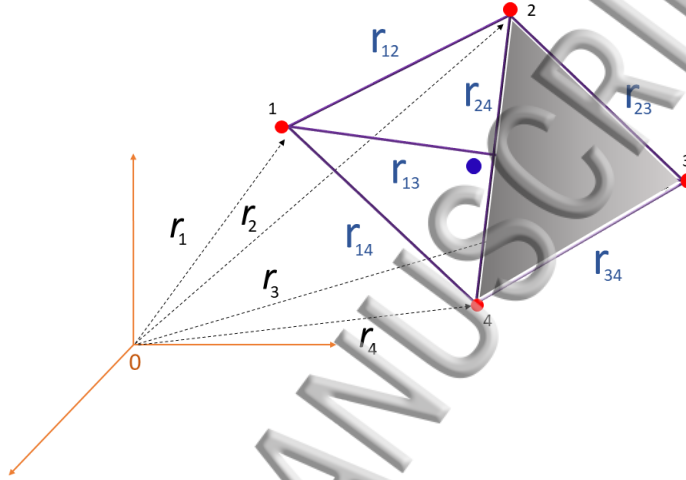


FIG. 1. 4-body system: at $d = 3$, the coordinate vectors \mathbf{r}_i mark positions of vertices of the tetrahedron of interaction with sides r_{ij} . For illustration one of the faces of this tetrahedron (shaded triangle) and the center-of-mass (blue large bubble) are marked.

The center-of-mass motion described by vectorial coordinate

$$\mathbf{R}_{CM} = \frac{1}{\sqrt{4}} \sum_{k=1}^4 \mathbf{r}_k, \quad (4)$$

can be separated out; this motion is described by a d -dimensional plane wave, $\sim e^{i\mathbf{k}\cdot\mathbf{R}_{CM}}$.

The spectral problem is formulated in the space of relative motion $\mathfrak{R}_{rel} \equiv \mathbb{R}^{3d}$; it is of the form,

$$\mathcal{H}_{rel} \Psi(x) \equiv \left(-\frac{1}{2} \Delta_{rel}^{(3d)} + V(r_{ij}) \right) \Psi(x) = E \Psi(x), \quad \Psi \in L_2(\mathfrak{R}_{rel}), \quad (5)$$

where $\Delta_{rel}^{(3d)}$ is the flat-space Laplacian in the space of relative motion.

If the space of relative motion \mathfrak{R}_{rel} is parameterized by 3, d -dimensional vectorial Jacobi

coordinates, then

$$\mathbf{q}_j = \frac{1}{\sqrt{j(j+1)}} \sum_{k=1}^j k (\mathbf{r}_{k+1} - \mathbf{r}_k), \quad j = 1, 2, 3, \quad (6)$$

and the flat-space $3d$ -dimensional Laplacian in the space of relative motion becomes diagonal

$$\Delta_{rel}^{(3d)} = \sum_{j=1,2} \frac{\partial^2}{\partial \mathbf{q}_j \partial \mathbf{q}_j}. \quad (7)$$

Thus, \mathbf{q}_j plays the role of the Cartesian coordinate vector in the space of relative motion.

The cases $d = 2$ (4 bodies on a plane) and $d = 1$ (4 bodies on a line) are special. For $d = 2$ the *tetrahedron of interaction* degenerates either into a quadrangle with 4 external vertices or a triangle with 3 external vertices and 1 internal (in both cases the volume of the tetrahedron vanishes, it plays the role of a constraint). For $d = 1$ the tetrahedron degenerates into an interval: the vertices of the tetrahedron correspond to 2 endpoints and 2 marked points inside the interval, and the volume of tetrahedron as well as the areas of all faces (triangles) are equal to zero identically. This implies that on the line ($d = 1$) the relative variables obey 3 constraints

$$x_{12} + x_{31} + x_{23} = 0, \quad x_{13} + x_{41} + x_{34} = 0, \quad x_{12} + x_{24} + x_{41} = 0, \quad (8)$$

where it is assumed that x_i denotes the position of the i th body and $x_{ij} = x_i - x_j$. Hence, the 6 relative distances are related and only 3 of them are independent. Therefore, see [4]

$$\Delta_{rel}^{(3)} = 2 \left(\frac{\partial^2}{\partial x_{12}^2} + \frac{\partial^2}{\partial x_{13}^2} + \frac{\partial^2}{\partial x_{14}^2} + \frac{\partial^2}{\partial x_{12} \partial x_{13}} + \frac{\partial^2}{\partial x_{12} \partial x_{14}} + \frac{\partial^2}{\partial x_{13} \partial x_{14}} \right). \quad (9)$$

cf. (7). The configuration space \mathfrak{R}_{rel} is $0 < x_{12} < x_{13} < x_{14} < \infty$. Now,

Observation [5]:

There exists a family of eigenstates of the Hamiltonian (1), including the ground state, which depends on 6 relative distances $\{r_{ij}\}$ only. The same is correct for the n body problem: there exists a family of the eigenstates, including the ground state, which depends on relative distances only.

This observation is presented for the case of scalar particles, bosons. It can be generalized to the case of fermions, namely:

In the case of 4 fermions there exists a family of the eigenstates of the Hamiltonian (1), including the ground state, in which the coordinate functions depend on 6 relative distances $\{r_{ij}\}$ only. The same is correct for the n body problem, see [3]: there exists a family of the eigenstates, including the ground state, in which the coordinate functions depend on relative distances only.

Our primary goal is to find the differential operator, in the form of the Hamiltonian with positive-definite kinetic energy, in the space of relative distances $\{r_{ij}\}$ for which these states are eigenstates. In other words, to find a differential equation depending only on $\{r_{ij}\}$ for which these states are solutions. This implies a study of the evolution of the tetrahedron of interaction with fixed center-of-mass. We consider the case of 4 spinless particles.

I. GENERALITIES

As a first step let us change the variables (6) in the space of relative motion \mathfrak{R}_{rel} :

$$(\mathbf{q}_j) \leftrightarrow (\{r_{ij}\}, \{\Omega\}) . \quad (10)$$

This is a generalization of the Euler coordinates; where for $d > 2$ the number of (independent) relative distances $\{r_{ij}\}$ is equal to 6 and $\{\Omega\}$ is a collection of $(3d - 6)$ angular variables. Thus, we split \mathfrak{R}_{rel} into a combination of the space of relative distances \mathfrak{R}_{radial} and a space parameterized by angular variables, essentially those on the sphere $S^{3(d-2)}$. There are known several ways to introduce variables in \mathfrak{R}_{rel} , see e.g. [6]. In particular, unlike [6], for the space of relative distances \mathfrak{R}_{radial} we take the relative (mutual) distances r_{ij} .

A key observation is that in new coordinates $(\{r_{ij}\}, \{\Omega\})$ the flat-space Laplace operator, the kinetic energy operator in (5), takes the form of the sum of 2 second-order differential operators

$$\frac{1}{2}\Delta_{rel}^{(3d)} = \Delta_{radial}^{(6)}(r_{ij}, \partial_{ij}) + \Delta_{\Omega}^{(3d-6)}(r_{ij}, \Omega, \partial_{ij}, \partial_{\Omega}) , \quad \partial_{ij} \equiv \frac{\partial}{\partial r_{ij}} , \quad (11)$$

($d > 2$) where the first operator depends on relative distances *only*. (Hence, it contains derivatives w.r.t. relative distances while the coefficient functions do not depend on angles.) The second operator depends on angular derivatives in such a way that it annihilates any angle-independent function Ψ , namely

$$\Delta_{\Omega}^{(3d-6)}(r_{ij}, \Omega, \partial_{ij}, \partial_{\Omega}) \Psi(r_{ij}) = 0 . \quad (12)$$

Hereafter, we omit the superscripts in $\Delta_{radial}^{(6)}$, $\Delta_{rel}^{(3d)}$ and $\Delta_{\Omega}^{(3d-6)}$.

The special cases $d = 1$ and $d = 2$ will be considered separately in section IV. In particular, for $d = 1$ the operator Δ_{Ω} is absent (no angular variables occur), thus

$$\Delta_{radial}(r_{ij}, \partial_{ij}) = \frac{1}{2} \Delta_{rel}(r_{ij}, \partial_{ij}) , \quad (13)$$

see (9). For $d > 2$, the commutator $[\Delta_{radial}, \Delta_{\Omega}] \neq 0$.

Now, if we look for angle-independent solutions of (5), due to the decomposition (11) the general spectral problem (5) is reduced to a particular spectral problem

$$\mathcal{H}_r \Psi(r_{ij}) \equiv \left(-\Delta_{radial}(r_{ij}, \partial_{ij}) + V(r_{ij}) \right) \Psi(r_{ij}) = E \Psi(r_{ij}) , \quad \Psi \in L_2(\mathfrak{R}_{radial}) , \quad (14)$$

where $\mathfrak{R}_{radial} \subset \mathfrak{R}_{rel}$ is the space of relative distances. Clearly, we can write

$$\Delta_{radial}(r_{ij}, \partial_{ij}) = g^{\mu\nu}(r) \partial_{\mu} \partial_{\nu} + b^{\mu}(r) \partial_{\mu} , \quad (\mu, \nu = 1, 2, 3, 4, 5, 6) , \quad (15)$$

where $g^{\mu\nu}(r)$ is a 6×6 matrix whose entries are the coefficients in front of the second derivatives $\partial_{\mu} \partial_{\nu}$, and $b^{\mu}(r)$ is a column vector; both are r -dependent. In (15), we made the identifications $1 \rightarrow r_{12}$, $2 \rightarrow r_{13}$, $3 \rightarrow r_{14}$, $4 \rightarrow r_{23}$, $5 \rightarrow r_{24}$, $6 \rightarrow r_{34}$ for μ and ν .

For any $d > 2$ one can find the d -dependent gauge factor $\Gamma = \Gamma(r_{ij})$ such that $\Delta_{radial}(r_{ij}, \partial_{ij})$ takes the form of the Schrödinger operator,

$$\Gamma^{-1} \Delta_{radial}(r_{ij}, \partial_{ij}) \Gamma = \Delta_{LB}(r_{ij}) - V_{eff}(r_{ij}) \equiv \Delta_{r,\Gamma} . \quad (16)$$

Here $\Delta_{LB}(r)$ is the six-dimensional Laplace-Beltrami operator with contravariant, d -independent metric $g^{\mu\nu}(r)$, on a non-flat, (non-constant curvature) manifold. This makes sense of the kinetic energy. The function $V_{eff}(r)$ is the d -dependent *effective potential*. The potential $V_{eff}(r)$ becomes singular at the boundary of the configuration space, where the determinant $D(r) = \det g^{\mu\nu}(r)$ vanishes. It can be checked that the operator Δ_r is Hermitian with measure $D(r)^{-\frac{1}{2}}$. Thus, we arrive at the spectral problem for the Hamiltonian

$$H_{LB}(r) = -\Delta_{LB}(r) + V_{eff}(r) + V(r) , \quad (17)$$

with $d > 2$ and with a d -independent Laplace-Beltrami operator $\Delta_{LB}(r)$. It is easy to see that at $d = 2$, as a consequence of the vanishing volume of the tetrahedron of interaction, the operator $\Delta_{LB}(r)$ becomes degenerate: $D(r) = \det g^{\mu\nu}(r) = 0$. The configuration space

$D \geq 0$ (equivalently, the space of relative coordinates) at $d > 2$ shrinks to its boundary $D = 0$ for $d = 2$.

The connection between the kinetic energy ($\Delta^{(4d)}$) in the original Hamiltonian (1) and that of the Hamiltonian (17) can be summarized as follows,

$$\Delta^{(4d)} \xrightarrow{\text{removal of } \mathbf{R}_{CM}} \Delta^{(3d)}_{rel} \xrightarrow{\text{angle-independent solutions}} \Delta_{radial} \xrightarrow{\text{gauge transformation } \Gamma} \Delta_{LB} . \quad (18)$$

Consequently, we reduce the original $4d$ -dimensional problem to a 6 dimensional one. As for the potential, we simply add to the original V the effective potential V_{eff} arising from the d -dependent gauge transformation Γ . Again, the case $d = 1$ is special, the gauge factor is trivial, $\Gamma = 1$, and

$$\Delta_{LB}(r) = \Delta_{radial}(r) = \Delta_{rel}(r) . \quad (19)$$

Following the *de-quantization* procedure [1]-[3] of replacement of the quantum momentum (derivative) by the classical momentum $-i\partial \rightarrow P$, one can get a classical analogue of the Hamiltonian (17),

$$H_{LB}^{(c)}(r) = g^{\mu\nu}(r) P_\mu P_\nu + V(r) + V_{eff}(r) . \quad (20)$$

It describes the internal motion of a 6-dimensional body with tensor of inertia $(g^{\mu\nu})^{-1}$ with center of mass fixed.

The Hamiltonians (17), (20) are the main objects of study of this paper.

II. CASE $d = 1$: DETAILED RESULTS

For the 1 dimensional case $d = 1$, we introduce the S_4 invariant symmetric polynomials

$$\begin{aligned} \sigma_1(x) &= x_1 + x_2 + x_3 + x_4 \\ \sigma_2(x) &= x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4 \\ \sigma_3(x) &= x_1 x_2 x_3 + x_1 x_3 x_4 + x_2 x_3 x_4 + x_1 x_2 x_4 \\ \sigma_4(x) &= x_1 x_2 x_3 x_4 . \end{aligned} \quad (21)$$

where it is assumed that x_i denotes the position of the i th body.

In the variables

$$Y = \sigma_1(x) , \quad \tau_k = \sigma_k(y(x)) , \quad (k = 2, 3, 4) , \quad (22)$$

$$y(x) = (y_1(x), y_2(x), y_3(x), y_4(x)) \quad \text{and} \quad y_i(x) = x_i - \frac{1}{4}Y, \quad (i = 1, 2, 3, 4), \quad (23)$$

are translational invariant, the original Laplacian (1) takes the algebraic form

$$\begin{aligned} -\sum_{i=1}^4 \frac{1}{2} \Delta_i^{(d=1)} &= -2 \partial_{Y,Y}^2 + \tau_2 \partial_{2,2}^2 + (2\tau_4 - \frac{1}{2}\tau_2^2) \partial_{3,3}^2 + (\tau_2 \tau_4 - \frac{3}{8}\tau_3^2) \partial_{4,4}^2 \\ &+ 3\tau_3 \partial_{2,3}^2 + 4\tau_4 \partial_{2,4}^2 - \frac{1}{2}\tau_2 \tau_3 \partial_{3,4}^2 + \frac{3}{2} \partial_2 + \frac{1}{4} \tau_2 \partial_4, \end{aligned} \quad (24)$$

which, upon the extraction of the center-of-mass motion, can be rewritten in terms of the generators of the algebra $sl(4, \mathbf{R})$. Moreover, it can be easily seen that this operator describes the kinetic energy of relative motion of the 4-body (A_3) rational Calogero model with potential

$$V_{A_3} = g \left(\frac{1}{x_{12}^2} + \frac{1}{x_{13}^2} + \frac{1}{x_{14}^2} + \frac{1}{x_{23}^2} + \frac{1}{x_{24}^2} + \frac{1}{x_{34}^2} \right), \quad (25)$$

in algebraic form, where g is the coupling constant and $x_{ij} \equiv x_i - x_j$.

Also, for $d = 1$ there exists another polynomial change of variables. In the space of relative distances the Laplace-Beltrami operator (19) is given by

$$\Delta_{LB} = \frac{\partial^2}{\partial x_{12}^2} + \frac{\partial^2}{\partial x_{13}^2} + \frac{\partial^2}{\partial x_{14}^2} + \frac{\partial^2}{\partial x_{12} \partial x_{13}} + \frac{\partial^2}{\partial x_{12} \partial x_{14}} + \frac{\partial^2}{\partial x_{13} \partial x_{14}}, \quad (26)$$

see (9). It corresponds to the 3-dimensional flat space Laplacian and is evidently an algebraic operator. Formally, it is not S_4 invariant unlike the original $4d$ -Laplacian in (1) with $d = 1$. However, the kinetic energy remains S_3 invariant. As a realization of this S_3 invariance in Δ_{LB} (19) let us introduce the natural variables

$$\xi_1 = x_{12} + x_{13} + x_{14}, \quad \xi_2 = x_{12} x_{13} + x_{12} x_{14} + x_{13} x_{14}, \quad \xi_3 = x_{12} x_{13} x_{14}, \quad (27)$$

which is a polynomial change of variables, so that (19) becomes

$$\begin{aligned} \Delta_{LB}(\xi) &= 6 \partial_{\xi_1}^2 + (3 \xi_1^2 - \xi_2) \partial_{\xi_2}^2 + (\xi_2^2 - \xi_1 \xi_3) \partial_{\xi_3}^2 + 8 \xi_1 \partial_{\xi_1, \xi_2}^2 \\ &+ 4 \xi_2 \partial_{\xi_1, \xi_3}^2 + 3(\xi_1 \xi_2 - \xi_3) \partial_{\xi_2, \xi_3}^2 + 3 \partial_{\xi_2} + \xi_1 \partial_{\xi_3}. \end{aligned} \quad (28)$$

The operator (28) is algebraic, it can be rewritten in terms of the generators of the maximal

affine subalgebra b_3 of the algebra $sl(4, \mathbf{R})$ in ξ -variables, c.f. below (52), see [4, 7].

$$\begin{aligned}
 \mathcal{J}_i^- &= \frac{\partial}{\partial \xi_i}, & i = 1, 2, 3, \\
 \mathcal{J}_{ij}^0 &= \xi_i \frac{\partial}{\partial \xi_j}, & i, j = 1, 2, 3, \\
 \mathcal{J}^0(N) &= \sum_{i=1}^3 \xi_i \frac{\partial}{\partial \xi_i} - N, \\
 \mathcal{J}_i^+(N) &= \xi_i \mathcal{J}^0(N) = \xi_i \left(\sum_{j=1}^3 \xi_j \frac{\partial}{\partial \xi_j} - N \right), & i = 1, 2, 3,
 \end{aligned} \tag{29}$$

where N is a parameter.

III. CASE $d > 2$: DETAILED RESULTS

A. r -representation

If we assume $d > 2$, after straightforward calculations the operator $\Delta_{radial}(r_{ij}, \partial_{ij})$ (in decomposition (11)) can be found to be

$$\begin{aligned}
 2 \Delta_{radial}(r_{ij}, \partial_{ij}) &= \left[2(\partial_{r_{12}}^2 + \partial_{r_{13}}^2 + \partial_{r_{14}}^2 + \partial_{r_{23}}^2 + \partial_{r_{24}}^2 + \partial_{r_{34}}^2) + \frac{2(d-1)}{r_{12}} \partial_{r_{12}} \right. \\
 &+ \frac{2(d-1)}{r_{13}} \partial_{r_{13}} + \frac{2(d-1)}{r_{14}} \partial_{r_{14}} + \frac{2(d-1)}{r_{23}} \partial_{r_{23}} + \frac{2(d-1)}{r_{24}} \partial_{r_{24}} + \frac{2(d-1)}{r_{34}} \partial_{r_{34}} \\
 &+ \frac{r_{12}^2 + r_{13}^2 - r_{23}^2}{r_{12} r_{13}} \partial_{r_{12}} \partial_{r_{13}} + \frac{r_{12}^2 + r_{14}^2 - r_{24}^2}{r_{12} r_{14}} \partial_{r_{12}} \partial_{r_{14}} + \frac{r_{13}^2 + r_{14}^2 - r_{34}^2}{r_{13} r_{14}} \partial_{r_{13}} \partial_{r_{14}} \\
 &+ \frac{r_{12}^2 + r_{23}^2 - r_{13}^2}{r_{12} r_{23}} \partial_{r_{12}} \partial_{r_{23}} + \frac{r_{12}^2 + r_{24}^2 - r_{14}^2}{r_{12} r_{24}} \partial_{r_{12}} \partial_{r_{24}} + \frac{r_{23}^2 + r_{24}^2 - r_{34}^2}{r_{23} r_{24}} \partial_{r_{23}} \partial_{r_{24}} \\
 &+ \frac{r_{13}^2 + r_{23}^2 - r_{12}^2}{r_{13} r_{23}} \partial_{r_{13}} \partial_{r_{23}} + \frac{r_{13}^2 + r_{34}^2 - r_{14}^2}{r_{13} r_{34}} \partial_{r_{13}} \partial_{r_{34}} + \frac{r_{23}^2 + r_{34}^2 - r_{24}^2}{r_{23} r_{34}} \partial_{r_{23}} \partial_{r_{34}} \\
 &\left. + \frac{r_{14}^2 + r_{24}^2 - r_{12}^2}{r_{14} r_{24}} \partial_{r_{14}} \partial_{r_{24}} + \frac{r_{14}^2 + r_{34}^2 - r_{13}^2}{r_{14} r_{34}} \partial_{r_{14}} \partial_{r_{34}} + \frac{r_{24}^2 + r_{34}^2 - r_{23}^2}{r_{24} r_{34}} \partial_{r_{24}} \partial_{r_{34}} \right].
 \end{aligned} \tag{30}$$

Notice the absence of the cross terms $\partial_{r_{12}} \partial_{r_{34}}$, $\partial_{r_{13}} \partial_{r_{24}}$ and $\partial_{r_{14}} \partial_{r_{23}}$; each of them involves 2 disconnected edges of the tetrahedron of interaction.

In general, the operator (30) does not depend on the choice of the angular variables Ω , but the operator $\Delta_{\Omega}(r_{ij}, \partial_{ij}, \Omega, \partial_{\Omega})$ in (11) does so. The configuration space in the space of relative distances is

$$0 < r_a < r_b + r_c < \infty, \quad 0 < r_b < r_a + r_c < \infty, \quad 0 < r_c < r_a + r_b < \infty, \tag{31}$$

($a \neq b \neq c = 12, 13, 14, 23, 24, 34$).

B. ρ -representation

Formally, the operator (30) is invariant under reflections $Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$,

$$r_{12} \Leftrightarrow -r_{12}, \quad r_{13} \Leftrightarrow -r_{13}, \quad r_{14} \Leftrightarrow -r_{14}, \quad r_{23} \Leftrightarrow -r_{23}, \quad r_{24} \Leftrightarrow -r_{24}, \quad r_{34} \Leftrightarrow -r_{34}. \quad (32)$$

If we introduce new reflection invariant variables,

$$r_{12}^2 = \rho_{12}, \quad r_{13}^2 = \rho_{13}, \quad r_{14}^2 = \rho_{14}, \quad r_{23}^2 = \rho_{23}, \quad r_{24}^2 = \rho_{24}, \quad r_{34}^2 = \rho_{34}, \quad (33)$$

the operator (30) becomes algebraic,

$$\begin{aligned} \Delta_{radial}(\rho_{ij}, \partial_{ij}) &= 4(\rho_{12} \partial_{\rho_{12}}^2 + \rho_{13} \partial_{\rho_{13}}^2 + \rho_{14} \partial_{\rho_{14}}^2 + \rho_{23} \partial_{\rho_{23}}^2 + \rho_{24} \partial_{\rho_{24}}^2 + \rho_{34} \partial_{\rho_{34}}^2) \\ &+ 2 \left((\rho_{12} + \rho_{13} - \rho_{23}) \partial_{\rho_{12}} \partial_{\rho_{13}} + (\rho_{12} + \rho_{14} - \rho_{24}) \partial_{\rho_{12}} \partial_{\rho_{14}} + (\rho_{13} + \rho_{14} - \rho_{34}) \partial_{\rho_{13}} \partial_{\rho_{14}} \right) \\ &+ 2 \left((\rho_{12} + \rho_{23} - \rho_{13}) \partial_{\rho_{12}} \partial_{\rho_{23}} + (\rho_{12} + \rho_{24} - \rho_{14}) \partial_{\rho_{12}} \partial_{\rho_{24}} + (\rho_{23} + \rho_{24} - \rho_{34}) \partial_{\rho_{23}} \partial_{\rho_{24}} \right) \\ &+ 2 \left((\rho_{13} + \rho_{23} - \rho_{12}) \partial_{\rho_{13}} \partial_{\rho_{23}} + (\rho_{13} + \rho_{34} - \rho_{14}) \partial_{\rho_{13}} \partial_{\rho_{34}} + (\rho_{23} + \rho_{34} - \rho_{24}) \partial_{\rho_{23}} \partial_{\rho_{34}} \right) \\ &+ 2 \left((\rho_{14} + \rho_{24} - \rho_{12}) \partial_{\rho_{14}} \partial_{\rho_{24}} + (\rho_{14} + \rho_{34} - \rho_{13}) \partial_{\rho_{14}} \partial_{\rho_{34}} + (\rho_{24} + \rho_{34} - \rho_{23}) \partial_{\rho_{24}} \partial_{\rho_{34}} \right) \\ &+ 2d(\partial_{\rho_{12}} + \partial_{\rho_{13}} + \partial_{\rho_{14}} + \partial_{\rho_{23}} + \partial_{\rho_{24}} + \partial_{\rho_{34}}). \end{aligned} \quad (34)$$

As a function of the ρ -variables, the operator (34) is not S_6 permutationally-invariant. Nevertheless, it remains S_4 invariant under the permutations of the particles (vertices of tetrahedron of interaction). For the 3-body case, where the number of ρ variables (relative distances) equals the number of particles, the corresponding operator Δ_{radial} is indeed S_3 permutationally-invariant.

From (31) and (33) it follows that the corresponding configuration space in ρ variables is given by the conditions

$$0 < \rho_A, \rho_B, \rho_C < \infty, \quad \rho_A < (\sqrt{\rho_B} + \sqrt{\rho_C})^2, \quad \rho_B < (\sqrt{\rho_A} + \sqrt{\rho_C})^2, \quad \rho_C < (\sqrt{\rho_A} + \sqrt{\rho_B})^2, \quad (35)$$

($A \neq B \neq C = 12, 13, 14, 23, 24, 34$). We remark that

$$S_{\Delta ABC}^2 \equiv \frac{2(\rho_A \rho_B + \rho_A \rho_C + \rho_B \rho_C) - (\rho_A^2 + \rho_B^2 + \rho_C^2)}{16} \geq 0, \quad (36)$$

because the left-hand side (l.h.s.) is equal to

$$\frac{1}{16}(r_A + r_B - r_C)(r_A + r_C - r_B)(r_B + r_C - r_A)(r_A + r_B + r_C) , \quad (37)$$

and conditions (31) hold. Therefore, from the Heron formula, $S_{\Delta ABC}^2$ is the square of the area of the *triangle of interaction* with sides r_A , r_B and r_C . The triangles of interaction are nothing but the faces of the tetrahedron.

The associated contravariant metric for the operator $\Delta_{radial}(\rho)$, defined by coefficients in front of second derivatives, is remarkably simple

$$g^{\mu\nu}(\rho) = \begin{pmatrix} 4\rho_{12} & \rho_{12} + \rho_{13} - \rho_{23} & \rho_{12} + \rho_{14} - \rho_{24} & \rho_{12} - \rho_{13} + \rho_{23} & \rho_{12} - \rho_{14} + \rho_{24} & 0 \\ \rho_{12} + \rho_{13} - \rho_{23} & 4\rho_{13} & \rho_{13} + \rho_{14} - \rho_{34} & \rho_{13} + \rho_{23} - \rho_{12} & 0 & \rho_{13} - \rho_{14} + \rho_{34} \\ \rho_{12} + \rho_{14} - \rho_{24} & \rho_{13} + \rho_{14} - \rho_{34} & 4\rho_{14} & 0 & \rho_{14} + \rho_{24} - \rho_{12} & \rho_{14} + \rho_{34} - \rho_{13} \\ \rho_{12} - \rho_{13} + \rho_{23} & \rho_{13} + \rho_{23} - \rho_{12} & 0 & 4\rho_{23} & \rho_{23} + \rho_{24} - \rho_{34} & \rho_{23} - \rho_{24} + \rho_{34} \\ \rho_{12} - \rho_{14} + \rho_{24} & 0 & \rho_{14} + \rho_{24} - \rho_{12} & \rho_{23} + \rho_{24} - \rho_{34} & 4\rho_{24} & \rho_{24} + \rho_{34} - \rho_{23} \\ 0 & \rho_{13} - \rho_{14} + \rho_{34} & \rho_{14} + \rho_{34} - \rho_{13} & \rho_{23} - \rho_{24} + \rho_{34} & \rho_{24} + \rho_{34} - \rho_{23} & 4\rho_{34} \end{pmatrix} . \quad (38)$$

It is linear in ρ -coordinates(!) with positive definite factorized determinant

$$D(\rho) = 36864 F_1 F_2 , \quad (39)$$

where

$$F_1 = V_4^2 , \quad (40)$$

$$F_2 = 36 \tilde{V}_1^2 V_4^2 - \tilde{V}_2^2 \tilde{V}_3^2 . \quad (41)$$

Here

- V_4^2 is the square of the volume of the tetrahedron of interaction.
- \tilde{V}_3^2 is the sum of the 4 areas (squared) of the faces (triangles) of the tetrahedron.
- \tilde{V}_2^2 is the sum of the 6 edges (squared) of the tetrahedron.
- By definition $\tilde{V}_1^2 \equiv 1$.

see (64)-(66). Hence, $F_{1,2}$ are of geometrical nature. They define the boundary of configuration space, $F_1 = 0$, $F_2 = 0$, where the determinant (39) degenerates, i.e., vanishes.

Following Conjecture 3 in [3], the operator $\Delta_{radial}(\rho)$ is self-adjoint with respect to the normalized radial measure dv_r of the form

$$dv_r = V_4^{d-4} d\rho_{12} d\rho_{13} d\rho_{14} d\rho_{23} d\rho_{24} d\rho_{34} . \quad (42)$$

C. Symmetry operators

The reduced radial Laplacian (34) admits a 3-dimensional symmetry algebra with elements of the type

$$\begin{aligned}
 L(a, b, c) = & (\rho_{13} a + \rho_{14} b - \rho_{23} a - \rho_{24} b) \partial_{\rho_{12}} + \left(\rho_{14} \left(\frac{3}{2} a + \frac{7}{2} b + 3c \right) - \rho_{12} \left(\frac{3}{2} a + \frac{3}{2} b + c \right) + \right. \\
 & \left. \rho_{23} \left(\frac{3}{2} a + \frac{3}{2} b + c \right) - \rho_{34} \left(\frac{3}{2} a + \frac{7}{2} b + 3c \right) \right) \partial_{\rho_{13}} + \left(\rho_{12} \left(\frac{1}{2} a - \frac{1}{2} b - c \right) \right. \\
 & \left. + \rho_{13} \left(-\frac{1}{2} a + \frac{1}{2} b + c \right) + \rho_{24} \left(\frac{5}{2} b + \frac{3}{2} a + 3c \right) - \rho_{34} \left(\frac{5}{2} b + \frac{3}{2} a + 3c \right) \right) \partial_{\rho_{23}} \\
 & + \left(c \rho_{12} - \rho_{13} (a + 3b + 3c) - \rho_{24} c + \rho_{34} (a + 3b + 3c) \right) \partial_{\rho_{14}} + (\rho_{12} (2b + c + a) - \\
 & \rho_{14} (2b + c + a) - \rho_{23} (2a + 3b + 3c) + \rho_{34} (2a + 3b + 3c)) \partial_{\rho_{24}} + \\
 & \left(\rho_{13} \left(\frac{1}{2} a + \frac{5}{2} b + 2c \right) - \rho_{14} \left(\frac{1}{2} a + \frac{5}{2} b + 2c \right) + \rho_{23} \left(\frac{3}{2} a + \frac{3}{2} b + 2c \right) \right. \\
 & \left. - \rho_{24} \left(\frac{3}{2} a + \frac{3}{2} b + 2c \right) \right) \partial_{\rho_{34}}, \tag{43}
 \end{aligned}$$

where a, b, c are parameters. Thus, the operator $L(a, b, c)$ commutes with $\Delta_{radial}(\rho)$. Out of (43) let us form the 3 linearly independent operators $\{J_1, J_2, J_3\}$,

$$\begin{aligned}
 J_1 & \equiv L \left(\frac{2\sqrt{35}}{35}, 0, -\frac{3\sqrt{35}}{35} \right), & J_2 & \equiv L \left(-\frac{17\sqrt{210}}{420}, \frac{35\sqrt{210}}{420}, -\frac{27\sqrt{210}}{420} \right), \tag{44} \\
 J_3 & \equiv L \left(\frac{5\sqrt{6}}{12}, \frac{\sqrt{6}}{12}, -\frac{\sqrt{6}}{4} \right).
 \end{aligned}$$

It can be checked that they satisfy the $so(3, \mathbf{R})$ commutation relations

$$[J_1, J_2] = J_3, \quad [J_2, J_3] = J_1, \quad [J_3, J_1] = J_2. \tag{45}$$

Thus, the symmetry algebra of $\Delta_{radial}(\rho)$ is isomorphic to $so(3, \mathbf{R})$.

As for the original 4-body problem (1) these integrals are *particular* integrals: they commute with the Hamiltonian (1) over the space of relative distances \mathfrak{R}_{radial} only

$$[\mathcal{H}, L] \mathfrak{R}_{radial} \rightarrow 0. \tag{46}$$

In general, \mathcal{H} and L do not commute.

The space of 2nd order symmetries of $\Delta_{radial}(\rho)$ is much more complicated. Due to space limitations we merely summarize our results. The space of 2nd order symmetries is the direct sum of the 6-dimensional space D_1 of symmetries whose 2nd order terms are homogeneous of order 1 in the ρ variables ($\dim D_1 = 6$) and the 21-dimensional space D_2 of symmetries whose 2nd order terms are homogeneous of order 2 in the ρ variables ($\dim D_2 = 21$). Under the adjoint action of the $so(3, \mathbf{R})$ 1st order symmetries, D_1 splits into the sum of 2 irreducible subspaces: one of dimension 1 (with the Hamiltonian as a basis) and one of dimension 5. To give a brief description of these elements it is convenient to use the complex basis $\{J^0, J^+, J^-\}$ typical for $sl(2, C)$,

$$J^0 = iJ_3, \quad J^+ = -J_2 + iJ_1, \quad J^- = J_2 + iJ_1. \quad (47)$$

The finite dimensional irreducible representations of $so(3, \mathbf{R})$ are indexed by a non-negative integer ℓ . The corresponding irreducible subspaces have a basis of $(2\ell + 1)$ elements $\{f_m^{(\ell)} : m = \ell, \ell - 1, \dots, -\ell\}$, such that the action of $so(3, \mathbf{R})$ is given by

$$J^0 f_m^{(\ell)} = m f_m^{(\ell)}, \quad J^\pm f_m^{(\ell)} = [(\ell \pm m + 1)(\ell \mp m)]^{1/2} f_{m\pm 1}^{(\ell)}. \quad (48)$$

For D_1 , the basis can be computed from $f_0^{(0)} = \Delta_{radial}(\rho)$, thus taking $\ell = 0$, and the 5 basis elements can be computed from

$$\begin{aligned}
f_2^{(2)} = & -2\rho_{13}\partial_{\rho_{13},\rho_{13}}^2 + (\rho_{13} + \rho_{34} - \rho_{14})\partial_{\rho_{34},\rho_{13}}^2 - \frac{1}{33}(63 + 46i\sqrt{6})\rho_{23}\partial_{\rho_{23},\rho_{23}}^2 \\
& - \frac{1}{6}(3 - 2i\sqrt{6})d\partial_{\rho_{12}} + \frac{1}{11}(5 + 4i\sqrt{6})(\rho_{13} + \rho_{12} - \rho_{23})\partial_{\rho_{13},\rho_{12}}^2 - \frac{1}{11}(13 + 6i\sqrt{6})\rho_{14}\partial_{\rho_{14},\rho_{14}}^2 \\
& + \frac{1}{11}(5 + 4i\sqrt{6})(\rho_{13} - \rho_{14} - \rho_{23} + \rho_{24})\partial_{\rho_{34},\rho_{12}}^2 + \frac{1}{11}(13 + 6i\sqrt{6})(\rho_{12} - \rho_{14} - \rho_{24})\partial_{\rho_{24},\rho_{14}}^2 \\
& + (\rho_{12} - \rho_{13} - \rho_{24} + \rho_{34})\partial_{\rho_{23},\rho_{14}}^2 - (\rho_{13} + \rho_{14} - \rho_{34})\partial_{\rho_{14},\rho_{13}}^2 - \frac{1}{66}(63 + 46i\sqrt{6})d\partial_{\rho_{23}} \\
& + \frac{1}{3}(4 + i\sqrt{6})(\rho_{12} - \rho_{14} - \rho_{23} + \rho_{34})\partial_{\rho_{24},\rho_{13}}^2 + \frac{1}{3}(3 - 2i\sqrt{6})\rho_{12}\partial_{\rho_{12},\rho_{12}}^2 - d\partial_{\rho_{13}} \\
& - \frac{1}{11}(3 - 2i\sqrt{6})(\rho_{23} - \rho_{34} - \rho_{24})\partial_{\rho_{34},\rho_{24}}^2 + \frac{1}{11}(27 + 4i\sqrt{6})(\rho_{12} - \rho_{13} - \rho_{23})\partial_{\rho_{23},\rho_{13}}^2 \\
& - \frac{1}{33}(15 + 34i\sqrt{6})(\rho_{23} + \rho_{24} - \rho_{34})\partial_{\rho_{24},\rho_{23}}^2 - \frac{1}{11}(13 + 6i\sqrt{6})\rho_{34}\partial_{\rho_{34},\rho_{34}}^2 \\
& + \frac{2}{11}(1 + 3i\sqrt{6})(\rho_{13} - \rho_{14} - \rho_{34})\partial_{\rho_{34},\rho_{14}}^2 + \frac{1}{3}(3 - 2i\sqrt{6})(\rho_{12} - \rho_{14} + \rho_{24})\partial_{\rho_{24},\rho_{12}}^2 \\
& - \frac{1}{33}(3 + 20i\sqrt{6})d\partial_{\rho_{24}} - \frac{4}{11}(4 + i\sqrt{6})(\rho_{23} + \rho_{34} - \rho_{24})\partial_{\rho_{34},\rho_{23}}^2 - \frac{1}{22}(13 + 6i\sqrt{6})d\partial_{\rho_{34}} \\
& - \frac{2}{33}(3 + 20i\sqrt{6})\rho_{24}\partial_{\rho_{24},\rho_{24}}^2 + \frac{2}{33}(9 - 17i\sqrt{6})(\rho_{12} - \rho_{13} + \rho_{23})\partial_{\rho_{23},\rho_{12}}^2 \\
& + \frac{1}{22}(13 + 6i\sqrt{6})d\partial_{\rho_{14}} ,
\end{aligned} \tag{49}$$

for $\ell = 2$, by using equations (48).

We can show that these 6 basis elements for D_1 are pairwise commutative and algebraically independent. Thus the free Hamiltonian system is integrable. However, the 6 basis symmetries fail to satisfy the algebraic conditions for a separable coordinate system [8]. Briefly, if the coefficients of a 2nd order symmetry operator in the ρ coordinates are given by $R^{\mu\nu}$, the eigenforms ω and eigenvalues λ_j are the solutions of the equation

$$\sum_{\nu=1}^6 (R^{\mu\nu} - \lambda g^{\mu\nu})\omega_\nu = 0, \quad \mu = 1, \dots, 6, \tag{50}$$

where the coefficients of the Hamiltonian are given by (38). For separability the 6 basis symmetries should pairwise commute, each should admit 6 eigenvalues and the symmetries should share the *same* 6 eigenforms. By a long computation one can show that the 6 basis symmetries do not have a common basis of eigenforms.

Under the adjoint action of the 1st order symmetries, D_2 splits into 5 irreducible subspaces, two of dimension 1 ($\ell = 0$), two of dimension 5 ($\ell = 2$) and one of dimension 9 ($\ell = 4$). The expressions for the basis symmetries are very lengthy and we do not list them

One of the dimension 1 subspaces has basis $\mathcal{J}_1 = \{J_1^2 + J_2^2 + J_3^2\}$ and one of the dimension 5 subspaces has basis

$$\mathcal{J}_2 = \{J_1^2 + J_2^2 - 2J_3^2, J_1^1 - 2J_2^2 + J_3^2, J_k J_\ell + J_\ell J_k, 1 \leq \ell < k \leq 3\}. \quad (51)$$

The basis symmetry for the other 1-dimensional irreducible subspace commutes with \mathcal{J}_1 and \mathcal{J}_2 . It appears that there are no more commutative sextuplets in this full 27-dimensional space, though we do not yet have a convincing proof. Thus, it appears that the free system is integrable, even superintegrable, but not separable.

D. The Representations of $sl(7, \mathbf{R})$

In the ρ -representation, the operator (34) is $sl(7, \mathbf{R})$ -Lie algebraic - it can be rewritten in terms of the generators of the maximal affine subalgebra b_7 of the algebra $sl(7, \mathbf{R})$, see e.g. [9, 10],

$$\begin{aligned} \mathcal{J}_i^- &= \frac{\partial}{\partial \lambda_i}, \quad i = 1, 2, \dots, 6, \\ \mathcal{J}_{ij}^0 &= \lambda_i \frac{\partial}{\partial \lambda_j}, \quad i, j = 1, 2, 3, \dots, 6, \\ \mathcal{J}^0(N) &= \sum_{i=1}^6 \lambda_i \frac{\partial}{\partial \lambda_i} - N, \\ \mathcal{J}_i^+(N) &= \lambda_i \mathcal{J}^0(N) = \lambda_i \left(\sum_{j=1}^6 \lambda_j \frac{\partial}{\partial \lambda_j} - N \right), \quad i = 1, 2, \dots, 6, \end{aligned} \quad (52)$$

where N is a parameter and

$$\lambda_1 \equiv \rho_{12}, \quad \lambda_2 \equiv \rho_{13}, \quad \lambda_3 \equiv \rho_{14}, \quad \lambda_4 \equiv \rho_{23}, \quad \lambda_5 \equiv \rho_{24}, \quad \lambda_6 \equiv \rho_{34}. \quad (53)$$

If N is a non-negative integer, a finite-dimensional representation space exists,

$$\mathcal{P}_N = \langle \lambda_1^{p_1} \lambda_2^{p_2} \lambda_3^{p_3} \lambda_4^{p_4} \lambda_5^{p_5} \lambda_6^{p_6} \mid 0 \leq p_1 + p_2 + p_3 + p_4 + p_5 + p_6 \leq N \rangle. \quad (54)$$

Explicitly, the operator (34) can be expressed as

$$\begin{aligned} \frac{1}{2} \Delta_{radial}(\mathcal{J}) &= 2(\mathcal{J}_{11}^0 \mathcal{J}_1^- + \mathcal{J}_{22}^0 \mathcal{J}_2^- + \mathcal{J}_{33}^0 \mathcal{J}_3^- + \mathcal{J}_{44}^0 \mathcal{J}_4^- + \mathcal{J}_{55}^0 \mathcal{J}_5^- + \mathcal{J}_{66}^0 \mathcal{J}_6^-) \\ &\quad + d(\mathcal{J}_1^- + \mathcal{J}_2^- + \mathcal{J}_3^- + \mathcal{J}_4^- + \mathcal{J}_5^- + \mathcal{J}_6^-) \end{aligned} \quad (55)$$

$$\begin{aligned}
& + \left[\mathcal{J}_{11}^0 (\mathcal{J}_2^- + \mathcal{J}_3^- + \mathcal{J}_4^- + \mathcal{J}_5^-) + \mathcal{J}_{22}^0 (\mathcal{J}_1^- + \mathcal{J}_3^- + \mathcal{J}_4^- + \mathcal{J}_6^-) \right. \\
& + \mathcal{J}_{33}^0 (\mathcal{J}_1^- + \mathcal{J}_2^- + \mathcal{J}_5^- + \mathcal{J}_6^-) + \mathcal{J}_{44}^0 (\mathcal{J}_1^- + \mathcal{J}_2^- + \mathcal{J}_5^- + \mathcal{J}_6^-) \\
& \left. + \mathcal{J}_{55}^0 (\mathcal{J}_1^- + \mathcal{J}_3^- + \mathcal{J}_4^- + \mathcal{J}_6^-) + \mathcal{J}_{66}^0 (\mathcal{J}_2^- + \mathcal{J}_3^- + \mathcal{J}_4^- + \mathcal{J}_5^-) \right] \\
& - 2 \left[\mathcal{J}_{12}^0 \mathcal{J}_4^- + \mathcal{J}_{13}^0 \mathcal{J}_5^- + \mathcal{J}_{21}^0 \mathcal{J}_4^- + \mathcal{J}_{23}^0 \mathcal{J}_6^- + \mathcal{J}_{31}^0 \mathcal{J}_5^- + \mathcal{J}_{32}^0 \mathcal{J}_6^- \right. \\
& \left. + \mathcal{J}_{41}^0 \mathcal{J}_2^- + \mathcal{J}_{45}^0 \mathcal{J}_6^- + \mathcal{J}_{54}^0 \mathcal{J}_6^- + \mathcal{J}_{62}^0 \mathcal{J}_3^- + \mathcal{J}_{64}^0 \mathcal{J}_5^- + \mathcal{J}_{51}^0 \mathcal{J}_3^- \right].
\end{aligned}$$

It acts on (54) as a filtration.

E. The Laplace Beltrami operator underlying geometry

The remarkable property of the algebraic operator $\Delta_{radial}(\rho)$ (34) is its gauge-equivalence to the Schrödinger operator: there is a gauge factor Γ such that

$$\Gamma^{-1} \Delta_{radial}(\rho) \Gamma = \Delta_{LB}(\rho) - V_{eff}(\rho), \quad (56)$$

where Δ_{LB} is the Laplace-Beltrami operator

$$\Delta_{LB}(\rho) = \sqrt{D(\rho)} \partial_\mu \frac{1}{\sqrt{D(\rho)}} g^{\mu\nu} \partial_\nu, \quad \partial_\nu \equiv \frac{\partial}{\partial \rho_\nu}, \quad (57)$$

see (38). It is given by

$$\Gamma = (F_1 F_2)^{-1/4} (V_4^2)^{1-d/4} = F_1^{\frac{3-d}{4}} F_2^{-1/4}, \quad (58)$$

see (39), (40), (41), and the *effective* potential is

$$V_{eff} = \frac{3 \tilde{V}_2^4 + 112 \tilde{V}_3^2}{32 F_2} + \frac{(d-5)(d-3) \tilde{V}_3^2}{72 F_1}. \quad (59)$$

Therefore Γ is of geometric nature: it can be rewritten in terms of volumes (equivalently, volume variables, see below). The effective potential becomes singular at the boundary of the configuration space.

Thus, taking into account the gauge rotation (56) we can arrive at the 6-dimensional Hamiltonian

$$\mathcal{H}_{LB}(r) = -\Delta_{LB}(r) + V(r) + V_{eff}(r), \quad (60)$$

ing the space of r -relative distances, or

$$\mathcal{H}_{LB}(\rho) = -\Delta_{LB}(\rho) + V(\rho) + V_{eff}(\rho) , \quad (61)$$

in ρ -space. The Hamiltonians (60) and (61) describe the 6-dimensional quantum particle moving in the curved space with metric $g^{\mu\nu}$ and kinetic energy Δ_{LB} , and in particular, in ρ -space with metric $g^{\mu\nu}(\rho)$ (38) and kinetic energy $\Delta_{LB}(\rho)$.

Making the de-quantization of (61) we arrive at a 6-dimensional classical system which is characterized by the Hamiltonian,

$$\mathcal{H}_{LB}^{(c)}(\rho) = g^{\mu\nu}(\rho) P_\mu P_\nu + V(\rho) + V_{eff}(\rho) , \quad (62)$$

where P_μ , $\mu = 12, 13, 14, 23, 24, 34$ are classical canonical momenta in ρ -space and $g^{\mu\nu}(\rho)$ is given by (38). This operator (62) is suitable for investigating special configurations (trajectories) for the classical 4-body system. It is worth mentioning that even in the planar case, the dynamics of the classical 4-body problem is very rich [11]-[13].

IV. REDUCTION TO LOWER DIMENSIONS: $d = 1, 2$

At $d = 2$ (planar systems) and $d = 1$ (a system on the line) the number of independent ρ -variables reduces from 6 to 5 and 3, respectively, and the expression (34) for the operator Δ_{radial} ceases to be valid. In particular, the determinant of the metric defined by the coefficients of the 2nd order derivatives in (34) vanishes. This makes the cases $d = 2$ and $d = 1$ quite distinct from $d \geq 3$.

In particular, one can ask the question: *do there exist variables for which Δ_{radial} is an algebraic operator at $d = 2$?* In this section we provide a partial answer to this question. To this end, in addition to the ρ -representation we will introduce 2 new representations in terms of purely geometric variables (see below) obtained from the *tetrahedron of interaction*. We call them *volume*-variables and *u*-variables, respectively. More importantly, the volume-representation can be easily extended to the general n -body case.

A. Volume variables representation

Let us consider, assuming $d \geq 3$, the following change of variables

$$(\rho_{12}, \rho_{13}, \rho_{14}, \rho_{23}, \rho_{24}, \rho_{34},) \Rightarrow (\mathcal{V}, S, P, q_1, q_2, q_3), \quad (63)$$

where

$$\begin{aligned} \mathcal{V} \equiv V_4^2 = \frac{1}{144} & \left[[(\rho_{13} + \rho_{14} + \rho_{23} + \rho_{24}) \rho_{34} - (\rho_{13} - \rho_{14})(\rho_{23} - \rho_{24}) - \rho_{34}^2] \rho_{12} \right. \\ & - \rho_{13}^2 \rho_{24} - \rho_{34} \rho_{12}^2 + \rho_{23} [(\rho_{14} - \rho_{24}) \rho_{34} - \rho_{14}(\rho_{14} + \rho_{23} + \rho_{24})] \\ & \left. + \rho_{13} [\rho_{14}(\rho_{23} + \rho_{24} - \rho_{34}) + \rho_{24}(\rho_{23} - \rho_{24} + \rho_{34})] \right], \end{aligned} \quad (64)$$

is the square of the volume of the *tetrahedron of interaction*, the variable

$$S \equiv \tilde{V}_3^2 = S_1 + S_2 + S_3 + S_4, \quad (65)$$

is the sum of the areas squared of its 4 faces (see (36)), and the variable

$$P \equiv \tilde{V}_2^2 = \rho_{12} + \rho_{13} + \rho_{14} + \rho_{23} + \rho_{24} + \rho_{34}, \quad (66)$$

is the sum of all the 6 edges (squared). This variable is nothing but the square of the hyper-radius in the space of relative motion, or in other words, in the space of relative distances.

These 3 variables (\mathcal{V}, S, P) are purely geometric; they are homogeneous polynomials in ρ -variables of dimension 3, 2 and 1, respectively. Notice that these quantities define the effective potential V_{eff} (59). We call them *volume variables*. Clearly, \mathcal{V}, S and P are S_4 -invariant under the permutations of the 4-body positions (vertices of the tetrahedron). However, only the variable P is S_6 invariant under the permutations of the 6 ρ -variables (edges of the tetrahedron). The remaining 3 variables (q_1, q_2, q_3) can be chosen as $q_1 = \rho_{12}, q_2 = \rho_{13}$ and $q_3 = \rho_{14}, d > 1$. The specific form of the q -variables is irrelevant for our purposes, see below.

In the above mentioned variables, Δ_{radial} (34) can be further decomposed into the sum of 2 operators

$$\Delta_{radial} = \Delta_g + \Delta_{q,g}, \quad (67)$$

with the following properties:

- $\Delta_g = \Delta_g(\mathcal{V}, S, P)$ is an algebraic operator for any d . It depends only on *volume*

variables \mathcal{V} , S , and P and their derivatives,

$$\begin{aligned} \Delta_g = & \frac{2}{9} \mathcal{V} S \partial_{\mathcal{V},\mathcal{V}}^2 + (54 \mathcal{V} + \frac{1}{2} S P) \partial_{S,S}^2 + 8 P \partial_{P,P}^2 \\ & + 32 S \partial_{S,P}^2 + 2 \mathcal{V} (P \partial_{\mathcal{V},S}^2 + 24 \partial_{\mathcal{V},P}^2) + \frac{1}{9} (d-2) S \partial_{\mathcal{V}} \\ & + \frac{1}{2} (d-1) P \partial_S + 12 d \partial_P . \end{aligned} \quad (68)$$

- $\Delta_{q,g} = \Delta_{q,g}(\mathcal{V}, S, P, q_1, q_2, q_3)$, for arbitrary d . This operator annihilates any volume-variables dependent function, namely $\Delta_{q,g} f(\mathcal{V}, S, P) = 0$. We do not give its explicit form.
- $[\Delta_g, \Delta_{q,g}] \neq 0$.

The operator (68) is $sl(4, \mathbf{R})$ -Lie-algebraic, see e.g. [10], and it is gauge-equivalent to a 3-dimensional Schrödinger operator in curved space with d -independent metric (see below). For this operator Δ_g , the reduction from $d \geq 3$ to $d = 2$ simply corresponds to imposing the condition $\mathcal{V} = 0$ together with $d = 2$. In turn, the reduction to $d = 1$ occurs when 2 conditions are imposed: $\mathcal{V} = S = 0$ together with $d = 1$. Both limits to $d = 2$ and $d = 1$ are geometrically transparent and, more importantly, Δ_g remains algebraic.

The d -independent metric of Δ_g is given by

$$g^{\mu\nu}(\mathcal{V}, S, P) = \begin{pmatrix} \frac{2}{9} \mathcal{V} S & \mathcal{V} P & 24 \mathcal{V} \\ \mathcal{V} P & 54 \mathcal{V} + \frac{1}{2} S P & 16 S \\ 24 \mathcal{V} & 16 S & 8 P \end{pmatrix}, \quad (69)$$

and its determinant factorizes

$$D_g(\mathcal{V}, S, P) \equiv \text{Det } g^{\mu\nu} = G_1 G_2, \quad (70)$$

where

$$\begin{aligned} G_1 &= \frac{8}{9} \mathcal{V}, \\ G_2 &= S^2 (P^2 - 64 S) - 9 P \mathcal{V} (P^2 - 72 S) - 34992 \mathcal{V}^2. \end{aligned} \quad (71)$$

The boundary of the configuration space is defined by $\mathcal{V} = 0$. Using the gauge factor

$$\Gamma_g = G_1^{\frac{3-d}{4}} G_2^{-\frac{1}{4}}, \quad (72)$$

to make a gauge rotation of the operator Δ_g we arrive at the Schrödinger operator

$$\Gamma_g^{-1} \Delta_g \Gamma_g = \Delta_{LB}(\mathcal{V}, S, P) - \tilde{V}_g, \quad (73)$$

with effective potential

$$\tilde{V}_g = (d-5)(d-3) \frac{S}{81 G_1} + \frac{(P^2 - 48 S)(324 \mathcal{V} - P S)}{8 G_2}, \quad (74)$$

where the 1st term vanishes at $d = 3, 5$, and Δ_{LB} is the Laplace-Beltrami operator

$$\Delta_{LB}(\mathcal{V}, S, P) = \sqrt{D_g} \partial_\mu \frac{1}{\sqrt{D_g}} g^{\mu\nu} \partial_\nu. \quad (75)$$

Here ν, μ labels the variables \mathcal{V}, S, P , and $g^{\mu\nu}$ is given by (69).

Thus, for the original 4-body problem (14) in the space of relative motion, provided that the potential depends only on the volume variables, and taking into account the gauge rotation Γ_g (72), we arrive at the gauge-equivalent Hamiltonian

$$\mathcal{H}_{LB}(\mathcal{V}, S, P) = -\Delta_{LB}(\mathcal{V}, S, P) + \tilde{V}_g(\mathcal{V}, S, P) + V(\mathcal{V}, S, P), \quad (76)$$

in the space of volume variables. The Hamiltonian (76) describes a 3-dimensional quantum particle moving in the curved space parametrized by \mathcal{V}, S, P with metric $g^{\mu\nu}$ (69) and kinetic energy Δ_{LB} . The form of (76) implies the possible existence of a subfamily of eigenfunctions in the form of a multiplicative factor times an inhomogeneous polynomial in the variables (\mathcal{V}, S, P) . The volume variables can be generalized to the case of non equal masses (see Appendix B).

1. Towards $d = 2$

Let us assume that $V = V(\mathcal{V}, S, P)$ in (14). In this case, we can ignore the operator $\Delta_{g,g}$ in (67). Now, for $d = 2$ the volume of the tetrahedron of interaction vanishes identically: $\mathcal{V} = 0$. Therefore, the operator Δ_g (68) reduces to

$$\Delta_g|_{d=2} = \frac{1}{2} S P \partial_{S,S}^2 + 8 P \partial_{P,P}^2 + 32 S \partial_{S,P}^2 + \frac{1}{2} P \partial_S + 24 \partial_P. \quad (77)$$

Thus, in the limit $d \rightarrow 2$, Δ_g remains algebraic (more precisely $sl(3, \mathbf{R})$ -Lie-algebraic). The corresponding metric of $\Delta_g|_{d=2}$ takes the form

$$g^{\mu\nu}(S, P) = \begin{pmatrix} \frac{1}{2} S P & 16 S \\ 16 S & 8 P \end{pmatrix}, \quad (78)$$

and its determinant factorizes as

$$D_k(S, P) \equiv \text{Det } g^{\mu\nu} = K_1 K_2 , \quad (79)$$

where

$$K_1 = S , \quad K_2 = P^2 - 64 S . \quad (80)$$

The boundary of the configuration space is defined by $S = 0$. Using the gauge factor

$$\Gamma_{gk} = (K_1 K_2)^{-\frac{1}{4}} , \quad (81)$$

to gauge-rotate the restricted operator $\Delta_g|_{d=2}$, we obtain

$$\Gamma_{gk}^{-1} \Delta_g|_{d=2} \Gamma_{gk} = \Delta_{LB}(S, P) - \tilde{V}_{gk} . \quad (82)$$

Here the effective potential reads

$$\tilde{V}_{gk} = \frac{P^3}{32 S (P^2 - 64 S)} . \quad (83)$$

Thus, for the original 4-body problem (14) in the space of relative motion, provided that the potential depends only on the 2 volume variables (S, P) , and taking into account the gauge rotation Γ_{gk} (82), we arrive at the gauge-equivalent 2-dimensional Hamiltonian

$$\mathcal{H}_{LB}(S, P) = -\Delta_{LB}(S, P) + \tilde{V}_{gk}(S, P) + V(S, P) . \quad (84)$$

The Hamiltonian (84) describes a 2-dimensional quantum particle moving in the curved space with metric $g^{\mu\nu}$ (78). The form of (84) suggests the possible existence of a subfamily of eigenfunctions in the form of a multiplicative factor times a polynomial in the variables (S, P) .

2. Towards $d = 1$

For $d = 1$, both the volume variable \mathcal{V} and the area variable S vanish identically. In this case the algebraic operator (68) depends on the variable P alone, it has the form

$$\Delta_g|_{d=1} = 8 P \partial_{P,P}^2 + 12 \partial_P , \quad (85)$$

which after a suitable gauge rotation and upon the addition of an harmonic potential $\sim \omega^2 P$ becomes the Laguerre operator. We again point out that the form of the operator $\Delta_g|_{d=1}$

implies the existence of a subfamily of solutions of the original 4-body problem (14) in the space of relative motion, which depend only on the variable P .

Let us clarify the space "degeneration" from $d = 3$ to $d = 1$ with a concrete example. For $d = 1$, the number of functionally independent variables (degrees of freedom) in Δ_{radial} (67) is 3 and the 1st operator Δ_g in (67) solely depends on the variable P , see (59). Therefore, the operator $\Delta_{q,g}$ must involve 2 q -variables only.

Now, without loss of generality, let us choose

$$q_1 = \rho_{12}, \quad q_2 = \rho_{23}, \quad q_3 = \sqrt{\rho_{12}} + \sqrt{\rho_{23}} - \sqrt{\rho_{13}}, \quad (86)$$

as the q -variables for $d > 1$. For $d = 1$, $\rho_{ij} \equiv (x_i - x_j)^2$ and $\infty > x_1 > x_2 > x_3 > x_4 > 0$, the variable q_3 vanishes identically and

$$P = 3(\rho_{12} + \rho_{34}) + 4\sqrt{\rho_{23}}(\sqrt{\rho_{12}} + \sqrt{\rho_{23}} + \sqrt{\rho_{34}}) + 2\sqrt{\rho_{12}}\sqrt{\rho_{34}}. \quad (87)$$

For $d \geq 2$, the operator $\Delta_{q,g}$ (67) reads

$$\begin{aligned} \Delta_{q,g} = & 4q_1 \partial_{q_1, q_1}^2 + 4q_2 \partial_{q_2, q_2}^2 - 2(q_3^2 - 2\sqrt{q_1}q_3 - 2\sqrt{q_2}q_3 + 2\sqrt{q_1}\sqrt{q_2}) \partial_{q_1, q_2}^2 \\ & + \frac{q_3(q_3^2 - 3\sqrt{q_1}q_3 - 4\sqrt{q_2}q_3 + 4\sqrt{q_1}\sqrt{q_2} + 2q_1 + 4q_2)}{\sqrt{q_2}(\sqrt{q_1} + \sqrt{q_2} - q_3)} \partial_{q_1, q_3}^2 + 2d(\partial_{q_1} + \partial_{q_2}) \\ & + \frac{q_3(q_3^2 - 4\sqrt{q_1}q_3 - 3\sqrt{q_2}q_3 + 4\sqrt{q_1}\sqrt{q_2} + 4q_1 + 2q_2)}{\sqrt{q_1}(\sqrt{q_1} + \sqrt{q_2} - q_3)} \partial_{q_2, q_3}^2 \\ & + \frac{(d-1)(\sqrt{q_1}\sqrt{q_2} - q_3\sqrt{q_2} + q_1 + q_2 - \sqrt{q_1}q_3)}{\sqrt{q_1}\sqrt{q_2}(\sqrt{q_1} + \sqrt{q_2} - q_3)} \partial_{q_3} \\ & + 8\mathcal{V} \left[\partial_{\mathcal{V}, q_1}^2 + \partial_{\mathcal{V}, q_2}^2 + \frac{\sqrt{q_1}\sqrt{q_2} - q_3\sqrt{q_2} + q_1 + q_2 - \sqrt{q_1}q_3}{32\sqrt{q_1}\sqrt{q_2}(\sqrt{q_1} + \sqrt{q_2} - q_3)} \partial_{\mathcal{V}, q_3}^2 \right] \\ & + \sum_{i=1}^3 A_i \partial_{S, q_i}^2 + 8(2q_1 \partial_{P, q_1}^2 + 2q_2 \partial_{P, q_2}^2 + q_3 \partial_{P, q_3}^2), \end{aligned} \quad (88)$$

where the coefficients A_i are functions of $(\mathcal{V}, S, P, q_1, q_2, q_3)$. At $d = 1$ they vanish: $A_i = 0$. Also all terms involving derivatives $\partial_{\mathcal{V}}$ vanish. Thus, in the limit $d \rightarrow 1$ we end up with

$$\begin{aligned} \Delta_{q,g} |_{d=1} = & 4q_1 \partial_{q_1, q_1}^2 + 4q_2 \partial_{q_2, q_2}^2 - 4\sqrt{q_1}\sqrt{q_2} \partial_{q_1, q_2}^2 + 2(\partial_{q_1} + \partial_{q_2}) \\ & + 16(q_1 \partial_{P, q_1}^2 + q_2 \partial_{P, q_2}^2). \end{aligned} \quad (89)$$

We see that the operator $\Delta_{q,g}$ depends on 2 q -variables only. Finally, for Δ_{radial} we arrive

and the well defined 3-dimensional operator

$$\begin{aligned}\Delta_{radial} |_{d=1} &= \Delta_g |_{d=1} + \Delta_{q,g} |_{d=1} \\ &= 8P \partial_{P,P}^2 + 12 \partial_P + 4q_1 \partial_{q_1,q_1}^2 + 4q_2 \partial_{q_2,q_2}^2 \\ &\quad - 4\sqrt{q_1}\sqrt{q_2} \partial_{q_1,q_2}^2 + 2(\partial_{q_1} + \partial_{q_2}) + 16(q_1 \partial_{P,q_1}^2 + q_2 \partial_{P,q_2}^2),\end{aligned}\tag{90}$$

which, after a suitable change of variables, becomes the algebraic operator (9).

B. u-variables representation

It is worth mentioning another decomposition of the operator Δ_{radial} (34), assuming $d \geq 3$, in the variables

$$(\rho_{12}, \rho_{13}, \rho_{14}, \rho_{23}, \rho_{24}, \rho_{34},) \Rightarrow (u_1, u_2, u_3, q_1, q_2, q_3),\tag{91}$$

where

$$u_1 = \rho_{12} + \rho_{34}; \quad u_2 = \rho_{13} + \rho_{24}; \quad u_3 = \rho_{23} + \rho_{14},\tag{92}$$

are nothing but the sum of 2 disconnected edges (squared). Thus, without common vertices, of the *tetrahedron of interaction*, they are geometrical-type variables. They also are S_4 invariant under the permutations of the 4-body positions.

For simplicity we can choose $q_1 = \rho_{12}$, $q_2 = \rho_{13}$ and $q_3 = \rho_{14}$, ($d \geq 3$). In the new variables (91), the operator Δ_{radial} (34) is decomposed in the sum of two operators

$$\Delta_{radial} = \Delta_u + \Delta_{q,u},\tag{93}$$

with the following properties:

- $\Delta_u = \Delta_u(u_1, u_2, u_3)$ is an algebraic operator for any d and involves the u -variables and its derivatives only

$$\begin{aligned}\frac{1}{2}\Delta_u &= 2u_1 \partial_{u_1,u_1}^2 + 2u_2 \partial_{u_2,u_2}^2 + 2u_3 \partial_{u_3,u_3}^2 \\ &\quad + 2(u_1 + u_2 - u_3) \partial_{u_1,u_2}^2 + 2(u_1 + u_3 - u_2) \partial_{u_1,u_3}^2 + 2(u_2 + u_3 - u_1) \partial_{u_2,u_3}^2 \\ &\quad + 2d(\partial_{u_1} + \partial_{u_2} + \partial_{u_3}).\end{aligned}\tag{94}$$

- $\Delta_{q,u} = \Delta_{q,u}(u_1, u_2, u_3, q_1, q_2, q_3)$: for any d , it annihilates any u -dependent function, namely $\Delta_{q,u} f(u_1, u_2, u_3) = 0$.

$$[\Delta_u, \Delta_{q,u}] \neq 0 .$$

If the original 4-body potential depends on u -variables only the decomposition (93) implies the further reduction of the already reduced spectral problem (14) to

$$(-\Delta_u + V(u))\Psi(u) = E\Psi(u) . \quad (95)$$

The operator Δ_u (94) is $sl(4, \mathbf{R})$ -Lie-algebraic with a flat d -independent metric

$$g^{\mu\nu}(u) = \begin{pmatrix} 4u_1 & 2(u_1 + u_2 - u_3) & 2(u_1 + u_3 - u_2) \\ 2(u_1 + u_2 - u_3) & 4u_2 & 2(u_2 + u_3 - u_1) \\ 2(u_1 + u_3 - u_2) & 2(u_2 + u_3 - u_1) & 4u_3 \end{pmatrix} , \quad (96)$$

and with a rather simple factorizable expression for its determinant

$$D(u) \equiv \text{Det } g^{\mu\nu}(u) = 32(u_1 + u_2 - u_3)(u_1 + u_3 - u_2)(u_2 + u_3 - u_1) . \quad (97)$$

The boundary of the configuration space is defined by $D(u) = 0$. Moreover, using the gauge factor

$$\Gamma_u = D(u)^{\frac{1-d}{4}} , \quad (98)$$

for gauge rotation of the operator Δ_u we obtain a gauge-equivalent 3-dimensional Schrödinger operator

$$\Gamma_u^{-1} \Delta_u(u) \Gamma_u = \Delta_{LB}(u) - \tilde{V}_u(u) , \quad (99)$$

with the effective potential of the form

$$\tilde{V}_u(u) = (d-1)(d-3) \frac{(u_1^2 + u_2^2 + u_3^2 - 2(u_1 u_2 + u_1 u_3 + u_2 u_3))}{2(u_1 - u_2 - u_3)(u_1 + u_2 - u_3)(u_1 - u_2 + u_3)} . \quad (100)$$

Finally, for the original 4-body problem (14) in the space of relative motion, provided that the potential only depends on the u -variables, taking into account the gauge rotation Γ_u (99) and assuming the u -dependent solutions are studied only, we arrive at the gauge-equivalent 3-dimensional Hamiltonian

$$\mathcal{H}_{LB}(u_1, u_2, u_3) = -\Delta_{LB}(u_1, u_2, u_3) + \tilde{V}_u(u_1, u_2, u_3) + V(u_1, u_2, u_3) , \quad (101)$$

in the space of u -variables. The Hamiltonian (101) also describes a 3-dimensional quantum particle moving in the flat space parametrized by u_1, u_2, u_3 with metric $g^{\mu\nu}$ (96) and kinetic energy $\Delta_{LB}(u)$. The form of (101) implies the possible existence of a subfamily of eigenfunctions in the form of a multiplicative factor times an inhomogeneous polynomial in the variables (u_1, u_2, u_3) . The u -variables do not admit a generalization to the case of non equal masses.

Toward $d = 2$ and $d = 1$

Unlike the volume variables \mathcal{V} and S , the u -variables (91) are not subject to any constraint at $d = 2$ ($\mathcal{V} = 0$) and $d = 1$ ($\mathcal{V} = S = 0$). Moreover, for the operator Δ_u (94) the passage to lower dimensions is non-singular. Upon reduction to $d = 1, 2$, only the overall multiplicative factor in front of the first derivative terms in (94) changes.

However, for $d = 2$ the number of variables (degrees of freedom) in Δ_{radial} (67) is reduced to 5. Therefore, in this case the operator $\Delta_{q,u}$ in (93) must involve only 2 q -variables:

$$\Delta_{q,u}|_{d=2} = \sum_{i+j=1}^2 Y_{i,j} \partial_{q_1}^i \partial_{q_2}^j, \quad (102)$$

with coefficients $Y_{i,j} = Y_{i,j}(u_1, u_2, u_3, q_1, q_2)$. In general, $\Delta_{q,u}|_{d=2}$ is not an algebraic operator.

At $d = 1$ the operator $\Delta_{q,u}$ vanishes, $\Delta_{q,u} = 0$, while the algebraic operator Δ_u (94), after a suitable gauge rotation and change of variables, describes the kinetic energy of relative motion of the 4-body (A_3) rational Calogero model with potential (25), see [4].

C. P-variable representation

Note that in (68), the coefficients in front of the 2nd and the 1st derivative in P (66) do not involve the volume variables \mathcal{V} (64) and S (65). Furthermore, the variable

$$P = u_1 + u_2 + u_3 = \rho_{12} + \rho_{13} + \rho_{14} + \rho_{23} + \rho_{24} + \rho_{34}. \quad (103)$$

is nothing but the sum of the u -variables (91), which appear in the algebraic operator Δ_u (94) for any d . Based on these 2 facts let us make, assuming $d \geq 3$, the change of variables

$$(\rho_{12}, \rho_{13}, \rho_{14}, \rho_{23}, \rho_{24}, \rho_{34},) \Rightarrow (P, q_1, q_2, q_3, q_4, q_5). \quad (104)$$

We call this the P -representation. It is worth noting that P is, up to an overall constant factor, the unique linear combination of ρ -variables that is both S_4 -invariant under the permutations of the 4-body positions, as well as S_6 -invariant under the permutations of the 6 ρ 's. At the same time the q -variables form a set of well-defined quantities such that the Jacobian of the transformation (104) is non-singular.

In the variables (104), the operator Δ_{radial} (34) admits the decomposition

$$\Delta_{radial} = \Delta_P + \Delta_{q,P}, \quad (105)$$

with the following properties:

- $\Delta_P = \Delta_P(P)$ is an algebraic operator for any d and involves the P -variable and its derivatives only

$$\Delta_P = 8 P \partial_{P,P}^2 + 12 d \partial_P . \quad (106)$$

- $\Delta_{q,P} = \Delta_{q,P}(P, q_1, q_2, q_3, q_4, q_5)$: For any d , it annihilates any P -dependent function, namely $\Delta_{q,P} f(P) = 0$.

- $[\Delta_P, \Delta_{q,P}] \neq 0$.

Using a gauge factor

$$\Gamma_P = P^{\frac{1-3d}{4}} , \quad (107)$$

for gauge rotation of the operator Δ_P we obtain the gauge-equivalent, 1-dimensional Schrödinger operator

$$\Gamma_P^{-1} \Delta_P(P) \Gamma_P = \Delta_{LB}(P) - \tilde{V}_P(P) , \quad (108)$$

with Laplace-Beltrami operator

$$\Delta_{LB}(P) = 4(2 P \partial_{P,P}^2 + \partial_P) , \quad (109)$$

and metric

$$g^{11} = 8 P , \quad (110)$$

and an effective potential of the form

$$\tilde{V}_P(P) = \frac{3(d-1)(3d-1)}{2P} . \quad (111)$$

In conclusion, for the original 4-body problem (14) in the space of relative motion, provided that the potential depends on the P -variable only, and taking into account the gauge rotation Γ_P (108), we obtain the gauge-equivalent 1-dimensional Hamiltonian

$$\mathcal{H}_{LB}(P) = -\Delta_{LB}(P) + \tilde{V}_P(P) + V(P) . \quad (112)$$

The form of (112) implies the possible existence of a subfamily of eigenfunctions in the form of a P -dependent multiplicative factor times an inhomogeneous polynomial in P . For $d = 1$, this remarkable property was previously pointed out in [17]. It is evident that the P -variable admits a generalization to the case of non equal masses.

Towards $d = 2$ and $d = 1$

For the operator Δ_P (106) the passage to lower dimensions is non-singular. In the limit $d = 1, 2$ only the overall multiplicative factor in front of the first derivative term in (106) changes.

As for the operator $\Delta_{q,P}$ in (105), in the case $d = 2$ it must involve only 4 q -variables:

$$\Delta_{q,P}|_{d=2} = \sum_{i+j+k+\ell=1}^2 Y_{i,j,k,\ell} \partial_{q_1}^i \partial_{q_2}^j \partial_{q_3}^k \partial_{q_4}^\ell, \quad (113)$$

with certain coefficients $Y_{i,j,k,\ell} = Y_{i,j,k,\ell}(P, q_1, q_2, q_3, q_4)$. In general, $\Delta_{q,P}|_{d=2}$ is not an algebraic operator.

For $d = 1$, the operator $\Delta_{q,P}$ depends on 2 q -variables alone:

$$\Delta_{q,P}|_{d=1} = \sum_{i+j=1}^2 Z_{i,j} \partial_{q_1}^i \partial_{q_2}^j, \quad (114)$$

Here $Z_{i,j} = Z_{i,j}(P, q_1, q_2)$. Again, in general $\Delta_{q,P}|_{d=1}$ is not algebraic.

V. (QUASI)-EXACT-SOLVABILITY

In this section, for $d \geq 3$ we describe in more detail the exact and quasi-exactly solvable (QES) models for the 4-body problem in the ρ -representation (space of relative distances).

A. QES in ρ -variables, $d \geq 3$

(I). *Quasi-Exactly-Solvable problem in ρ -variables.*

Let us take the d -independent function

$$\Psi_0(\rho) \equiv F_2^{\frac{1}{4}} F_1^{\frac{\gamma}{2}} e^{-\omega P - \frac{A}{2} P^2}, \quad (115)$$

where $\gamma, \omega > 0$ and $A \geq 0$ and for $\omega = 0, A > 0$ are constants. Here P is given by (66) and

$$F_1 = \mathcal{V}, \quad (116)$$

$$F_2 = 36 \mathcal{V} - P S, \quad (117)$$

ing written in terms of the volume variables (64)-(66). We look for the potential for which the function (115) is the ground state function for the Hamiltonian $H_{LB}(\rho)$ (61) of the 6-dimensional quantum particle. This potential can be found immediately by calculating the ratio

$$\frac{\Delta_{LB}(\rho)\Psi_0}{\Psi_0} = V_0 - E_0, \quad (118)$$

where $\Delta_{LB}(\rho)$ is given by (57) with metric (38). The result is

$$V_0(\rho) = \frac{3P^2 + 112S}{32F_2} + \gamma(\gamma-1)\frac{S}{18F_1} + 8\omega^2 P + 4AP(4\omega P - 6\gamma - 11) + 8A^2 P^3, \quad (119)$$

which is d -independent. This includes both the effective potential V_{eff} and many-body potential V with the energy of the ground state

$$E_0 = 12\omega(3 + 2\gamma). \quad (120)$$

Now, we take the Hamiltonian $H_{LB,0} \equiv -\Delta_{LB}(\rho) + V_0$ with potential (119), subtract E_0 (120) and make the gauge rotation with Ψ_0 (115). As a result we obtain the $sl(7, \mathbf{R})$ -Lie-algebraic operator with additional potential ΔV_N

$$\begin{aligned} \Psi_0^{-1}(-\Delta_{LB}(\rho) + V_0 - E_0)\Psi_0 &\equiv h^{(ges)}(J) + \Delta V_N = -\Delta_R(\mathcal{J}) \\ &+ 2(d-3-2\gamma)(\mathcal{J}_1^- + \mathcal{J}_2^- + \mathcal{J}_3^- + \mathcal{J}_4^- + \mathcal{J}_5^- + \mathcal{J}_6^-) \\ &+ 16A(\mathcal{J}_1^+(N) + \mathcal{J}_2^+(N) + \mathcal{J}_3^+(N) + \mathcal{J}_4^+(N) + \mathcal{J}_5^+(N) + \mathcal{J}_6^+(N)) \\ &+ 16\omega(\mathcal{J}_{11}^0 + \mathcal{J}_{22}^0 + \mathcal{J}_{33}^0 + \mathcal{J}_{44}^0 + \mathcal{J}_{55}^0 + \mathcal{J}_{66}^0) + \Delta V_N, \end{aligned} \quad (121)$$

see (55), where

$$\Delta V_N = 16ANP = 16AN(\rho_{12} + \rho_{13} + \rho_{14} + \rho_{23} + \rho_{24} + \rho_{34}). \quad (122)$$

It is evident that if the parameter N takes integer values, the d -independent operator $h^{(ges)}(J)$ has a finite-dimensional invariant subspace \mathcal{P}_N , (54) with $\dim \mathcal{P}_N \sim N^3$ at large N . Finally, we arrive at the quasi-exactly-solvable, d -independent, single particle Hamiltonian in the space of relative distances ρ ,

$$H_{LB}^{(ges)}(\rho) = -\Delta_{LB}(\rho) + V_N^{(ges)}(\rho), \quad (123)$$

cf.(17), where

$$V_N^{(ges)} = \frac{3P^2 + 112S}{32F_2} + \frac{\gamma(\gamma-1)S}{18F_1} + 8\omega^2 P + 4AP(4\omega P - 6\gamma - 11 - 4N) + 8A^2 P^3, \quad (124)$$

is a QES potential. Its configuration space is defined by $F_1 \geq 0$, while if it is fulfilled then $F_2 \geq 0$.

For this potential $\sim N^3$ eigenstates can be found by algebraic means. They have the factorized form of the polynomial in ρ multiplied by Ψ_0 (115),

$$\text{Pol}_N(\rho_{12}, \rho_{13}, \rho_{14}, \rho_{23}, \rho_{24}, \rho_{34}) \Psi_0(F_1, F_2, P). \quad (125)$$

These polynomials are the eigenpolynomials of the quasi-exactly-solvable, d -independent, algebraic operator

$$\begin{aligned} \frac{1}{2} h^{(qes)}(\rho) = & -2(\rho_{12} \partial_{\rho_{12}}^2 + \rho_{13} \partial_{\rho_{13}}^2 + \rho_{14} \partial_{\rho_{14}}^2 + \rho_{23} \partial_{\rho_{23}}^2 + \rho_{24} \partial_{\rho_{24}}^2 + \rho_{34} \partial_{\rho_{34}}^2) \\ & - ((\rho_{12} + \rho_{13} - \rho_{23}) \partial_{\rho_{12}} \partial_{\rho_{13}} + (\rho_{12} + \rho_{14} - \rho_{24}) \partial_{\rho_{12}} \partial_{\rho_{14}} + (\rho_{13} + \rho_{14} - \rho_{34}) \partial_{\rho_{13}} \partial_{\rho_{14}}) \\ & - ((\rho_{12} + \rho_{23} - \rho_{13}) \partial_{\rho_{12}} \partial_{\rho_{23}} + (\rho_{12} + \rho_{24} - \rho_{14}) \partial_{\rho_{12}} \partial_{\rho_{24}} + (\rho_{23} + \rho_{24} - \rho_{34}) \partial_{\rho_{23}} \partial_{\rho_{24}}) \\ & - ((\rho_{13} + \rho_{23} - \rho_{12}) \partial_{\rho_{13}} \partial_{\rho_{23}} + (\rho_{13} + \rho_{34} - \rho_{14}) \partial_{\rho_{13}} \partial_{\rho_{34}} + (\rho_{23} + \rho_{34} - \rho_{24}) \partial_{\rho_{23}} \partial_{\rho_{34}}) \\ & - ((\rho_{14} + \rho_{24} - \rho_{12}) \partial_{\rho_{14}} \partial_{\rho_{24}} + (\rho_{14} + \rho_{34} - \rho_{13}) \partial_{\rho_{14}} \partial_{\rho_{34}} + (\rho_{24} + \rho_{34} - \rho_{23}) \partial_{\rho_{24}} \partial_{\rho_{34}}) \\ & - (2\gamma + 3)(\partial_{\rho_{12}} + \partial_{\rho_{13}} + \partial_{\rho_{14}} + \partial_{\rho_{23}} + \partial_{\rho_{24}} + \partial_{\rho_{34}}) \\ & + 8\omega(\rho_{12} \partial_{\rho_{12}} + \rho_{13} \partial_{\rho_{13}} + \rho_{14} \partial_{\rho_{14}} + \rho_{23} \partial_{\rho_{23}} + \rho_{24} \partial_{\rho_{24}} + \rho_{34} \partial_{\rho_{34}}) \\ & + 8AP(\rho_{12} \partial_{\rho_{12}} + \rho_{13} \partial_{\rho_{13}} + \rho_{14} \partial_{\rho_{14}} + \rho_{23} \partial_{\rho_{23}} + \rho_{24} \partial_{\rho_{24}} + \rho_{34} \partial_{\rho_{34}} - N), \end{aligned} \quad (126)$$

or, equivalently, of the quasi-exactly-solvable $sl(7, \mathbf{R})$ -Lie-algebraic operator

$$\begin{aligned} \frac{1}{2} h^{(qes)}(J) = & -2(\mathcal{J}_{11}^0 \mathcal{J}_1^- + \mathcal{J}_{22}^0 \mathcal{J}_2^- + \mathcal{J}_{33}^0 \mathcal{J}_3^- + \mathcal{J}_{44}^0 \mathcal{J}_4^- + \mathcal{J}_{55}^0 \mathcal{J}_5^- + \mathcal{J}_{66}^0 \mathcal{J}_6^-) \\ & - \left[\mathcal{J}_{11}^0 (\mathcal{J}_2^- + \mathcal{J}_3^- + \mathcal{J}_4^- + \mathcal{J}_5^-) + \mathcal{J}_{22}^0 (\mathcal{J}_1^- + \mathcal{J}_3^- + \mathcal{J}_4^- + \mathcal{J}_6^-) \right. \\ & + \mathcal{J}_{33}^0 (\mathcal{J}_1^- + \mathcal{J}_2^- + \mathcal{J}_5^- + \mathcal{J}_6^-) + \mathcal{J}_{44}^0 (\mathcal{J}_1^- + \mathcal{J}_2^- + \mathcal{J}_5^- + \mathcal{J}_6^-) \\ & \left. + \mathcal{J}_{55}^0 (\mathcal{J}_1^- + \mathcal{J}_3^- + \mathcal{J}_4^- + \mathcal{J}_6^-) + \mathcal{J}_{66}^0 (\mathcal{J}_2^- + \mathcal{J}_3^- + \mathcal{J}_4^- + \mathcal{J}_5^-) \right] \\ & + 2 \left[\mathcal{J}_{12}^0 \mathcal{J}_4^- + \mathcal{J}_{13}^0 \mathcal{J}_5^- + \mathcal{J}_{21}^0 \mathcal{J}_4^- + \mathcal{J}_{23}^0 \mathcal{J}_6^- + \mathcal{J}_{31}^0 \mathcal{J}_5^- + \mathcal{J}_{32}^0 \mathcal{J}_6^- \right. \\ & \left. + \mathcal{J}_{41}^0 \mathcal{J}_2^- + \mathcal{J}_{45}^0 \mathcal{J}_6^- + \mathcal{J}_{54}^0 \mathcal{J}_6^- + \mathcal{J}_{62}^0 \mathcal{J}_3^- + \mathcal{J}_{64}^0 \mathcal{J}_5^- + \mathcal{J}_{51}^0 \mathcal{J}_3^- \right] \\ & - (3 + 2\gamma)(\mathcal{J}_1^- + \mathcal{J}_2^- + \mathcal{J}_3^- + \mathcal{J}_4^- + \mathcal{J}_5^- + \mathcal{J}_6^-) \\ & + 8A(\mathcal{J}_1^+(N) + \mathcal{J}_2^+(N) + \mathcal{J}_3^+(N) + \mathcal{J}_4^+(N) + \mathcal{J}_5^+(N) + \mathcal{J}_6^+(N)) \\ & + 8\omega(\mathcal{J}_{11}^0 + \mathcal{J}_{22}^0 + \mathcal{J}_{33}^0 + \mathcal{J}_{44}^0 + \mathcal{J}_{55}^0 + \mathcal{J}_{66}^0), \end{aligned} \quad (127)$$

(121).

As for the original many-body problem (14) in the space of relative distances

$$\mathcal{H}_r \Psi(r) \equiv \left(-\Delta_{radial}(r) + V(r) \right) \Psi(r) = E \Psi(r), \quad \Psi \in L_2(\mathfrak{R}_{radial}), \quad (128)$$

the potential for which quasi-exactly-solvable, polynomial solutions occur in the form

$$\text{Pol}_N(\rho_{12}, \rho_{13}, \rho_{14}, \rho_{23}, \rho_{24}, \rho_{34}) \Gamma(F_1, F_2) \Psi_0(F_1, F_2, P), \quad (129)$$

where $\Gamma \sim D^{-1/4} F_1^{\frac{4-d}{4}}$, see (58), is given by

$$V_{relative,N}^{(qes)} = V_N^{(qes)} - V_{eff} = \frac{4\gamma(\gamma-1) - (d-5)(d-3)}{72} \frac{S}{F_1} + 8\omega^2 P + 4AP(4\omega P - 6\gamma - 11 - 4N) + 8A^2 P^3, \quad (130)$$

c.f. (124). It does not depend on F_2 and does not contain a singular term $\sim 1/F_2$.

(II). *Exactly-Solvable problem in ρ -variables.*

If the parameter A vanishes in (115), (124) and (121), (127) we have the exactly-solvable problem where Ψ_0 (115) at $A = 0$ plays the role of the ground state function,

$$\Psi_0(\rho_{12}, \rho_{13}, \rho_{23}) = F_2^{\frac{1}{4}} F_1^{\frac{\gamma}{2}} e^{-\omega P}. \quad (131)$$

The $sl(7, \mathbf{R})$ -Lie-algebraic operator (127) contains no raising generators $\{\mathcal{J}^+(N)\}$ and becomes

$$h^{(exact)} = -\Delta_{\mathbf{R}}(\mathcal{J}) + 2(d-3-2\gamma)(\mathcal{J}_1^- + \mathcal{J}_2^- + \mathcal{J}_3^- + \mathcal{J}_4^- + \mathcal{J}_5^- + \mathcal{J}_6^-) + 16\omega(\mathcal{J}_{11}^0 + \mathcal{J}_{22}^0 + \mathcal{J}_{33}^0 + \mathcal{J}_{44}^0 + \mathcal{J}_{55}^0 + \mathcal{J}_{66}^0), \quad (132)$$

see (55), and, hence, preserves the infinite flag of finite-dimensional invariant subspaces \mathcal{P}_N (54) at $N = 0, 1, 2, \dots$. The single particle potential (124) becomes

$$V^{(es)}(\rho) = \frac{3P^2 + 112S}{32F_2} + \gamma(\gamma-1)\frac{S}{18F_1} + 8\omega^2 P. \quad (133)$$

Thus, we arrive at the exactly-solvable single particle Hamiltonian in the space of relative distances

$$H_{LB}^{(es)}(\rho) = -\Delta_{LB}(\rho) + V^{(es)}(\rho), \quad (134)$$

where the spectrum of energies

$$E_N = 12\omega(N + 3 + 2\gamma), \quad N = 0, 1, 2, \dots, \quad (135)$$

is equidistant. Its degeneracy is equal to the number of partitions of

$$N = n_1 + n_2 + n_3 + n_4 + n_5 + n_6 . \quad (136)$$

All eigenfunctions have the factorized form of a polynomial in ρ multiplied by Ψ_0 (131),

$$\text{Pol}_N(\rho_{12}, \rho_{13}, \rho_{14}, \rho_{23}, \rho_{24}, \rho_{34}) \Psi_0(F_1, F_2, P) , \quad N = 0, 1, 2, \dots . \quad (137)$$

Note that these polynomials are eigenpolynomials of the exactly-solvable, d -independent, algebraic operator (127) with $A = 0$,

$$h^{(exact)}(\rho) = h^{(ges)}(\rho) |_{A=0} . \quad (138)$$

The polynomials Pol_N are orthogonal w.r.t. Ψ_0^2 (131) in the domain given by (36). To the best of our knowledge these orthogonal polynomials have not been studied in the literature.

The Hamiltonian with potential (133) can be considered as a type of a d -dimensional generalization of the 4-body Calogero model [14] with loss of the property of pairwise interaction only. Now the potential of interaction contains 2-, 3- and 4-body interactions. If $\gamma = 0, 1$ in (133) we obtain the celebrated harmonic oscillator potential in the space of relative distances, see e.g. [15]-[16] for the 3-body case. In turn, in the space of relative motion this potential contains no singular terms at all and becomes,

$$V_{harmonic} = 8\omega^2 P = 8\omega^2 (\rho_{12} + \rho_{13} + \rho_{14} + \rho_{23} + \rho_{24} + \rho_{34}) . \quad (139)$$

Thus, we arrive at the (non-singular) harmonic oscillator potential $V_{harmonic}$. The potential (133) is a d -dimensional generalization of the harmonic oscillator in the space of relative motion rather than a potential of a generalized 4-body (rational) Calogero model.

VI. CONCLUSIONS

In this paper we studied the quantum 4 body problem in a d -dimensional space. Based on the change of variables from individual Cartesian coordinates $\{\mathbf{r}_i\}$ to center-of-mass vector coordinate \mathbf{R}_{CM} , mutual relative distances between bodies $\{r_{ij}\}$ and angles $\{\Omega\}$,

$$(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) \Leftrightarrow (\mathbf{R}_{CM}, \{r_{ij}\}, \{\Omega\}) , \quad (140)$$

kinetic energy given by the original flat diagonal Laplace operator decomposes naturally into the sum of 3 operators

$$\sum_{i=1}^4 \frac{1}{2} \Delta_i^{(d)} = \Delta_{\mathbf{R}_{CM}} + \Delta_{radial} + \Delta_{\Omega}, \quad (141)$$

where $\Delta_{\mathbf{R}_{CM}}$ is the center of mass Laplacian, the operator Δ_{radial} depends on the mutual distances (equivalently, the radial variables) only, $\rho_{ij} = r_{ij}^2$, and Δ_{Ω} annihilates any function of the radial variables alone. The operator $\Delta_{radial}(\rho)$ is self-adjoint, it does not depend on how angular variables Ω are introduced. It is positive-definite. Also it is an $sl(7, R)$ -Lie-algebraic operator, see (34) and (55).

On the subspace of the Hilbert space of angle-independent eigenfunctions, the above-mentioned change of variables implies that the original multi-dimensional spectral problem,

$$\mathcal{H} \Psi = E \Psi, \quad (142)$$

is reduced to a much simpler, *restricted* one,

$$\left(-\Delta_{radial}(\rho) + V(\rho) \right) \psi = E \psi. \quad (143)$$

This restricted spectral problem depends on 6 variables solely. Moreover, the ground state function, if it exists, should be an eigenfunction of such restricted spectral problem as was predicted by Ter-Martirosyan [5].

It was shown that there exists a gauge factor Γ such that the l.h.s. in (143) is gauge-equivalent to the Hamiltonian of a 6-dimensional quantum particle in a curved space with external potential,

$$\mathcal{H}_{LB} \equiv \Gamma^{-1} \left(-\Delta_{radial}(\rho) + V(\rho) \right) \Gamma = -\Delta_{LB} + V_{eff}(\rho) + V(\rho). \quad (144)$$

Here Δ_{LB} is the Laplace-Beltrami operator with contravariant metric $g^{\mu\nu}$ (38), and $V_{eff}(\rho)$ (59) is the effective potential which emerged as a result of the Γ -gauge rotation. The boundary of the configuration space for \mathcal{H}_{LB} is defined by the condition $\det g^{\mu\nu} = 0$.

For the case (34) and $d \geq 3$ we determined the 1st and 2nd order symmetry operators for the free Hamiltonian and showed that the system was integrable and superintegrable but, apparently, nonseparable.

The (Lie)-algebraic form of the operator $\Delta_{radial}(\rho)$ suggests a way to find the exact solutions of both the restricted and the original spectral problems. In particular, adding

$\Delta_{radial}(\rho)$ the terms linear in derivatives, $A_{ij} \rho_{ij} \partial_{ij}$, and then gauging them away with factor $\sim \exp(-\tilde{A}_{ij} \rho_{ij})$ leads to the anisotropic harmonic oscillator potential in the space of relative distances,

$$V^{(ex)} = \sum_{i<j}^6 \omega_{ij}^2 \rho_{ij} , \quad (145)$$

which is an exactly-solvable potential for the restricted problem and perhaps, quasi-exactly-solvable for the original problem.

A novel result was the introduction of 2 different representations for the operator Δ_{radial} in (143). They involve pure geometrical variables defined by the tetrahedron of interaction. In particular, the volume-variables representation allows us a better understanding of the degeneration from $d \geq 3$ to lower dimensions $d = 2$ and $d = 1$. In this limiting process, a Lie-algebraic sector of the problem is preserved. For the restricted problem (143) in the volume-variables representation we obtain, provided that the original potential only depends on the volume variables, the gauge-equivalent Hamiltonian

$$\mathcal{H}_{LB}(\mathcal{V}, S, P) = -\Delta_{LB}(\mathcal{V}, S, P) + \tilde{V}_g(\mathcal{V}, S, P) + V(\mathcal{V}, S, P) , \quad (146)$$

which describes a 3-dimensional quantum particle moving in a curved space.

Interestingly, in the u -variables representation there exists another gauge-equivalent Hamiltonian

$$\mathcal{H}_{LB}(u_1, u_2, u_3) = -\Delta_{LB}(u_1, u_2, u_3) + \tilde{V}_u(u_1, u_2, u_3) + V(u_1, u_2, u_3) , \quad (147)$$

in the space of u -variables which describes a 3-dimensional quantum particle moving not in a curved but in a flat space. For $d = 1$ the operator (147), after a suitable gauge rotation and change of variables, reduces to the 4-body (A_3) rational Calogero-Sutherland model.

For any d , in the P -variable representation we have the remarkable property of the existence of a family of eigenfunctions of the 4-body problem that only depend on the P -variable.

Consequently, exactly- and quasi-exactly-solvable models can be constructed for any d . This reveals interesting links between exact solvability and polyhedra which, more importantly, set up the basis towards the *geometrization* of the n -body problem. The question about the existence of a representation in which the whole operator Δ_{radial} in (143) remains algebraic at $d = 2$ is still open.

Also, the case of non-equal masses is presented in the Appendix A. The operator $\Delta_{radial} \rightarrow \Delta'_{radial}$ (A2) admits a simple limit to the atomic (say, $m_1 \rightarrow \infty$) and molecular (say, $m_{1,\dots,p} \rightarrow \infty$) situations. In the atomic case, for the operator $\Delta'_{radial}(\rho)$ (A2) all 2nd order cross derivatives $\partial_{\rho_{1j}}\partial_{\rho_{1k}}$ disappear, while other terms remain. The number of variables in this case remains unchanged. In the molecular case, not only cross derivatives $\partial_{\rho_{qj}}\partial_{\rho_{qk}}, q = 1, \dots, p$ but also the derivatives w.r.t. $\rho_{ij}, 1 \leq i < j \leq p$ vanish. Thus, in general the operator Δ'_{radial} depends on $6 - \frac{p(p-1)}{2}$ variables. Other variables which may appear in the potential $V(\rho)$ are external parameters. This corresponds to the so-called Bohr-Oppenheimer approximation (of zero order) in molecular physics.

In Appendix B, we introduce the volume variables for the case of arbitrary masses. In the Appendix C, the generalization of the volume-variables to the n -body case is presented as well.

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Appendix A: ρ -representation for non equal masses

Consider the general case of 4 particles located at points $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4$ of masses m_1, m_2, m_3, m_4 , respectively. The analogue of decomposition of kinetic energy of relative motion $\Delta_{rel}^{(3d)}$, see (1),(5), in variables (r_{ij}, Ω) exists,

$$\frac{1}{2}\Delta_{rel}^{(3d,m)} = \Delta_{radial}^{(6,m)}(r_{ij}, \partial_{ij}) + \Delta_{\Omega}^{(3d-6,m)}(r_{ij}, \Omega, \partial_{ij}, \partial_{\Omega}), \quad \partial_{ij} \equiv \frac{\partial}{\partial r_{ij}}, \quad (A1)$$

(1). Explicitly, the operator $\Delta_{radial}^{(6,m)}(r_{ij}, \partial_{ij})$ becomes (in terms of the relative coordinates $\rho_{ij} = r_{ij}^2$), see [3],

$$\begin{aligned} \Delta'_{radial}(\rho_{ij}, \partial_{ij}) = & 2 \left(\frac{1}{\mu_{12}} \rho_{12} \partial_{\rho_{12}}^2 + \frac{1}{\mu_{13}} \rho_{13} \partial_{\rho_{13}}^2 + \frac{1}{\mu_{14}} \rho_{14} \partial_{\rho_{14}}^2 + \frac{1}{\mu_{23}} \rho_{23} \partial_{\rho_{23}}^2 + \frac{1}{\mu_{24}} \rho_{24} \partial_{\rho_{24}}^2 \right. \\ & \left. + \frac{1}{\mu_{34}} \rho_{34} \partial_{\rho_{34}}^2 \right) + d \left(\frac{1}{\mu_{12}} \partial_{\rho_{12}} + \frac{1}{\mu_{13}} \partial_{\rho_{13}} + \frac{1}{\mu_{14}} \partial_{\rho_{14}} + \frac{1}{\mu_{23}} \partial_{\rho_{23}} + \frac{1}{\mu_{24}} \partial_{\rho_{24}} + \frac{1}{\mu_{34}} \partial_{\rho_{34}} \right) \\ & + \frac{2}{m_1} \left((\rho_{12} + \rho_{13} - \rho_{23}) \partial_{\rho_{12}} \partial_{\rho_{13}} + (\rho_{12} + \rho_{14} - \rho_{24}) \partial_{\rho_{12}} \partial_{\rho_{14}} + (\rho_{13} + \rho_{14} - \rho_{34}) \partial_{\rho_{13}} \partial_{\rho_{14}} \right) \\ & + \frac{2}{m_2} \left((\rho_{12} + \rho_{23} - \rho_{13}) \partial_{\rho_{12}} \partial_{\rho_{23}} + (\rho_{12} + \rho_{24} - \rho_{14}) \partial_{\rho_{12}} \partial_{\rho_{24}} + (\rho_{23} + \rho_{24} - \rho_{34}) \partial_{\rho_{23}} \partial_{\rho_{24}} \right) \\ & + \frac{2}{m_3} \left((\rho_{13} + \rho_{23} - \rho_{12}) \partial_{\rho_{13}} \partial_{\rho_{23}} + (\rho_{13} + \rho_{34} - \rho_{14}) \partial_{\rho_{13}} \partial_{\rho_{34}} + (\rho_{23} + \rho_{34} - \rho_{24}) \partial_{\rho_{23}} \partial_{\rho_{34}} \right) \\ & + \frac{2}{m_4} \left((\rho_{14} + \rho_{24} - \rho_{12}) \partial_{\rho_{14}} \partial_{\rho_{24}} + (\rho_{14} + \rho_{34} - \rho_{13}) \partial_{\rho_{14}} \partial_{\rho_{34}} + (\rho_{24} + \rho_{34} - \rho_{23}) \partial_{\rho_{24}} \partial_{\rho_{34}} \right), \end{aligned} \quad (A2)$$

where

$$\frac{1}{\mu_{ij}} = \frac{m_i + m_j}{m_i m_j}, \quad (A3)$$

is the reduced mass for particles i and j . (Compare this with (34) for the case of equal masses $m_1 = m_2 = m_3 = m_4 = 1$.) This operator has the same algebraic structure as $\Delta_{radial}(\rho_{ij})$ but lives on a different manifold in general. It can be rewritten in terms of the generators of the maximal affine subalgebra b_7 of the algebra $sl(7, \mathbf{R})$, see (52), c.f. (55). The contravariant metric tensor, obtained from the coefficients in front of the second derivatives in (A2), does not depend on d and its determinant is

$$D_m = \det g^{\mu\nu} = 9216 c_m V_4^2 \left[\left(\sum V_{2,m} \right) \left(\sum V_{3,m} \right) - 9 (m_1 + m_2 + m_3 + m_4) V_4^2 \right], \quad (A4)$$

and is positive definite, where $c_m = \frac{m_1 + m_2 + m_3 + m_4}{m_1^2 m_2^2 m_3^2 m_4^2}$, V_4^2 given by (64),

$$\sum V_{2,m} = m_1 m_2 r_{12}^2 + m_1 m_3 r_{13}^2 + m_1 m_4 r_{14}^2 + m_2 m_3 r_{23}^2 + m_2 m_4 r_{24}^2 + m_3 m_4 r_{34}^2, \quad (A5)$$

is the weighted sum of square of sides and diagonals of the tetrahedron of interaction,

$$\sum V_{3,m} = \frac{1}{m_1} S^2(r_{23}, r_{24}, r_{34}) + \frac{1}{m_2} S^2(r_{13}, r_{14}, r_{34}) + \frac{1}{m_3} S^2(r_{12}, r_{14}, r_{24}) + \frac{1}{m_4} S^2(r_{12}, r_{13}, r_{23}), \quad (A6)$$

is the weighted sum of squares of areas, and $S^2(a, b, c)$ is the square of the area of the triangle of interaction with sizes a, b, c . Hence, D_m is still proportional to the square of the volume of tetrahedron V_4^2 being of pure geometrical nature!

Making the gauge transformation of (A2) with determinant (A4) as the factor,

$$\Gamma = D_m^{-\frac{1}{4}} V_4^{1-\frac{d}{4}} , \quad (\text{A7})$$

we find that

$$\Gamma^{-1} \Delta'_{radial}(\rho_{ij}) \Gamma = \Delta'_{LB}(\rho) - V_{eff} , \quad (\text{A8})$$

is the Laplace-Beltrami operator plus the effective potential

$$\begin{aligned} V_{eff} = & \frac{3 (\sum V_{2,m}^2)^2 + 28 (m_1 + m_2 + m_3 + m_4) m_1 m_2 m_3 m_4 \sum V_{3,m}^2}{32 m_1 m_2 m_3 m_4 ((\sum V_{2,m}^2) \sum V_{3,m}^2 - 9 (m_1 + m_2 + m_3 + m_4) V_4^2)} \\ & + \frac{(d-5)(d-3) \sum V_{3,m}^2}{72 V_4^2} , \end{aligned} \quad (\text{A9})$$

where its 2nd term is absent for $d = 3, 5$. The Laplace-Beltrami operator plays a role of the kinetic energy of a 6-dimensional quantum particle moving in curved space. While V_{eff} can be considered as the centrifugal potential.

Appendix B: volume-variables representation for non-equal masses

For arbitrary masses (m_1, m_2, m_3, m_4) , the analogue of decomposition (67) for modified by arbitrary masses Δ'_{radial} can be written and the analogue of the operator Δ_g (68) can be derived in modified volume variables,

$$\begin{aligned} \Delta'_g = & \frac{2}{9} \mathcal{V} \tilde{S} \partial_{\mathcal{V}, \mathcal{V}}^2 + \left(\frac{27M}{2m} \mathcal{V} + \frac{1}{2m} \tilde{S} \tilde{P} \right) \partial_{\tilde{S}, \tilde{S}}^2 + 2M \tilde{P} \partial_{\tilde{P}, \tilde{P}}^2 \\ & + 8M \tilde{S} \partial_{\tilde{S}, \tilde{P}}^2 + 2\mathcal{V} \left(\frac{1}{m} \tilde{P} \partial_{\mathcal{V}, \tilde{S}}^2 + 6M \partial_{\mathcal{V}, \tilde{P}}^2 \right) + \frac{1}{9} (d-2) \tilde{S} \partial_{\mathcal{V}} \\ & + \frac{1}{2m} (d-1) \tilde{P} \partial_{\tilde{S}} + 3M d \partial_{\tilde{P}} , \end{aligned} \quad (\text{B1})$$

where $M = m_1 + m_2 + m_3 + m_4$, $m = m_1 m_2 m_3 m_4$, and

$$\begin{aligned} \mathcal{V} & \equiv V_4^2 , \\ \tilde{P} & \equiv \sum V_{2,m} = m_1 m_2 r_{12}^2 + m_1 m_3 r_{13}^2 + m_1 m_4 r_{14}^2 + m_2 m_3 r_{23}^2 + m_2 m_4 r_{24}^2 + m_3 m_4 r_{34}^2 , \\ \tilde{S} & \equiv \sum V_{3,m} = \frac{1}{m_1} S^2(r_{23}, r_{24}, r_{34}) + \frac{1}{m_2} S^2(r_{13}, r_{14}, r_{34}) \\ & + \frac{1}{m_3} S^2(r_{12}, r_{14}, r_{24}) + \frac{1}{m_4} S^2(r_{12}, r_{13}, r_{23}) . \end{aligned} \quad (\text{B2})$$

The contravariant metric tensor obtained from (B1) does not depend on d . Its determinant is

$$D_{gm} = 2 M \mathcal{V} \left[\frac{m \left(162 M \tilde{P} \tilde{S} \mathcal{V} - 2187 M^2 \mathcal{V}^2 + \tilde{P}^2 \tilde{S}^2 \right) - 16 m^2 M \tilde{S}^3 - 9 \tilde{P}^3 \mathcal{V}}{9 m^2} \right]. \quad (\text{B3})$$

Making the gauge transformation of (B1) with determinant (B3) and volume of tetrahedron as the factor:

$$\Gamma = D_{gm}^{-1/4} \mathcal{V}^{1-d/4}, \quad (\text{B4})$$

we find that

$$\Gamma^{-1} \Delta'_g(\tilde{P}, \tilde{S}, \mathcal{V}) \Gamma = \Delta'_{g, LB}(\tilde{P}, \tilde{S}, \mathcal{V}) - V_{eff}, \quad (\text{B5})$$

is the Laplace-Beltrami operator with the effective potential

$$V_{eff} = \frac{(\tilde{P}^2 - 12 m M \tilde{S})(81 M \mathcal{V} - \tilde{P} \tilde{S})}{8 \left(2187 m M^2 \mathcal{V}^2 + m \tilde{S}^2 \left(16 m M \tilde{S} - \tilde{P}^2 \right) + 9 \tilde{P} \mathcal{V} \left(\tilde{P}^2 - 18 m M \tilde{S} \right) \right)} + (d-5)(d-3) \frac{\tilde{S}}{72 \mathcal{V}}, \quad (\text{B6})$$

where the 2nd term is absent for $d = 3, 5$. The Laplace-Beltrami operator plays the role of the kinetic energy of a 3-dimensional quantum particle moving in curved space.

Appendix C: Geometrical variables for the n -body system

Based on concrete results for $n = 2, 3, 4, 5$ we introduce geometrical variables for the n -body system in d -dimensional space $d \geq n - 1$. They allow us to study the degeneration of the system from $d \geq n - 1$ to lower dimensions.

1. volume-variables representation for the n -body system

For equal masses $m_i = 1$ ($i = 1, 2, \dots, n$), we introduce the set of $(n - 1)$ volume variables $\{\mathcal{V}_k\}$, $k = 2, 3, \dots, n$, where \mathcal{V}_n is the volume (squared) of the n -vertex polytope of interaction (whose vertices correspond to the positions of the particles) and \mathcal{V}_k is the sum over the squares of the contents (volumes of faces) of fixed dimension k . In these

variables, the operator $\Delta_{n,radial}$ [3] which depends solely on the $\frac{n(n-1)}{2}$ relative distances between particles can be decomposed as the sum of 2 operators

$$\Delta_{n,radial} = \Delta_{n,g} + \Delta_{n,q} , \quad (C1)$$

($[\Delta_{n,g}, \Delta_{n,q}] \neq 0$) with the following properties

- $\Delta_{n,g} = \Delta_{n,g}(\{\mathcal{V}_k\})$ is an algebraic operator for any d . It involves volume variables $\{\mathcal{V}_k\}$, $k = 2, 3, \dots, n$, alone. Explicitly,

$$\begin{aligned} \Delta_{n,g} = & \mathcal{V}_n \sum_{i=2}^{n-1} a_i \mathcal{V}_i \partial_{i+1,n}^2 + \sum_{i=2}^n b_i \mathcal{V}_i \partial_{i,2}^2 + \sum_{i=0}^{n-2} e_i (d-i) \mathcal{V}_{i+1} \partial_{i+2} \\ & + \sum_{j=1}^{n-3} \sum_{i=1}^j \left(c_{i,j} \mathcal{V}_{n+1-i} \mathcal{V}_{n-j-2} + f_{i,j} \mathcal{V}_{n-i} \mathcal{V}_{n-j-1} \right) \partial_{n-i,n-j}^2 . \end{aligned} \quad (C2)$$

($n > 2$) where

$$\mathcal{V}_0 \equiv 0 \quad , \quad \mathcal{V}_1 \equiv 1 \quad , \quad \partial_i \equiv \partial_{\mathcal{V}_i} \quad , \quad \partial_{i,j}^2 \equiv \partial_{\mathcal{V}_i} \partial_{\mathcal{V}_j}$$

and $a_i, b_i, c_{i,j}, f_{i,j}, e_i$ are constants that can depend on n . In particular,

$$a_{n-1} = \frac{2}{(n-1)^2} \quad , \quad b_2 = 2/n \quad , \quad e_0 = n(n-1) \quad , \quad e_{j-2} = \frac{n-j+1}{(j-1)^2} . \quad (C3)$$

- $\Delta_{n,q} = \Delta_{n,q}(\{\mathcal{V}_k\}, q_1, q_2, \dots, q_w)$, $w = (n-1)(n-2)/2$ for arbitrary d . This operator annihilates any volume-like function, namely, $\Delta_{n,q} f(\{\mathcal{V}_k\}) = 0$. We were unable to find explicitly other constants for arbitrary n .

The operator (C2) is $sl(n, \mathbf{R})$ -Lie-algebraic and is gauge-equivalent to a $(n-1)$ -dimensional Schrödinger operator in a curved space. For this operator $\Delta_{n,g}$, the reduction from $d = n-1$ to $d = n-2$ simply corresponds to the condition $\mathcal{V}_n = 0$ while the reduction to $d = n-3$ occurs when $\mathcal{V}_n = \mathcal{V}_{n-1} = 0$ and so on. All the limits from $d \geq n-1$ to $d = \tilde{d} < n-1$ are geometrically transparent and, more importantly, $\Delta_{n,g}$ remains algebraic. The form of (C2) implies the existence of a subset of eigenfunctions in the form of a global factor times a polynomial solution in the variables $\{\mathcal{V}_k\}$. These geometrical variables can be generalized to the case of non equal masses.

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