Complete sets of commuting operators and $O(3)$ scalars in the enveloping algebra of $SU(3)$

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We consider the "missing label" problem for basis vectors of $SU(3)$ representations in a basis corresponding to the group reduction $SU(3)\supset O(3)\supset O(2)$. We prove that only two independent $O(3)$ scalars exist in the enveloping algebra of $SU(3)$, in addition to the obvious ones, namely the angular momentum L^2 and the two $SU(3)$ Casimir operators $C^{(2)}$ and $C^{(3)}$. Any one of these two operators (of third and fourth order in the generators) can be added to $C^{(2)}$, $C^{(3)}$, L^2, and L_3 to form a complete set of commuting operators. The eigenvalues of the third and fourth order scalars $X^{(3)}$ and $X^{(4)}$ are calculated analytically or numerically for many cases of physical interest. The methods developed in this article can be used to resolve a missing label problem for any semisimple group G, when reduced to any semisimple subgroup H.

1. INTRODUCTION

The general problem that we touch upon in this article is that of providing a complete labeling for the states transforming under an irreducible representation of a given Lie group G. In a certain sense this problem has been completely solved for the classical semisimple groups,[1] corresponding to the Cartan algebras A_n, B_n, C_n, and D_n. Indeed the Gel'fand–Tseitlin patterns[2] provide us precisely with such a set of labels, and the corresponding "canonical basis" consists of a complete nondegenerate set of orthonormal basis functions. The basis functions are the common set of eigenfunctions of a complete set of commuting operators, consisting of the Casimir operators of the group G and of all the Casimir operators of a "canonical" chain of subgroups of G. Thus, e.g., for the group $SU(n)$ the canonical chain is

$$SU(n)\supset S[U(n-1)\times U(1)]\supset S[U(n-2)\times U(1)\times U(1)] \supset\cdots\supset S[U(1)\times\cdots\times U(1)\times U(1)] \quad (1)$$

so that the complete set of commuting operators consists of all the Casimir operators of $SU(n)$, $SU(n-1)$, $\ldots, SU(2)$ and of the $(n-1)$ linear operators (the Cartan subalgebra), corresponding to the $U(1)$ subgroups. Similarly, the problem is solved for the orthogonal and symplectic groups (and also for some of the noncompact groups, corresponding to the same algebras[3]).

Unfortunately, in physics one is often interested in other operators, which may correspond to subgroups, not figuring in the canonical reduction, or may lie in the enveloping algebra of the Lie algebra of G, without being Casimir operators of any subgroup of G. Hence it is important to study other bases and indeed to perform a systematic study of possible bases for representations of various Lie groups.

In this article we restrict ourself to a very simple case, which is, however, of considerable physical interest, namely the group $SU(3)$. The standard application of $SU(3)$ in particle physics, namely the "eightfold way"[4] does indeed make use of the canonical chain of subgroups $SU(3)\supset S[U(2)\times U(1)]\supset S[U(1)\times U(1)]$. However, in nuclear physics[5–7] and more generally in group theoretical treatments of the many-body problem,[8] the quantity of prime interest is angular momentum, associated with the group $O(3)$ that is imbedded into $SU(3)$ in an irreducible manner [this $O(3)$ is the intersection of $SU(3)$ and $SL(3,R)$]. The corresponding chain of subgroups is

$$SU(3)\supset O(3)\supset O(2). \quad (2)$$

Basis functions of $SU(3)$, corresponding to the reduction (2) are eigenfunctions of the second $C^{(2)}$ and third $C^{(3)}$ order Casimir operators of $SU(3)$ and of the angular momentum operators L^2 and L_3. There is one label missing to characterize the states completely and indeed there can be more than one state, characterized by given $O(3)$ quantum numbers (l,m) within a given representation (k_1,k_2) of $SU(3)$. Several different methods have been proposed to resolve this degeneracy problem, and they can be divided into two classes.

The first type of solution leads to a simple labeling of the states (by integers), but to nonorthogonal basis functions that are not eigenfunctions of any complete set of commuting operators.[5,6,9] The other type of solution of the degeneracy problem for $O(3)$ states in $SU(3)$ representations leads to orthonormal states, that are eigenfunctions of $C^{(2)}$, $C^{(3)}$, L^2, L_3 and an additional Hermitian operator X in the enveloping algebra of $SU(3)$.[6,10] The eigenvalues of X provide the missing label for the state vectors; they are, however, not integer numbers and must in general be obtained by solving certain algebraic equations. What is more, Racah has proven[10] that it is not possible to construct any operator in the enveloping algebra of $SU(3)$ that would resolve this missing label problem and have integer eigenvalues.

The purpose of this article is to investigate further the second of the above approaches, that is, in general to study all possible complete sets of commuting operators, the eigenfunctions of which will provide an orthonormal basis for the representations of the group G [in this case $G=SU(3)$]. Investigations along these lines have been carried out,[11] e.g., for the rotation groups $O(3)$ and $O(4)$, the Euclidean groups $E(2)$ and $E(3)$, and the Lorentz groups $O(2,1)$ and $O(3,1)$. Each nonequivalent complete set of commuting operators (consisting of operators from the enveloping algebra of the given algebra that may or may not be Casimir operators of subalgebras, and possibly of some further reflection type operators) provides us with a different set of basis functions. In particular the "nonsubgroup" type operators lead to the appearance of many new types of special functions in group theoretical studies[11,12] (e.g., Lamé and Heun functions).

In this article we consider the reduction of $SU(3)$ to $O(3)$ as in Eq. (2) and study the complete set of commuting operators

$$C^{(2)}, C^{(3)}, L^2, L_3, \text{ and } X, \tag{3}$$

where X is the additional "degeneracy lifting" operator, supplying the label missing in the reduction (2). In order to commute with L^2 and L_3, the operator X must be an $O(3)$ scalar. We shall search for X in the enveloping algebra of $SU(3)$—hence it will automatically commute with the $SU(3)$ Casimir operators $C^{(2)}$ and $C^{(3)}$.

Our main result is that we have shown that only a very small number of independent $O(3)$ scalars X exists in the enveloping algebra of $SU(3)$. Indeed only one third order $X^{(3)}$ and one fourth order $X^{(4)}$ independent operator of this type can be found. All other $O(3)$ scalars can then be written as polynomials in $C^{(2)}$, $C^{(3)}$, L^2, $X^{(3)}$, and $X^{(4)}$ (this result was probably well known, e.g., to Racah, but we are not aware of any general proof).

In Sec. 2 we show for an arbitrary connected Lie group G and an arbitrary (compact or semisimple) Lie subgroup $H \subset G$ that the number of independent scalars with respect to H in the enveloping algebra of G is finite. We then identify G with $SU(3)$, H with $O(3)$, and derive a generating function for the number of $O(3)$ scalars of each order. Finally we present the independent $O(3)$ scalars explicitly. At this stage it is appropriate to stress that the method presented for deriving the generating function for the number of subgroup scalars of a definite order in the enveloping algebra of a given group is quite general and can be applied to many cases of physical interest.

In Sec. 3 we discuss the operator $X^{(3)}$ in detail, derive formulas for its eigenvalues for the cases when the $O(3)$ representation J occurs at most twice in the representation (k_1, k_2). We present a numeric method, making use of the Gel'fand–Tseitlin states, for calculating the $X^{(3)}$ and $X^{(4)}$ eigenvalues for arbitrary representations. The method, which turns out to be quite simple, is then applied to calculate the eigenvalues on a computer for a large number of representations. The results are presented in Tables I and II. A different method for calculating the eigenvalues of $X^{(3)}$ was quite recently presented by Hughes.[13] For those four representations that he considered our results coincide (up to a normalization factor equal to $2\sqrt{6}$). His operator Q_l^0 differs from $X^{(4)}$ by an algebraic combination of the lower order $O(3)$ scalar operators so that the eigenvalues cannot be easily compared. Still another method for calculating these eigenvalues was essentially contained in the by now classical articles of Bargmann and Moshinsky.[6]

2. SUBGROUP INVARIANTS IN THE ENVELOPING ALGEBRA OF THE GROUP

A. Proof that the algebra of invariants is finitely generated

Let H be a connected Lie group, compact or semisimple, with Lie algebra $\mathcal{H}$ and let the matrices $T(h)$, $h \in H$, be an $n \times n$ matrix representation of H. The mapping $\mathbf{x} \to T(h)\mathbf{x}$, where $\mathbf{x} = (x_1, \ldots, x_n)$ is a column vector, induces a representation of H in the space $P[\mathbf{x}]$ of all polynomials in the indeterminants $x_1, \ldots, x_n$ over the complex field. Clearly, the subspaces $P_m[\mathbf{x}]$ consisting of homogeneous polynomials of degree m in the x_j are invariant under the group action, $m = 0, 1, 2, \cdots$.

An *invariant* in $P[\mathbf{x}]$ is a polynomial $p(\mathbf{x})$ which is fixed under the group action: $p(T(g)\mathbf{x}) = p(\mathbf{x})$ for all $h \in H$. Clearly, the invariants in $P[\mathbf{x}]$ form an associative algebra $I[\mathbf{x}]$. In particular $a_1p_1(\mathbf{x}) + a_2p_2(\mathbf{x}) \in I[\mathbf{x}]$ and $p_1(\mathbf{x})p_2(\mathbf{x}) \in I[\mathbf{x}]$ for any invariants $p_1, p_2 \in I[\mathbf{x}]$ and constants $a_1, a_2 \in C$. Furthermore, $I[x] = \sum_{m=0}^{\infty} I_m[\mathbf{x}]$, where $I_m[\mathbf{x}] = I[\mathbf{x}] \cap P_m[\mathbf{x}]$.

A fundamental fact about $I[\mathbf{x}]$ is that it is finitely generated. That is, there exists a finite set $i_1, \ldots, i_q$ of nonconstant invariants such that for every $p(\mathbf{x}) \in I[\mathbf{x}]$ it is possible to find a polynomial $h(y_1, \ldots, y_q)$ with the property $p(\mathbf{x}) \equiv h(i_1(\mathbf{x}), \ldots, i_q(\mathbf{x}))$. Clearly one can choose $i_1, \ldots, i_q$ as homogeneous polynomials in the x_j. Furthermore, if one of the generators, say i_q, can be expressed as a polynomial in the remaining generators, then we can remove it and $i_1, \ldots, i_{q-1}$ will still generate $I[\mathbf{x}]$.

Proceeding in this way, we eventually obtain a minimal set of nonconstant homogeneous polynomial invariants $i'_1, \ldots, i'_{q'}$ which generate $I[\mathbf{x}]$. Such a minimal generating set for $I[\mathbf{x}]$ is called an *integrity basis*. A proof of the existence of a finite integrity basis can be obtained by a slight modification of that given by Weyl,[14] and will not be repeated here.

Let G be a connected Lie group containing H as a Lie subgroup. Then $\mathcal{H}$ is a subalgebra of the Lie algebra $\mathcal{G}$ of G. Let $\mathcal{U}$ be the universal enveloping algebra[1] of $\mathcal{G}$. If $X_1, \ldots, X_n$ is a basis for $\mathcal{G}$, it follows from the Poincaré–Birkhoff–Witt (PBW) theorem[1] that as a vector space $\mathcal{U} \approx \sum_{m=0}^{\infty} \oplus \mathcal{U}_m$, where $\mathcal{U}_0 = C$, $\mathcal{U}_1 = \mathcal{G}$, and $\mathcal{U}_m$ is the space of all symmetric polynomials $p(X_1, \ldots, X_n)$ in the Lie algebra generators which are homogeneous of degree m (see Ref. 1). Furthermore, H (and $\mathcal{H}$) act on $\mathcal{U}$ by means of the adjoint representation, and the subspaces $\mathcal{U}_m$ are invariant under this action. In this paper we are interested in computing the elements in $\mathcal{U}$ which are fixed under the adjoint action of H. If we denote the set of all such elements by $\mathcal{J}$, we see easily that $\mathcal{J}$ is an associative algebra and $\mathcal{J} \approx \sum_{m=0}^{\infty} \oplus \mathcal{J}_m$, where $\mathcal{J}_m \subseteq \mathcal{U}_m$.

Note that as a vector space $\mathcal{U}$ is isomorphic to $P[\mathbf{x}]$. Indeed, by the PBW theorem every $p \in \mathcal{U}$ can be written uniquely as $p = \sum_{m=0}^{\infty} p_m(X_1, \ldots, X_n)$, $p_m \in \mathcal{U}_m$. Moreover, the assignment $p_m(X_1, \ldots, X_n) \leftrightarrow p_m(x_1, \ldots, x_n)$ yields an isomorphism of $\mathcal{U}_m$ and $P_m[\mathbf{x}]$. Finally, if we define the $n \times n$ matrix representation T of H to be that induced by the adjoint action of H on the basis $X_1, \ldots, X_n$ of $\mathcal{G}$, we see that there is a one-to-one correspondence between invariants in $\mathcal{U}$ and polynomial invariants in $P[\mathbf{x}]$.

We can define the notion of an integrity basis for the invariants $\mathcal{J}$ in $\mathcal{U}$ in exact analogy with the definition for the invariants $I[\mathbf{x}]$ in $P[\mathbf{x}]$. An *integrity basis* for $\mathcal{J}$ is a finite set $\{i_1, \ldots, i_q\}$ such that: (1) Each $i_j \in \mathcal{J}$ is homogeneous of degree $m_j \geq 1$ and symmetric in

$X_1, \ldots, X_n$, i.e., each $i_j \in \mathcal{J}_{m_j}$. (2) Every $i \in \mathcal{J}$ can be expressed as a polynomial in $i_1, \ldots, i_q$. (Here we must take into account the fact that the X_k hence the i_j may not commute.) (3) No one of the i_k may be expressed as a polynomial in the remaining i_j, $j \neq k$.

Due to the noncommutativity of the X_j it is not immediately obvious that $\mathcal{J}$ has a finite integrity basis. However, the following holds.

Theorem: If $i_1(x), \ldots, i_q(\mathbf{x})$ is an integrity basis of homogeneous polynomials for $I(\mathbf{x})$, then $i_1(X_1, \ldots, X_n), \ldots, i_q(X_1, \ldots, X_n)$ contains an integrity basis for $\mathcal{J}$. Here, $i_j(X_1, \ldots, X_n)$ is the homogeneous symmetric polynomial in $\mathcal{U}$ corresponding to $i_j(X_1, \ldots, X_n)$.

Proof: We will show that any $C \in \mathcal{J}$ can be expressed as a polynomial in $i_1, \ldots, i_q$. Without loss of generality we can assume $C = C_m \in \mathcal{J}_m$. The proof now proceeds by induction on m. The case $m = 0$ is obvious. Suppose C_m can be expressed as a polynomial in $i_1, \ldots, i_q$ for any $m < m_0$ and consider some $C_{m_0} \in \mathcal{J}_{m_0}$. Since $\{i_j(\mathbf{x})\}$ is an integrity basis for $I[\mathbf{x}]$, if follows that the polynomial $C_{m_0}(\mathbf{x}) \in \mathcal{J}_{m_0}[\mathbf{x}]$ can be expressed as a polynomial in the $i_j(\mathbf{x})$.

Suppose for example that $C_{m_0}(\mathbf{x}) = i_1(\mathbf{x}) i_2(\mathbf{x})$ where $i_1 \in I_{m_1}[x]$, $i_2 \in I_{m_2}[x]$, and $m_0 = m_1 + m_2$. Now consider the elements $C_{m_0}(X_j)$ and $i_1(X_j) i_2(X_j)$ in $\mathcal{U}$. We have $C_{m_0}(X_j) \in \mathcal{J}_{m_0}$ while in general

$$i_1(X_j) i_2(X_j) \subseteq \sum_{m=0}^{m_0} \oplus \mathcal{J}_m.$$

However, it is easy to see that the component of $i_1 i_2$ in $\mathcal{J}_{m_0}$ is just $C_{m_0}(X_j)$. Thus,

$$C_{m_0}(X_j) - i_1(X_j) i_2(X_j) = \sum_{m=0}^{m_0 - 1} C_m(X_j).$$

Since each $C_m(X_j)$ for $m < m_0$ can be expressed as a polynomial in the invariants $i_1, \ldots, i_q$, the induction step is complete. Our example easily extends to the general case. QED

In general $i_1(X_j), \ldots, i_q(X_j)$ is not an integrity basis for $\mathcal{J}$ but rather a subset $i_1', \ldots, i_q'$ is an integrity basis. This is because there may exist algebraic relations between $i_1(X_j), \ldots, i_q(X_j)$ in $\mathcal{J}$ which have no counterpart in $I(\mathbf{x})$. Such relations are consequences of the commutation relations of $\mathcal{G}$. Indeed, if $i_1(X_j)$ and $i_2(X_j)$ do not commute, then $i(X_j) = [i_1(X_j), i_2(X_j)]$ is also an invariant and the relation $i = i_1 i_2 - i_2 i_1$ is not obtainable from $I(\mathbf{x})$.

In conclusion: To find an integrity basis for $\mathcal{J}$ we first find an integrity basis $i_1, \ldots, i_q$ for $I(\mathbf{x})$. Then, forming all possible commutators $[i_s(X_j), i_p(X_j)]$, we determine a minimal subset of the i_k which are independent.

B. Generating function for the number of $O(3)$ invariants of arbitrary finite order in the enveloping algebra of $SU(3)$

In this paper we are concerned with the example $G = SU(3)$, $H = O(3)$.

Under the adjoint representation of $O(3)$ the eight-dimensional Lie algebra $SU(3)$ splits into a direct sum of the irreducible three- and five-dimensional representations of $O(3)$. The elements L_j, T_{ij}, $1 \leq i,j \leq 3$ form a basis for $SU(3)$ where the vector L_j transforms according to the three-dimensional representation D_1 and the symmetric traceless tensor T_{ij} transforms according to the five-dimensional representation D_2 of $O(3)$.

In more physical terms we can identify $L = \{L_i\}$ and $T = \{T_{ik}\}$ with the angular momentum and quadrupole moment operators, putting

$$\begin{aligned} L_j &= \epsilon_{jlk} x_l p_k, \\ T_{jk} &= \tfrac{1}{2}(p_j p_k + x_j x_k) - \tfrac{1}{6}(\vec{p}^2 + \vec{x}^2)\delta_{jk}, \end{aligned} \qquad (4)$$

where x_j are the coordinates of a particle and $p_j = -i\,\partial/\partial x_j$ its momentum. These operators satisfy the $SU(3)$ commutation relations

$$\begin{aligned} [L_j, L_k] &= i\epsilon_{jkl} L_l, \\ [L_j, T_{kl}] &= i\epsilon_{jkm} T_{lm} + i\epsilon_{jlm} T_{km}, \\ [T_{jk}, T_{lm}] &= \tfrac{1}{4} i(\delta_{jl}\epsilon_{kmn} + \delta_{jm}\epsilon_{kln} + \delta_{kl}\epsilon_{jmn} + \delta_{km}\epsilon_{jln}) L_n. \end{aligned} \qquad (5)$$

In the defining representation of $SU(3)$ these generators can be identified as follows:

$$\begin{aligned}
&L_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_2 = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \\
&L_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
&T_{11} = \frac{1}{6}\begin{pmatrix} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{pmatrix}, \quad T_{22} = \frac{1}{6}\begin{pmatrix} -1 & 0 & -3 \\ 0 & 2 & 0 \\ -3 & 0 & -1 \end{pmatrix}, \\
&T_{33} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
&T_{12} = \frac{i}{2}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_{23} = \frac{i}{2\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \\
&T_{31} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.
\end{aligned} \qquad (6)$$

By our theorem, to find an $O(3)$ integrity basis for the enveloping algebra of $SU(3)$ it is enough to find an integrity basis for the space of all polynomials in the eight indeterminants l_i, t_{jk}, $1 \leq i,j,\ k \leq 3$, where $t_{jk} = t_{kj}$ and $t_{jj} = 0$. Here the l_i transform under $O(3)$ according to D_1 and the t_{jk} according to D_2. In this case it is clear that the subspace P_{nm} of polynomials homogeneous of degree n in the l_i and degree m in the t_{jk} is invariant under the group action. Thus we can classify polynomial invariants $C^{(n,m)}$ in terms of their degrees of homogeneity n, m.

It is very easy to construct examples of polynomial invariants, e.g.,

$$\begin{aligned} &C^{(2,0)} = l_i l_i, \quad C^{(2,1)} = l_i t_{ij} l_j, \\ &C^{(0,2)} = t_{ij} t_{ij}, \quad C^{(0,3)} = t_{ij} t_{jk} t_{ki}, \\ &C^{(2,2)} = l_i t_{ij} t_{jk} l_k, \quad C^{(3,3)} = \epsilon_{abc} t_{bk} t_{cj} t_{jh} l_a l_k l_h. \end{aligned} \qquad (7)$$

The basic problem is to find all such independent invariants, or more specifically, to construct an integrity basis. We will show below that the above list of six invariants is in fact an integrity basis and thus solve our problem.

First of all it is useful to apply Lie's theory of invariants to this problem, e.g., Ref. 15. It follows easily from this theory that the action of the three-dimensional algebra $so(3)$ on eight-parameter functions $F(l_i, t_{jk})$ implies the existence of exactly five functionally independent invariants $h_a(l_i, t_{jk})$, $a=1,\ldots,5$. By this we mean that there exist five invariant functions analytic (but not necessarily polynomials) in the variables l_i, t_{jk} such that every other invariant is an analytic function of these five. Furthermore, no one of the h_a can be expressed as an analytic function of the remaining four.

By inspection one can show that the invariants $C^{(2,0)}$, $C^{(2,1)}, C^{(0,2)}, C^{(0,3)}, C^{(2,2)}$ are functionally independent, so that all other invariants must be analytic functions of these five. However, the remaining invariants would have to be expressible as *polynomials* in these five invariants for them to be an integrity basis. $C^{(3,3)}$ is not so expressible. Indeed a direct computation yields

$$\begin{aligned}[C^{(3,3)}]^2 &= C^{(2,0)}C^{(2,1)}C^{(0,3)}C^{(2,2)} + \tfrac{1}{2}C^{(2,0)}C^{(0,2)}[C^{(2,2)}]^2 \\ &\quad - \tfrac{1}{4}C^{(2,0)}[C^{(0,2)}]^2[C^{(2,1)}]^2 - \tfrac{1}{9}[C^{(2,0)}]^3[C^{(0,3)}]^2 \\ &\quad - \tfrac{1}{3}[C^{(2,0)}]^2C^{(0,2)}C^{(2,1)}C^{(0,3)} + \tfrac{1}{2}C^{(0,2)}[C^{(2,1)}]^2C^{(2,2)} \\ &\quad - \tfrac{1}{3}[C^{(2,1)}]^3C^{(0,3)} - [C^{(2,2)}]^3,\end{aligned} \tag{8}$$

i.e., $[C^{(3,3)}]^2$ is a polynomial in the first five invariants but $C^{(3,3)}$ is not.

To show explicitly that we have found an integrity basis we generalize a technique found in Ref. 14, p. 181, and Ref. 16, to derive a generating function for the number of invariants of rank (n,m). For this we recall that the irreducible representations of $O(3)$ can be denoted by D_j, $j=0,1,2,\cdots$, and that the character $\chi_j(\theta)$ of D_j corresponding to a rotation through the angle θ is

$$\chi_j(\theta) = \sum_{k=-j}^{j} \exp(ik\theta). \tag{9}$$

By choosing a weight basis it is straightforward to check that the character $\chi_{n,m}(\theta)$ of $O(3)$ acting on the subspace $P_{n,m}$ is

$$\chi_{n,m}(\theta) = \sum_{a,\ldots,h} \exp[i\theta(a-c+2d+e-g-2h)], \tag{10}$$

where the sum is taken over all nonnegative integers $a,\ldots,h$ such that $a+b+c=n$, $d+e+f+g+h=m$. It follows from this that

$$\begin{aligned}F[\exp(i\theta),P,D] &\equiv [(1-\exp(i\theta)P)(1-P)(1-\exp(-i\theta)P)(1-\exp(2i\theta)D) \\ &\quad \times(1-\exp(i\theta)D)(1-D)(1-\exp(-i\theta)D)(1-\exp(-2i\theta)D)]^{-1} \\ &= \sum_{n,m=0}^{\infty} \chi_{n,m}(\theta)P^nD^m,\end{aligned} \tag{11}$$

i.e., $F[\exp(i\theta),P,D]$ is a generating function for the character $\chi_{n,m}(\theta)$. Note that the number of invariants of degree (n,m) is just the multiplicity of the identity representation D_0 of $O(3)$ in $P_{n,m}$. Thus, using the orthogonality relations

$$\frac{1}{\pi}\int_0^{2\pi} \chi_n(\theta)\overline{\chi_m(\theta)}\sin^2\frac{\theta}{2}\,d\theta = \delta_{nm}, \tag{12}$$

we find

$$\frac{1}{\pi}\int_0^{2\pi} \sin^2\frac{\theta}{2}\,F(\exp(i\theta),P,D)\,d\theta = \sum_{n,m=0}^{\infty} N_{n,m}P^nD^m, \tag{13}$$

where the integer $N_{n,m}$ is the number of linearly independent $O(3)$ invariants of rank (n,m). Setting $\exp(i\theta)=\lambda$, we can regard the left-hand side as a contour integral about a unit circle in the complex λ plane. Evaluating the integral by residues and employing some tedious algebra, we finally obtain

$$\frac{1+P^3D^3}{(1-P^2)(1-D^2)(1-D^3)(1-P^2D^2)(1-P^2D)} = \sum_{n,m=0}^{\infty} N_{n,m}P^nD^m. \tag{14}$$

It is illuminating to compare this expression with our earlier results. Since $C^{(2,0)}$, $C^{(2,1)}$, $C^{(0,2)}$, $C^{(0,3)}$, and $C^{(2,2)}$ are functionally independent, we can construct invariants of the form $[C^{(2,0)}]^a[C^{(2,1)}]^b[C^{(0,2)}]^c[C^{(0,3)}]^d \times[C^{(2,2)}]^e$, where $a,\ldots,e$ run over the nonnegative integers and the set of all such invariants is linearly independent. If these were all possible invariants, then the generating function (14) would be

$$\frac{1}{(1-P^2)(1-D^2)(1-D^3)(1-P^2D)(1-P^2D^2)}. \tag{15}$$

However, the actual $N_{n,m}$ is in general larger than that predicted by (15) which shows that there are additional invariants. Indeed $N_{3,3}=1$, while it is impossible to construct a $(3,3)$ invariant out of $C^{(2,0)},\ldots,C^{(2,2)}$. Thus, there must exist a new $(3,3)$ invariant. This new invariant is clearly $C^{(3,3)}$. We can now obtain new invariants of the form $C^{(3,3)}[C^{(2,0)}]^a,\ldots,[C^{(2,2)}]^e$. This accounts for all terms in (14) and completely solves the problem of finding all $O(3)$ invariants. (It is not possible to obtain independent invariants by taking higher power of $C^{(3,3)}$ because $[C^{(3,3)}]^2$ can be expressed as a polynomial in $C^{(2,0)},\ldots,C^{(2,2)}$.)

C. The $O(3)$ invariants and the $SU(3)\supset O(3)$ reduction

It was shown above that there are at most six algebraicly independent $O(3)$ scalars in the enveloping algebra of $SU(3)$. They can easily be expressed in terms of the generators L_i and T_{ik} of Eqs. (4)–(6) and indeed they are given by Eq. (7) with l_i and t_{ij} replaced by the operators L_i and T_{ij}.

The two Casimir operators[17] $C^{(2)}$ and $C^{(3)}$ of $SU(3)$ are, of course, also $O(3)$ scalars and must be contained among those found. Indeed, it is easy to check that we have

$$\begin{aligned}C^{(2)} &= (\tfrac{3}{4})^2(L^2+2T^2) = (\tfrac{3}{4})^2(L_iL_i+2T_{ik}T_{ik}), \\ \text{const } C^{(3)} &= LTL - \tfrac{4}{3}TTT = L_iT_{ik}L_k - \tfrac{4}{3}T_{ik}T_{kl}T_{li}.\end{aligned} \tag{16}$$

It is also easy to verify that the operator

$$X^{(6)} = \epsilon_{abc}T_{bd}T_{ce}T_{ef}L_aL_dL_f$$

can be expressed in terms of the commutator of the two operators

$$X^{(3)} = L_a T_{ab} L_b \text{ and } X^{(4)} = L_a T_{ab} T_{bc} L_c \tag{17}$$

and lower order terms.

In addition to the angular momentum L^2 and the two Casimir operators $C^{(2)}$ and $C^{(3)}$ we thus only have two new independent $O(3)$ invariants $X^{(3)}$ and $X^{(4)}$ [see (17)].

Note that the scalars of this section do not quite coincide with those listed in Eq. (7) because they are not all symmetrized. However, they do agree in the highest order terms and they provide an alternative integrity basis which is computationally easier to deal with.

Let us note here that the operator $X^{(3)}$ is equivalent to an operator used in a similar context by Bargmann and Moshinsky.[6]

Returning to the problem of representations in the $SU(3) \supset O(2)$ basis, we see that the basis functions of irreducible representations of $SU(3)$ can be chosen to be eigenstates of the operators $C^{(2)}$, $C^{(3)}$, L^2, L_3, and X, where X is in principle an arbitrary function of the operators (17).

If we make the natural restriction that X be an operator of a definite order in the enveloping algebra of $SU(3)$, we find that only one third-order and one fourth-order are available. Some physical implications of this fact will be discussed in the final section.

In conclusion, the operators L^2, $C^{(2)}$, $C^{(3)}$, $X^{(3)}$, and $X^{(4)}$ form an integrity basis for the $O(3)$ scalars in the enveloping algebra of $SU(3)$.

3. SPECTRUM OF THE $O(3)$—SCALAR OPERATORS

The purpose of this section is to calculate the spectrum of the third and fourth order operators $X^{(3)}$ and $X^{(4)}$ and to demonstrate some of their general properties. Indeed, for any practical use of the present state labeling method it is essential to know the spectrum of the operators for all $SU(3)$ representations likely to appear in applications.

The $SU(3) \supset O(3)$ case is only the simplest of many group—subgroup pairs of physical interest where some labels are missing. Higher order operators can resolve these labeling problems not only in principle, but in our opinion are the most practical way to approach the problem. It is therefore natural to perform the (computer) calculations of the spectra in a way which is not limited to the $SU(3) \supset O(3)$ case but can readily be extended to cases like $SU(4) \supset SU(2) \times SU(2)$, $G_2 \supset O(3)$, and others. The basis we use for deriving the secular equations is that of Gel'fand and Tseitlin,[2] with $U(3)$ generators E_{ik} satisfying the commutation relations

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}, \tag{18}$$

where δ_{jk} is the Kronecker delta. An explicit form of the matrix elements of the $U(3)$ generators can be found in the second example of Ref. 18 [Eq. (22)]; correspondence between the notations in the present paper and Ref. 18 is established by putting $E_{ik} \equiv C_i^k$ and $m_{ik} \equiv h_{ik}$, where m_{ik} are the elements of each pattern-basis vector.

It is convenient to replace the generators L_1, L_2, and L_3 of (6) by equivalent ones:

$$L_1 = E_{12} + E_{23}, \quad L_0 = E_{11} - E_{33}, \quad L_{-1} = E_{21} + E_{32}, \tag{19}$$

whose commutations relations

$$[L_1, L_{-1}] = L_0, \quad [L_0, L_1] = L_1, \quad [L_0, L_{-1}] = -L_{-1} \tag{20}$$

follow from (18). The generators (19) can be realized as 3×3 matrices:

$$L_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad L_{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{21}$$

With the choice (19) the five components of the operator T_{ik} then can be taken as

$$\begin{aligned} &T_2 = E_{13}, \quad T_1 = E_{12} - E_{23}, \quad T_0 = E_{11} - 2E_{22} + E_{33}, \\ &T_{-2} = E_{31}, \quad T_{-1} = E_{21} - E_{32}. \end{aligned} \tag{22}$$

Realized as 3×3 matrices, these are

$$\begin{aligned} &T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ &T_{-2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}. \end{aligned} \tag{23}$$

Using (18), one readily verifies that T_i indeed is the rank two $O(3)$-tensor operator:

$$\begin{aligned} &[T_2, L_1] = 0, \quad [T_2, L_0] = -2T_2, \quad [T_2, L_{-1}] = T_1, \\ &[T_1, L_1] = 2T_2, \quad [T_1, L_0] = -T_1, \quad [T_1, L_{-1}] = T_0, \\ &[T_0, L_1] = 3T_1, \quad [T_0, L_0] = 0, \quad [T_0, L_{-1}] = -3T_{-1}, \text{ etc.} \end{aligned} \tag{24}$$

The second order operators $C^{(2)}$, L^2, and T^2 are then

$$\begin{aligned} &L^2 = L_1 L_{-1} + L_{-1} L_1 + L_0^2, \\ &T^2 = T_2 T_{-2} + T_{-2} T_2 + \tfrac{1}{2}(T_1 T_{-1} + T_{-1} T_1) + \tfrac{1}{6} T_0^2, \\ &C^{(2)} = \sum_{i,k=1}^{3} E_{ik} E_{ki} = (\tfrac{3}{4})^2 (L^2 + 2T^2). \end{aligned} \tag{25}$$

The labeling operators then are

$$\begin{aligned} X^{(3)} = {} & 3(L_1 T_{-2} L_1 + L_{-1} T_2 L_{-1}) \\ & + \tfrac{3}{2}(L_{-1} T_1 L_0 + L_0 T_1 L_{-1} + L_1 T_{-1} L_0 + L_0 T_{-1} L_1) \\ & - \tfrac{1}{2}(L_1 T_0 L_{-1} + L_{-1} T_0 L_1) + L_0 T_0 L_0 \end{aligned} \tag{26}$$

and

$$\begin{aligned} X^{(4)} = {} & 2T_0 L_0 L_0 T_0 + (-T_0 L_1 L_{-1} T_0 + \tfrac{3}{2} T_0 L_1 L_0 T_{-1} + \tfrac{3}{2} T_0 L_0 L_1 T_{-1} \\ & - 6T_0 L_1 L_1 T_{-1} + 9T_1 L_{-1} L_{-1} T_1 + \tfrac{3}{2} T_1 L_{-1} L_0 T_0 \\ & + \tfrac{3}{2} T_1 L_0 L_{-1} T_0 - \tfrac{3}{2} T_1 L_{-1} L_1 T_{-1} + 3T_1 L_0 L_0 T_{-1} \\ & - \tfrac{3}{2} T_1 L_1 L_{-1} T_{-1} + 9T_1 L_1 L_0 T_{-2} + 9T_1 L_0 L_1 T_{-2} \\ & - 6T_2 L_{-1} L_{-1} T_0 + 9T_2 L_{-1} L_0 T_{-1} + 9T_2 L_0 L_{-1} T_{-1} \\ & + 6T_2 L_{-1} L_1 T_{-2} - 12 T_2 L_0 L_0 T_{-2} + 6T_2 L_1 L_{-1} T_{-2}) \\ & + (\cdots), \end{aligned} \tag{26'}$$

where $(\cdots)$ stands for terms with signs of indices opposite to those in the first bracket. Here $X^{(3)}$ and $X^{(4)}$ are normalized so that their eigenvalues are integers whenever possible. The operators $X^{(4)}$ in (17) and (26′) differ by $O(3)$-scalars of order lower than four. By a straightforward calculation one verifies that $X^{(i)}$, indeed, are $O(3)$ scalars:

$$[X^{(i)}, L_1] = [X^{(i)}, L_0] = [X^{(i)}, L_{-1}] = 0, \quad i = 3 \text{ or } 4. \tag{27}$$

An irreducible representation of $U(3)$ is denoted by integers (m_{13}, m_{23}, m_{33}) such that $m_{13} \geq m_{23} \geq m_{33}$. If $m_{33} = 0$, a $U(3)$ representation reduces to that of $SU(3)$ with $p = m_{13} - m_{23}$ and $q = m_{23}$. The patterns

$$\left|\begin{matrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \\ & & m_{11} & & \end{matrix}\right\rangle,$$

$$m_{i,k+1} \geq m_{ik} \geq m_{i+1,k}, \quad m_{ik} \text{ integers}, \tag{28}$$

transformed by the generators E_{ik} according to (22) of Ref. 18, form an orthonormal basis in a space in which an irreducible unitary representation of the group $U(3)$ acts. If $m_{33} = 0$, the space is irreducible with respect to $SU(3)$. Since m_{13}, m_{23}, and m_{33} are fixed throughout an irreducible representation of $U(3)$, we shall omit them when writing the patterns.

The $C^{(2)}$ and $C^{(3)}$ operators are[17] diagonal in the basis (28) because they are the Casimir operators of $U(3)$ [and $SU(3)$]. Since E_{11}, E_{22}, and E_{33} are diagonal in (28) too, L_0 and T_0 are also diagonal. One has, in particular

$$L_0\left|\begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix}\right\rangle = (m_{11} + m_{12} + m_{22} - m_{13} - m_{23} - m_{33})\left|\begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix}\right\rangle \tag{29a}$$

and

$$T_0\left|\begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix}\right\rangle = (m_{13} + m_{23} + m_{33} + 3m_{11} - 3m_{12} - 3m_{22})\left|\begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix}\right\rangle. \tag{29b}$$

An arbitrary $SU(3)$ pattern for a given representation is a linear combination of $O(3)$ states $|JM\rangle$:

$$\left|\begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix}\right\rangle = \sum_{JK} a_J |JMK\rangle. \tag{30}$$

Here a_J are some coefficients,

$$M = m_{11} + m_{12} + m_{22} - m_{13} - m_{23} - m_{33} \tag{31}$$

is the eigenvalue of L_0, J denotes an $O(3)$-irreducible subspace, and K are the eigenvalues of $X^{(i)}$ which we want to find. The values of J for any $U(3)$ representation are well known.[19,20] The summation in (30) extends over all $J \geq M$ which occur in the $SU(3)$ space labeled by m_{13} and m_{23} ($m_{33} = 0$). There is no summation over M in (30) because both the Gel'fand—Tseitlin and $|JMK\rangle$ states are eigenvectors of the $O(2)$ generator L_0. When $X^{(i)}$ acts on both sides of (30), one gets

$$\sum_{m_{11}+m_{12}+m_{22}=M+m_{13}+m_{23}} x_m(m_{12}, m_{22}, m_{11})\left|\begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix}\right\rangle = \sum_{J,K} a_J K |JMK\rangle. \tag{32}$$

The coefficients x_M are matrix elements of $X^{(i)}$ between the patterns with the same value (31) of M. They are calculated using (20), (22), (26), and (22) of Ref. 18. For example, the diagonal matrix element of $X^{(3)}$ is

$$\begin{aligned}\left\langle\begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix}\right| X^{(3)} \left|\begin{matrix} m_{12} & & m_{22} \\ & m_{11} & \end{matrix}\right\rangle = \frac{3}{m_{12} - m_{22} + 1} \Bigg(& \frac{(m_{13} - m_{12})(m_{12} - m_{23} + 1)(m_{12} - m_{33} + 2)(m_{12} - m_{11} + 1)[2(m_{11} - m_{22}) - 2M - N/3]}{(m_{12} - m_{22} + 2)} \\ & + \frac{(m_{13} - m_{22} + 1)(m_{23} - m_{22})(m_{22} - m_{33} + 1)(m_{11} - m_{22})[2(m_{11} - m_{12} - 1) - 2M - N/3]}{(m_{12} - m_{22})}\Bigg) \\ & + 3(m_{12} - m_{11})(m_{11} - m_{22} + 1)\Bigg(2M - \frac{N}{3} + 2\frac{(m_{13} - m_{12} + 1)(m_{12} - m_{23})(m_{12} - m_{33} + 1)}{(m_{12} - m_{22} + 1)(m_{12} - m_{22})} \\ & - 2\frac{(m_{13} - m_{22} + 2)(m_{23} - m_{22} + 1)(m_{22} - m_{33})}{(m_{12} - m_{22} + 1)(m_{12} - m_{22} + 2)}\Bigg) + \tfrac{1}{2}N(M + 1)(2M + 3), \end{aligned} \tag{33}$$

where M is given by (31) and N is the eigenvalue of T_0:

$$N = 3(m_{11} - m_{12} - m_{22}) + m_{13} + m_{23} + m_{33}. \tag{34}$$

Substituting (30) into the left side of (32), and comparing the coefficients of the linearly independent vectors $|JMK\rangle$, we arrive at the secular equation

$$|x_M(m_{11}, m_{12}, m_{22}) - K| = 0. \tag{35}$$

The roots $K_1, K_2, \cdots$ of (35) are real because $X^{(i)}$ is Hermitian. The value of M in (30) is a fixed parameter. Hence we have secular equation (35) for every value of M which occurs in the $U(3)$ representation (m_{13}, m_{23}, m_{33}). [For $SU(3)$ we still have $m_{33} = 0$.] Equation (35) is of the first order when M equals its highest (smallest) value within the inequalities (28), i.e., $M = m_{13} - m_{33}$ ($M = m_{33} - m_{13}$). Then indeed, there is only one pattern, namely $m_{11} = m_{12} = m_{13}$, $m_{22} = m_{23}$ ($m_{11} = m_{22} = m_{33}$, $m_{12} = m_{23}$). Consequently, (30) has the form

$$\left|\begin{matrix} m_{13} & & m_{23} \\ & m_{13} & \end{matrix}\right. = |m_{13} - m_{33}, m_{13} - m_{33}, K\rangle. \tag{36}$$

The order of Eq. (35) increases, in general, when the absolute value $|M|$ diminishes, and for $M = 0$, (35) is of the highest order. The order of (35) in this case equals to the number of different patterns (28) with $M = 0$, or, what is the same, it equals the number of $O(3)$ representations contained in (m_{13}, m_{23}, m_{33}).

From the property

$$X^{(i)}|JMK\rangle = K|JMK\rangle \quad \text{for } M = J, J - 1, \ldots, -J,$$

of $X^{(i)}$ it follows that the eigenvalue will occur as a root of the secular equation (35) for any M. Similarly, an eigenvalue, say K', calculated from (35) with $M = M'$, will be a root of every secular equation with $|M| \leq |M'|$. One has thus two alternative ways for computating the spectrum of $X^{(i)}$ for a given representation (m_{13}, m_{23}, m_{33}). First is solution of (high order) equation (35) for $M = 0$ in order to get all the eigenvalues K at once. The second way is the solving of equation (35) first for $M = m_{13} - m_{33}$, then for $M = m_{13} - m_{33} - 1$, $M = m_{13}$

$-m_{33}-2$, and so on. In this manner the order of the secular equation for a given M is drastically reduced because most of its roots are known from solving the secular equation for $M+1$. To illustrate this point, let us notice that, e.g., for the $SU(3)$ representation $(12,6,0)$ of dimension 343, the order of (35) at $M=0$ equals 25. Proceeding the second way, one would have to solve 4, 4, 3, and 1 secular equations of orders 1, 2, 3, and 4, respectively.

The tables contain the eigenvalues of $X^{(3)}$ and $X^{(4)}$ calculated by a computer for the lower $SU(3)$ representations. For pairs of contragredient representations [i.e., representations $(m_{13}, m_{23}, 0)$ and $(m_{13}, m_{13}-m_{23}, 0)$] the $O(3)$ branching rules coincide and the eigenvalues of $X^{(3)}$ differ by a sign, and those of $X^{(4)}$ are the same. Therefore, the tables contain only one representation of each pair. The computer time needed for construction of the tables was negligible. Thus in order to verify the eigenvalues we have obtained, the secular equation (35) was solved for all $M \geqslant 0$ for both $X^{(3)}$ and $X^{(4)}$.

The numerical results presented in the Table I for the $SU(3)$ representations

$$(k_1, k_2, 0) = (m_{12}-m_{33}, m_{23}-m_{33}, 0) \tag{37}$$

[note that $k_1 \geqslant k_2 \geqslant 0$ and k_1 and k_2 are the lengths of the first and second row in Young patterns for $SU(3)$] were obtained using the above algorithm, starting from Gel'fand—Tseitlin states. For the particular case considered in this article, i.e., the $SU(3) \supset O(3) \supset O(2)$ group—subgroup chain a different method could also be used for calculating the eigenvalues K. Indeed, Bargmann and Moshinsky[6] and Elliott[5] have calculated the matrix elements of the operator $X^{(3)} = LTL$ in certain nonorthogonal bases. All we have to do is take these matrices and diagonalize them. For analytic calculations (as opposed to computer ones) this procedure is simpler.

Since in many applications it is convenient to have explicit formulas for the eigenvalues K, rather than only numeric tables, we present below expressions for K in special cases, when the $O(3)$ representation J occurs in the $SU(3)$ representation (k_1, k_2) once (hence K is uniquely determined as a solution of a linear equation) or twice (then K is the solution of a quadratic equation).

To do this, we choose to make use of the Bargmann—Moshinsky basis vectors $P_{k_1 k_2 J q}$ in which we have

$$X^{(3)} P_{k_1 k_2 J q} = -3 \sum_{q'} \beta_{q' q} P_{k_1 k_2 J_{q'}} \tag{38}$$

[see formula (62) of the second of Refs. 6; the factor (-3) is due to a difference in the normalization of our $X^{(3)}$ and their operator Ω]. The matrix elements $\beta_{q'q}$ are given by formulas (66) and (67) of Ref. 6 and restrictions on the region of summation on (38) are given by their formula (59).

All we have to do is restrict ourselves to cases when only one or two values of the label q exist (no degeneracy or twofold degeneracy). If there is no degeneracy, then $K = -3\beta_{qq}$; if there is a degeneracy, then we obtain the eigenvalues K by diagonalizing the matrix $\beta_{q'q}$.

By inspecting the Bargmann—Moshinsky formulas, we see immediately that the representation J is contained in representation (k_1, k_2) at most once in any of the following cases:

$$J = 0, 1, k_1-1 \text{ or } k_1 \quad (k_1 \text{ and } k_2 \text{ arbitrary}), \tag{39}$$

or

$$k_2 = 0, 1, k_1-1, k_1 \quad (J \text{ arbitrary}). \tag{40}$$

The degeneracy is at most twofold if

$$J = 2, 3, k_1-2 \text{ or } k_1-3 \quad (k_1 \text{ and } k_2 \text{ arbitrary}), \tag{41}$$

or

$$k_2 = 2, 3, k_1-2 \text{ or } k_1-3 \quad (J \text{ arbitrary}). \tag{42}$$

Proceeding as described, we obtain the following expressions for the eigenvalues K in nondegenerate cases.

$J=0$: We have

$$K = 0 \tag{43}$$

for k_1 and k_2 both even [the representation $J=0$ is not contained in (k_1, k_2) otherwise].

$J=1$: We obtain

$$\begin{aligned} K &= -k_1 + 2k_2 && \text{for } k_1 \text{ even, } k_2 \text{ odd} \\ &= 2k_1 - k_2 + 3 && \text{for } k_1 \text{ odd, } k_2 \text{ even} \\ &= -(k_1+k_2+3) && \text{for } k_1 \text{ odd, } k_2 \text{ odd} \end{aligned} \tag{44}$$

($J=1$ is not present for k_1 even, k_2 even).

$J=k_1$: We have

$$K = \tfrac{1}{2}(k_1+1)(2k_1+3)(k_1-2k_2). \tag{45}$$

$J=k_1-1$: We have

$$K = \tfrac{1}{2}(k_1+3)(2k_1+1)(k_1-2k_2). \tag{46}$$

$k_2=0$: We have

$$K = \tfrac{1}{2}(2k_1+3)J(J+1) \quad \text{for } k_1-J \text{ even} \tag{47}$$

and J is not contained in $(k_1, 0)$ for k_1-J odd.

$k_2=1$: We have

$$\begin{aligned} K &= -3(k_1+1) + (k_1-\tfrac{1}{2})J(J+1) && \text{for } k_1-J \text{ even} \\ &= -3(k_1+1) + (k_1+\tfrac{5}{2})J(J+1) && \text{for } k_1-J \text{ odd.} \end{aligned} \tag{48}$$

$k_2=k_1$ and $k_2=k_1-1$: These are contragradient to $k_2=0$ and $k_2=1$; hence formulas (47) and (48) apply with reversed signs.

In the cases when at most a twofold degeneracy can occur, we obtain:

$J=2$: We have

$$\begin{aligned} K &= \pm 3[(2k_1+3)^2 - 4k_2(k_1-k_2)]^{1/2} \\ & && \text{for } k_1 \text{ even, } k_2 \text{ even, } 2 \leqslant k_2 \leqslant k_1-2, \\ &= -3(2k_1+3) && \text{for } k_1 \text{ even, } k_2 \text{ even, } k_1=k_2, \\ &= 3(2k_1+3) && \text{for } k_1 \text{ even, } k_2 \text{ even, } k_2=0, \\ &= 3(k_1-2k_2) && \text{for } k_1 \text{ even, } k_2 \text{ odd} \\ &= -3(2k_1-k_2+3) && \text{for } k_1 \text{ odd, } k_2 \text{ even} \\ &= 3(k_1+k_2+3) && \text{for } k_1 \text{ odd, } k_2 \text{ odd.} \end{aligned} \tag{49}$$

$J=3$: We have

TABLE I. Eigenvalues K of the third order operator $X^{(3)} = L\,T\,L$. The first column gives the representations of $SU(3)$, the rows give all possible values of the $O(3)$ label J and of K within the corresponding representation of $SU(3)$.

(0,0,0)	J	0											
	K	0											
(1,0,0)	J	1											
	K	5											
(2,0,0)	J	2	0										
	K	21	0										
(2,1,0)	J	2	1										
	K	0	0										
(3,0,0)	J	3	1										
	K	54	9										
(3,1,0)	J	3	2	1									
	K	18	21	−7									
(4,0,0)	J	4	2	0									
	K	110	33	0									
(4,1,0)	J	4	3	2	1								
	K	55	63	6	−2								
(4,2,0)	J	4	3	2	2	0							
	K	0	0	30.741	−30.741	0							
(5,0,0)	J	5	3	1									
	K	195	78	13									
(5,1,0)	J	5	4	3	2	1							
	K	117	132	36	27	−9							
(5,2,0)	J	5	4	3	3	2	1						
	K	39	44	79.573	−13.573	−33	11						
(6,0,0)	J	6	4	2	0								
	K	315	150	45	0								
(6 1,0)	J	6	5	4	3	2	1						
	K	210	234	89	81	12	−4						
(6,2,0)	J	6	5	4	4	3	2	2	0				
	K	105	117	160	22	0	41.677	−41.677	0				
(6,3,0)	J	6	5	4	4	3	3	2	1				
	K	0	0	58.864	−58.864	92.223	−92.223	0	0				
(7,0,0)	J	7	5	3	1								
	K	476	255	102	17								
(7,1,0)	J	7	6	5	4	3	2	1					
	K	340	375	171	166	54	33	−11					
(7,2,0)	J	7	6	5	5	4	3	3	2	1			
	K	204	225	277.903	82.097	60	100.723	−10.723	−45	15			
(7,3,0)	J	7	6	5	5	4	4	3	3	2	1		
	K	68	75	137.827	−17.827	186.231	−56.231	−108	30	39	−13		
(8,0,0)	J	8	6	4	2	0							
	K	684	399	190	57	0							
(8,1,0)	J	8	7	6	5	4	3	2	1				
	K	513	561	288	288	123	99	18	−6				
(8,2,0)	J	8	7	6	6	5	4	4	3	2	2	0	
	K	342	374	439.120	172.880	153	195.426	42.574	0	53.075	−53.075	0	
(8,3,0)	J	8	7	6	6	5	5	4	4	3	3	2	1
	K	171	187	256.966	49.034	321.259	8.741	87.827	−67.827	109.763	−121.763	6	−2
(8,4,0)	J	8	7	6	6	5	5	4	4	4	3	2	2
	K	0	0	−92.223	92.223	148.704	−148.704	214.075	−214.075	0	0	51.701	−51.701
	J	0											
	K	0											
(9,0,0)	J	9	7	5	3	1							
	K	945	588	315	126	21							
(9,1,0)	J	9	8	7	6	5	4	3	2	1			
	K	735	798	446	453	225	200	72	39	−13			
(9,2,0)	J	9	8	7	7	6	5	5	4	3	3	2	1
	K	525	570	649.542	300.458	285	331.966	124.034	76	122.795	−8.795	−57	19

TABLE I. (*continued*)

(9,3,0)	*J*	9	8	7	7	6	6	5	5	4	4	3	3
	K	315	342	422.117	147.883	503.385	108.615	182.223	−2.223	219.031	−69.031	−137.223	47.223
	J	2	1										
	K	45	−15										
(9,4,0)	*J*	9	8	7	7	6	6	5	5	5	4	4	3
	K	105	114	208.882	−18.882	284.014	−80.014	365.586	−167.567	56.981	−241.259	71.259	127.426
	J	3	2	1									
	K	−25.426	−51	17									
(10,0,0)	*J*	10	8	6	4	2	0						
	K	1265	828	483	230	69	0						
(10,1,0)	*J*	10	9	8	7	6	5	4	3	2	1		
	K	1012	1092	651	667	366	342	157	117	24	−8		
(10,2,0)	*J*	10	9	8	8	7	6	6	5	4	4	3	2
	K	759	819	915.101	470.899	462	239.837	516.163	189	231.906	62.094	0	64.692
	J	2	0										
	K	−64.692	0										
(10,3,0)	*J*	10	9	8	8	7	7	6	6	5	5	4	4
	K	506	546	639	285	738.624	249.376	320.882	93.118	373.216	16.784	119	−79
	J	3	3	2	1								
	K	−151.460	127.460	12	−4								
(10,4,0)	*J*	10	9	8	8	7	7	6	6	6	5	5	4
	K	253	273	374.279	87.721	468.452	25.548	568.366	149.892	−91.257	182.014	−182.014	247.375
	J	4	4	3	2	2	0						
	K	−269.112	21.737	0	−62.426	62.426	0						
(10,5,0)	*J*	10	9	8	8	7	7	6	6	6	5	5	5
	K	0	0	130.111	−130.111	213.169	−213.169	315.728	−315.728	0	409.805	−409.805	0
	J	4	4	3	3	2	1						
	K	92.223	−92.223	138.942	−138.942	0	0						
(11,0,0)	*J*	11	9	7	5	3	1						
	K	1650	1125	700	375	150	25						
(11,1,0)	*J*	11	10	9	8	7	6	5	4	3	2	1	
	K	1350	1449	909	936	552	531	279	234	90	45	−15	
(11,2,0)	*J*	11	10	9	9	8	7	7	6	5	5	4	3
	K	1050	1127	1241.755	690.245	690	753.821	396.179	345	387	165	92	145.426
	J	3	2	1									
	K	−7.426	−69	23									
(11,3,0)	*J*	11	10	9	9	8	8	7	7	6	6	5	5
	K	750	805	913.394	466.606	1032.968	437.032	223.721	510.279	578.506	141.494	230.023	9.977
	J	4	4	3	3	2	1						
	K	252.168	−82.168	−166.763	64.763	51	−17						
(11,4,0)	*J*	11	10	9	9	8	8	7	7	7	6	6	5
	K	450	483	594.225	233.775	708.135	173.865	828.332	287.538	18.131	339.169	−87.169	417.241
	J	5	5	4	4	3	3	2	1				
	K	−200.262	98.020	−296.648	86.648	147.906	−21.906	−63	21				
(11,5,0)	*J*	11	10	9	9	8	8	7	7	7	6	6	6
	K	150	161	292.172	−16.172	397.298	−103.298	517.824	−231.787	91.962	633.480	−345.871	111.391
	J	5	5	5	4	4	3	3	2	1			
	K	−454.366	205.257	−35.892	273.216	−83.216	−155.636	41.636	57	−19			
(12,0,0)	*J*	12	10	8	6	4	2	0					
	K	2106	1485	972	567	270	81	0					
(12,1,0)	*J*	12	11	10	9	8	7	6	5	4	3	2	1
	K	1755	1875	1226	1266	789	773	444	396	191	135	30	−10
(12,2,0)	*J*	12	11	10	10	9	8	8	7	6	6	5	4
	K	1404	1500	1635.477	964.523	975	599.201	1050.799	550	305.969	594.031	225	269.154
	J	4	3	2	2	0							
	K	80.846	0	76.426	−76.426	0							
(12,3,0)	*J*	12	11	10	10	9	9	8	8	7	7	6	6
	K	1053	1125	1251.147	698.853	1392.403	677.597	756.225	395.775	840.970	311.030	388.597	133.403
	J	5	5	4	4	3	3	2	1				
	K	425.641	24.359	151.231	−91.231	145.245	−181.245	18	−6				

TABLE I. (*continued*)

(12, 4, 0)	J	12	11	10	10	9	9	8	8	8	7	7	6
	K	702	750	874.359	425.641	1009.088	370.912	1151.439	477.878	164.683	549.298	48.702	642.696
	J	6	6	5	5	4	4	4	3	2	2	0	
	K	212.121	−95.817	216.187	−216.187	−324.526	281.695	42.831	0	73.546	−73.546	0	
(12, 5, 0)	J	12	11	10	10	9	9	8	8	8	7	7	7
	K	351	375	511.507	138.493	638.709	51.291	779.553	227.926	−110.479	918.293	270.150	−243.443
	J	6	6	6	5	5	5	4	4	3	3	2	1
	K	−377.824	371.787	48.038	−498.902	460.440	8.463	120.023	−100.023	−168.361	156.361	6	−2
(12, 6, 0)	J	12	11	10	10	9	9	8	8	8	7	7	7
	K	0	0	−172.049	172.049	−284.747	284.747	429.367	−429.367	−0	566.960	−566.960	0
	J	6	6	6	6	5	5	4	4	4	3	2	2
	K	−698.844	698.844	138.430	−138.430	213.169	−213.169	−301.257	301.257	0	0	72.560	−72.560
	J	0											
	K	0											

TABLE II. Eigenvalues K of the fourth order operator $X^{(4)} = TLLT$. The first column gives the representations of $SU(3)$, the rows give all possible values of the O(3) label J and of K within the corresponding representation of $SU(3)$.

(0, 0, 0)	J	0								
	K	0								
(1, 0, 0)	J	1								
	K	−35								
(2, 0, 0)	J	2	0							
	K	63	−840							
(2, 1, 0)	J	2	1							
	K	63	−315							
(3, 0, 0)	J	3	1							
	K	342	−1323							
(3, 1, 0)	J	3	2	1						
	K	222	−105	−1043						
(4, 0, 0)	J	4	2	0						
	K	898	−1881	−2352						
(4, 1, 0)	J	4	3	2	1					
	K	490	438	−1617	−1547					
(4, 2, 0)	J	4	3	2	2	0				
	K	354	270	−297	−1449	−2016				
(5, 0, 0)	J	5	3	1						
	K	1875	−2562	−3347						
(5, 1 0)	J	5	4	3	2	1				
	K	963	1458	−2466	−1881	−2691				
(5, 2, 0)	J	5	4	3	3	2	1			
	K	507	922	−2307.729	543.729	−1185	−2699			
(6, 0, 0)	J	6	4	2	0					
	K	3465	−3366	−4689	−4536					
(6, 1, 0)	J	6	5	4	3	2	1			
	K	1785	3147	−3638	−2154	−4065	−3419			
(6, 2, 0)	J	6	5	4	4	3	2	2	0	
	K	777	1995	−3678.851	2114.851	−1578	−3825	−1617	−3864	
(6, 3, 0)	J	6	5	4	4	3	3	2	1	
	K	441	1611	−3654	1530	−954	774	−3825	−3051	
(7, 0, 0)	J	7	5	3	1					
	K	5908	−4245	−6522	−6107					
(7, 1, 0)	J	7	6	5	4	3	2	1		
	K	3148	5745	−5133	−2318	−6114	−4425	−4979		
(7, 2, 0)	J	7	6	5	5	4	3	3	2	1
	K	1308	3681	−5596.600	4642.600	−2094	−5836.759	−1543.241	−2745	−5091

$$K=0 \quad \text{for } k_1 \text{ even, } k_2 \text{ even, } 2\le k_2\le k_1-2,$$
$$=-3\{k_1-2k_2\pm[(k_1-2k_2)^2+15(k_1+1)(k_1+3)]^{1/2}\}$$
$$\text{for } k_1 \text{ even, } k_2 \text{ odd, } 3\le k_2\le k_1-3,$$
$$=-9(k_1+3) \quad \text{for } k_1 \text{ even, } k_2 \text{ odd, } k_2=k_1-1,$$
$$=9(k_1+3) \quad \text{for } k_1 \text{ even, } k_2 \text{ odd, } k_2=1,$$
$$=3\{2k_1-k_2+3\pm[(2k_1-k_2+3)^2+15k_2(k_2+2)]^{1/2}\}$$
$$\text{for } k_1 \text{ odd, } k_2 \text{ even, } 2\le k_2\le k_1-3,$$
$$=-9(k_1-1) \quad \text{for } k_1 \text{ odd, } k_2 \text{ even, } k_2=k_1-1,$$
$$=6(2k_1+3) \quad \text{for } k_1 \text{ odd, } k_2 \text{ even, } k_2=0,$$
$$=-3\{k_1+k_2+3\pm[(k_1+k_2+3)^2+15(k_1-k_2)(k_1-k_2+2)]^{1/2}$$
$$\text{for } k_1 \text{ odd, } k_2 \text{ odd, } 3\le k_2\le k_1-2,$$
$$=-6(2k_1+3) \quad \text{for } k_1 \text{ odd, } k_2 \text{ odd, } k_1=k_2,$$
$$=9(k_1-1) \quad \text{for } k_1 \text{ odd, } k_2 \text{ odd, } k_2=1. \tag{50}$$

$J=k_1-2$: We have

$$K=-\tfrac{1}{2}\{(2k_1+1)(k_1+1)(2k_2-k_1)$$
$$\pm 6[-4k_1^2k_2(k_1-k_2)+k_1^4+2k_1^3-k_1^2-2k_1-1]^{1/2}\}, \tag{51}$$

valid for $2\le k_2\le k_1-2$ [if k_2 is outside these bounds, there is no degeneracy and we can use Eqs. (47) and (48)].

$J=k_1-3$: We have

$$K=-\tfrac{1}{2}\{(2k_1+1)(k_1+3)(2k_2-k_1)$$
$$\pm 6[-4k_1^2k_2(k_1-k_2)+k_1^4+6k_1^3-9k_1^2-6k_1+9]^{1/2}\}$$
$$\text{for } 3\le k_2\le k_1-3$$
$$=\pm\tfrac{1}{2}(2k_1+1)(k_1+1)(k_1-6) \quad \text{for } k_2=2 \text{ or } k_1-2. \tag{52}$$

For $k_2=0,1,k_1-1$ or k_1, see (47) and (48).

$k_2=2$: We have

$$K=\tfrac{1}{2}\{(2k_1+1)(J-2)(J+3)$$
$$\pm 6[J(J-1)(J+1)(J+2)+(2k_1+1)^2]^{1/2}\} \quad \text{for } k_1-J \text{ even}$$
$$=\tfrac{1}{2}(2k_2+1)[J(J+1)-12] \quad \text{for } k_1-J \text{ odd} \tag{53}$$

[the first formula holds for $2\le J\le k_1-2$; otherwise there is no degeneracy—see (43)—(46)].

$k_2=3$: We have

$$K=\tfrac{1}{6}\{30k_1-(2k_1-3)J(J+1)$$
$$\pm 6[16k_1^2-4k_1J(J+1)+J^4+(J-3)(J-1)(2J+3)]^{1/2}\}$$
$$\text{for } k_1-J \text{ even, } 3\le J\le k_1-3, \tag{54}$$
$$K=\tfrac{1}{6}\{30k_1-(2k_1+3)J(J+1)$$
$$\pm 6[16k_1^2+4k_1J(J+1)+J^4+(J-3)(J-1)(2J+3)]^{1/2}\}$$
$$\text{for } k_1-J \text{ odd, } 3\le J\le k_1-3.$$

For $J\le 2$ or $J\ge k_1-2$ see earlier formulas.

$k_2=k-2$ *and* k_1-3: These are contragredient to $k_2=2$ and $k_2=3$. Hence formulas (53) and (54) apply with reversed signs.

Further explicit formulas (for $J=4,5,k_1-4,k_1-5$, $k_2=4,5,k_1-4,k_1-5$) could be obtained by solving cubic equations (that may in some cases reduce to quadratic or linear ones), and we could proceed even further by solving quartic equations. We have, however, decided not to proceed in this direction.

Let us make a few further comments:

1. The eigenvalues of the operators $X^{(i)}$ coincide for the $U(3)$ representation (m_{13},m_{23},m_{33}) and the $SU(3)$ representation $(m_{13}-m_{33},m_{23}-m_{33})\equiv(k_1,k_2)$.

2. For any self-contragredient representation, i.e., such that $m_{13}-m_{33}=2(m_{23}-m_{33})$, and for any fixed value of J, the sum of all eigenvalues of $X^{(3)}$corresponding to J equals zero. More precisely, one has

$$\sum_{m_{11}+m_{12}+m_{23}=J+3m_{33}}\left\langle\begin{matrix}m_{12} & & m_{22}\\ & m_{11} & \end{matrix}\middle|X^{(3)}\middle|\begin{matrix}m_{12} & & m_{22}\\ & m_{11} & \end{matrix}\right\rangle=0. \tag{55}$$

This property is evidently connected to the automorphism $T_i\to -T_i$, $L_i\to L_i$, for which $X^{(3)}\to -X^{(3)}$.

3. A given eigenvector $|JMK_t\rangle$ of $X^{(i)}$ belonging to a representation space of (m_{13},m_{23},m_{33}) is readily constructed if one knows all eigenvalues K_j belonging to (m_{13},m_{23},m_{33}). Indeed,

$$|JMK_t\rangle\sim\prod_{j\ne t}(X^{(i)}-K_j)\psi, \tag{56}$$

where ψ is an arbitrary vector from the representation space of (m_{13},m_{23},m_{33}) such that

$$\langle\psi|JMK_t\rangle\ne 0.$$

4. CONCLUSIONS

The contents of this article can be summarized as follows:

1. We have shown that for an arbitrary semisimple group G and its semisimple subgroup H there exists only a finite number of independent scalars with respect to H in the enveloping algebra of G.

2. We have derived a generating function for the number of $O(3)$ invariants of any given order in the enveloping algebra of $SU(3)$. The method is quite general and can be applied to any (semisimple) group G and its (semisimple) subgroup H.

3. We have used the above results to prove that besides the Casimir operators of $SU(3)$ and angular momentum L^2 only two other independent $O(3)$ scalars exist in the enveloping algebra of $SU(3)$, namely $X^{(3)}=L_aT_{ab}L_b$ and $X^{(4)}=L_aT_{ab}T_{bc}L_c$ (both of these operators have already made an appearance in the literature[6,10,13]). Either of these operators (or an arbitrary nontrivial polynomial in $C^{(2)}$, $C^{(3)}$, L^2, $X^{(3)}$, $X^{(4)}$, and L_3) can be used to resolve the missing label problem in the $SU(3)\supset O(3)\supset O(2)$ reduction.

4. We consider the basis functions

$$|(m_{13},m_{23},m_{33})JMK\rangle \tag{57}$$

for irreducible representations of $U(3)$, where (m_{13},m_{23},m_{33}) label the $U(3)$ representation [$k_1=m_{13}-m_{33}$, $k_2=m_{23}-m_{33}$ for $SU(3)$], J is an eigenvalue of L^2, M of L_3, and K of $X^{(i)}$, i.e.,

$$X^{(i)}|(m_{13}m_{23}m_{33})JMK\rangle=K|(m_{13}m_{23}m_{33})JMK\rangle, \quad i=3 \text{ or } 4.$$

We also make use of the Gel'fand—Tseitlin formalism to derive a simple algorithm for evaluating K for any representation. The values of K are computed numerically for a large number of representations of $SU(3)$ and presented in the tables (containing all representations of known physical interest). In the case when the multiplicity of the $O(3)$ representation J in the $SU(3)$ representation (k_1, k_2) is 1 or 2, we give explicit formulas for the eigenvalues K of $X^{(3)}$, in terms of k_1, k_2, and J [see Eqs. (43)–(54)]. They are, of course, in agreement with Table I.

The eigenvalues K are integer whenever there is no degeneracy in J. If there are two or more multiplets with the same J, then the sum of the eigenvalues is integer, although the individual K's are solutions of algebraic equations of order equal to the multiplicity of J in the given $SU(3)$ representation. The eigenvalues K corresponding to the same J in contragredient representations of $SU(3)$ differ by a sign in the case of $X^{(3)}$ and remain unchanged for $X^{(4)}$. For self-contragradient representations the sum of all $X^{(3)}$ eigenvalues corresponding to $O(3)$ multiplets with the same J equals zero: if there is only one multiplet with a given $J = J_0$, then its K equals zero.

We have chosen to present a smaller number of computer calculated eigenvalues of $X^{(4)}$ in Table II, than for $X^{(3)}$ in Table I. The computer programs we have used are available on request and are suitable for arbitrary representations of $SU(3)$. Similarly we have running programs for explicit construction of eigenvectors of $X^{(3)}$ and $X^{(4)}$ as linear combinations of Gel'fand–Tseitlin patterns, and also a program for calculating matrix elements of any polynomial of $U(3)$ generators relative to both the basis of patterns and to the basis (57).

It should also be mentioned that a large amount of literature related *inter alia* to the $SU(3) \supset O(3) \supset O(2)$ missing label problem exists. Besides the articles already quoted we mention the work of Biedenharn,[21] the review by Louck and Galbraith[22] (containing numerous references) and the recent article by Asherova and Smirnov.[23]

Let us make a few comments on physical applications of the results of this paper.

1. The fact that the basis functions (57) form an orthonormal set is particularly helpful, e.g., if we are interested in calculating matrix elements of some operator Q (a Hamiltonian, a term in a Hamiltonian, a transition operator, etc.) that commutes with $X^{(i)}$ since we will then obtain selection rules with respect to K. Similarly, if some polynomial $P(X^{(3)}, X^{(4)}, C^{(2)}, C^{(3)}, L^2, L_3)$ commutes with Q, rather than $X^{(3)}$ itself, then this operator P should be used to provide the missing label. It is certainly of interest that the algebra of such polynomials is finitely generated.

2. Various $O(3)$ scalars in the enveloping algebra of $SU(3)$ have been succesfully used as models for two- and three-body forces.[7,24] One implication of the present results is the following:

The only "fundamental" forces that can be introduced in an $SU(3)$ scheme with an $O(3)$ invariant interaction are two-body forces involving $C^{(2)}$ and L^2, three-body forces involving $C^{(3)}$ and $X^{(3)}$, and four-body forces, involving $X^{(4)}$. Any other forces can be represented as polynomials in the fundamental ones.

3. The formalism developed in this article has an amusing application in elementary particle physics. Indeed, the problem of constructing a state vector for N identical pions in a state with definite isospin T can be solved by embedding an $O(3)$ group, related to the isospin, into an $U(3)$ group.[25,26] The N-pion state will be characterized by the $U(3)$ labels N_1, N_2, N_3 (with $N = N_1 + N_2 + N_3$, $N_1 \geq N_2 \geq N_3 \geq 0$), the isospin T, charge $Q = T_3$ and the degeneracy label K (the correspondence with the notations of the present article is $N_1 = m_{13}$, $N_2 = m_{23}$, $N_3 = m_{33}$, $T = L$, $Q = L_0$). If K is identified with the eigenvalue of operator $X^{(3)}$ as in this article, it is possible to obtain rigorous limits on the charge distribution of pions in N-pion production, following from isospin conservation and Bose statistics alone. This can then be done for arbitrary values of the isospin T; previous considerations[25,26] were restricted to $T = 0$ and 1, when no degeneracies occur. The results are presented in a separate article.[27]

Other group–subgroup chains of physical interest with a missing label problem are presently being considered. Work in progress on the Wigner supermultiplet scheme $SU(4) \supset SU(2) \times SU(2)$ (two missing labels) and also the schemes $SO(5) \supset SU(2) \times U(1)$ (one missing), $SO(5) \supset SO(3)$ (two missing), and $G_2 \supset SO(3)$ (four labels missing).

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