APPENDIX B.

Basic Properties of Special Functions

As a convenient reference we collect here some fundamental definitions and relations for those special functions which appear most frequently in this book. With the exception of the gamma and elliptic functions, all these functions arise as solutions of differential equations obtained by separating variables in the partial differential equations of mathematical physics. The notation used here is the same as that adopted in the Bateman project [36, 37], and the reader can find many additional properties of these functions in those references.

1. The Gamma Function

Defining integral:

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}\,dt,\qquad \operatorname{Re} z>0.$$

By analytic continuation $\Gamma(z)$ can be extended to a function analytic in the whole complex plane, with the exception of simple poles at $z=-n, n=0,1,2,\ldots$.

Functional equations:

$$\Gamma(z+1)=z\Gamma(z),\qquad \Gamma(z)\Gamma(1-z)=\pi/\sin\pi z.$$

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Special values:

$$\Gamma(n+1)=n!, \qquad n=0,1,2,\ldots; \quad \Gamma(\tfrac{1}{2})=\sqrt{\pi}\,.$$

The binomial coefficients are defined by

$$\binom{\mu}{n}=\mu(\mu-1)\cdots(\mu-n+1)/n!=\Gamma(\mu+1)/\Gamma(\mu-n+1)n!. \quad \text{(B.1)}$$

2. The Hypergeometric Function

The hypergeometric series, convergent for $|z|<1$, is given by

$$ {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle|z\right)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n \quad \text{(B.2)}$$

where

$$(a)_0=1, \qquad (a)_n=a(a+1)\cdots(a+n-1), \qquad n=1,2,\ldots, \quad \text{(B.3)}$$

is Pochhammer's symbol. By analytic continuation the ${}_2F_1$ can be extended to define a function analytic and single valued in the complex z plane cut along the positive real axis from $+1$ to $+\infty$.

Integral representation:

$$ {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle|z\right)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1+tz)^{-a}\,dt,$$

$$\operatorname{Re}c>\operatorname{Re}b>0, |\arg(1-z)|<\pi.$$

For fixed $z, {}_2F_1\left(\begin{smallmatrix}a,b\\c\end{smallmatrix}\middle|z\right)/\Gamma(c)$ is an entire function of the parameters a,b,c. If a or b is a negative integer and c is not a negative integer, then the hypergeometric series becomes a polynomial in z. Differential equation:

$$z(1-z)\frac{d^2u}{dz^2}+[c-(a+b+1)z]\frac{du}{dz}-abu=0. \quad \text{(B.4)}$$

This equation has the solution $u={}_2F_1\left(\begin{smallmatrix}a,b\\c\end{smallmatrix}\middle|z\right)$. For c not an integer the equation admits the linearly independent solution $u=$

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$z^{1-c}{}_2F_1\left(\begin{matrix}a-c+1,b-c+1\\2-c\end{matrix}\middle|z\right)$. Differential recurrence formulas:

$$\frac{d}{dz}\,{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle|z\right)=\frac{ab}{c}\,{}_2F_1\left(\begin{matrix}a+1,b+1\\c+1\end{matrix}\middle|z\right),$$

$$\left[z\frac{d}{dz}+a\right]{}_2F_1=a\,{}_2F_1\left(\begin{matrix}a+1,b\\c\end{matrix}\middle|z\right),$$

$$\left[z\frac{d}{dz}+c-1\right]{}_2F_1=(c-1)\,{}_2F_1\left(\begin{matrix}a,b\\c-1\end{matrix}\middle|z\right),$$

$$\left[z(1-z)\frac{d}{dz}-bz+c-a\right]{}_2F_1=(c-a)\,{}_2F_1\left(\begin{matrix}a-1,b\\c\end{matrix}\middle|z\right),$$

$$\left[(1-z)\frac{d}{dz}-(a+b-c)\right]{}_2F_1=(c-a)(c-b)c^{-1}\,{}_2F_1\left(\begin{matrix}a,b\\c+1\end{matrix}\middle|z\right),$$

$$\left[z(1-z)\frac{d}{dz}-bz+c-1\right]{}_2F_1=(c-1)\,{}_2F_1\left(\begin{matrix}a-1,b\\c-1\end{matrix}\middle|z\right),$$

$$\left[(1-z)\frac{d}{dz}-a\right]{}_2F_1=a(b-c)c^{-1}\,{}_2F_1\left(\begin{matrix}a+1,b\\c+1\end{matrix}\middle|z\right),$$

$$\left[z(1-z)\frac{d}{dz}-(b+a-1)z+c-1\right]{}_2F_1=(c-1)\,{}_2F_1\left(\begin{matrix}a-1,b-1\\c-1\end{matrix}\middle|z\right). \qquad \text{(B.5)}$$

Symmetry relation:

$${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle|z\right)={}_2F_1\left(\begin{matrix}b,a\\c\end{matrix}\middle|z\right).$$

Transformation formulas:

$$\begin{aligned}{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\middle|z\right)&=(1-z)^{-a}\,{}_2F_1\left(\begin{matrix}a,c-b\\c\end{matrix}\middle|\frac{z}{z-1}\right)\\&=(1-z)^{c-a-b}\,{}_2F_1\left(\begin{matrix}c-a,c-b\\c\end{matrix}\middle|z\right).\end{aligned}$$

Special cases. (i) Legendre polynomials:

$$P_n(x)={}_2F_1\left(\begin{matrix}n+1,-n\\1\end{matrix}\middle|\frac{1-x}{2}\right),\qquad n=0,1,2,\ldots;$$

(ii) Gegenbauer polynomials:

$$C_n^{\nu}(x)=\frac{\Gamma(2\nu+n)}{\Gamma(2\nu)n!}\,{}_2F_1\left(\begin{matrix}2\nu+n,-n\\\nu+\frac{1}{2}\end{matrix}\middle|\frac{1-x}{2}\right),\qquad n=0,1,2,\ldots;$$

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(iii) Jacobi polynomials:

$$P_n^{(\alpha,\beta)}(x)=\binom{n+\alpha}{n}{}_2F_1\left(\begin{matrix}n+\alpha+\beta+1,-n\\ \alpha+1\end{matrix}\middle|\frac{1-x}{2}\right),\qquad n=0,1,2,\ldots;$$

(iv) Legendre functions:

$$P_\nu^\mu(z)=\left(\frac{z+1}{z-1}\right)^{\mu/2}\frac{{}_2F_1\left(\begin{matrix}\nu+1,-\nu\\ 1-\mu\end{matrix}\middle|\frac{1-z}{2}\right)}{\Gamma(1-\mu)},$$

$$Q_\nu^\mu(z)=e^{i\mu\pi}2^{-\nu-1}\pi^{1/2}\Gamma(\nu+\mu+1)z^{-\nu-\mu-1}(z^2-1)^{\mu/2}$$

$$\times\frac{{}_2F_1\left(\begin{matrix}\nu/2+\mu/2+1,\nu/2+\mu/2+1/2\\ \nu+3/2\end{matrix}\middle|z^{-2}\right)}{\Gamma(\nu+3/2)}.\qquad\text{(B.6)}$$

3. The Confluent Hypergeometric Function.

The function ${}_1F_1\left(\begin{matrix}a\\ c\end{matrix}\middle|z\right)$ is defined by the series

$${}_1F_1\left(\begin{matrix}a\\ c\end{matrix}\middle|z\right)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{z^n}{n!}$$

convergent for all z.

Integral representation:

$${}_1F_1\left(\begin{matrix}a\\ c\end{matrix}\middle|z\right)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{zt}t^{a-1}(1-t)^{c-a-1}dt,\qquad \mathrm{Re}\,c>\mathrm{Re}\,a>0.$$

For fixed z, ${}_1F_1\left(\begin{matrix}a\\ c\end{matrix}\middle|z\right)/\Gamma(c)$ is an entire function of a and c. Differential equation:

$$z\frac{d^2u}{dz^2}+(c-z)\frac{du}{dz}-au=0.\qquad\text{(B.7)}$$

This equation has the solution $u={}_1F_1\left(\begin{matrix}a\\ c\end{matrix}\middle|z\right)$ and for c not an integer the equation admits the independent solution $u=z^{1-c}{}_1F_1\left(\begin{matrix}a-c+1\\ 2-c\end{matrix}\middle|z\right)$.

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Differential recurrence formulas:

$$\frac{d}{dz}\,{}_1F_1\left(\begin{matrix}a\\c\end{matrix}\Big|z\right)=\frac{a}{c}\,{}_1F_1\left(\begin{matrix}a+1\\c+1\end{matrix}\Big|z\right),$$

$$\left[z\frac{d}{dz}-z+c-1\right]{}_1F_1=(c-1)\,{}_1F_1\left(\begin{matrix}a-1\\c-1\end{matrix}\Big|z\right),$$

$$\left[z\frac{d}{dz}+a\right]{}_1F_1=a\,{}_1F_1\left(\begin{matrix}a+1\\c\end{matrix}\Big|z\right),$$

$$\left[z\frac{d}{dz}-z+c-a\right]{}_1F_1=(c-a)\,{}_1F_1\left(\begin{matrix}a-1\\c\end{matrix}\Big|z\right),$$

$$\left[\frac{d}{dz}-1\right]{}_1F_1=\frac{a-c}{c}\,{}_1F_1\left(\begin{matrix}a\\c+1\end{matrix}\Big|z\right),$$

$$\left[z\frac{d}{dz}+c-1\right]{}_1F_1=(c-1)\,{}_1F_1\left(\begin{matrix}a\\c-1\end{matrix}\Big|z\right). \qquad \text{(B.8)}$$

Transformation formula:

$${}_1F_1\left(\begin{matrix}a\\c\end{matrix}\Big|z\right)=e^z\,{}_1F_1\left(\begin{matrix}c-a\\c\end{matrix}\Big|-z\right).$$

Special cases. (i) Laguerre polynomials:

$$L_n^{(\alpha)}(x)=\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)n!}\,{}_1F_1\left(\begin{matrix}-n\\\alpha+1\end{matrix}\Big|x\right),\qquad n=0,1,2,\ldots;$$

(ii) Bessel functions:

$$J_\nu(x)=\frac{e^{-ix}(x/2)^\nu}{\Gamma(\nu+1)}\,{}_1F_1\left(\begin{matrix}\nu+1/2\\2\nu+1\end{matrix}\Big|2ix\right);$$

(iii) Parabolic cylinder functions:

$$D_\nu(x)=2^{1/2}\exp\left(\frac{-x^2}{4}\right)\left[\frac{\Gamma(1/2)}{\Gamma(1/2-\nu/2)}\,{}_1F_1\left(\begin{matrix}-\nu/2\\1/2\end{matrix}\Big|\frac{x^2}{2}\right)\right.$$

$$\left.+x2^{-1/2}\frac{\Gamma(-1/2)}{\Gamma(-\nu/2)}\,{}_1F_1\left(\begin{matrix}1/2-\nu/2\\3/2\end{matrix}\Big|\frac{x^2}{2}\right)\right]. \qquad \text{(B.9)}$$

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4. Parabolic Cylinder Functions

The function $u = D_\nu(x)$, (B.9iii), is a solution of the equation

$$\frac{d^2u}{dz^2} + \left(\nu + \frac{1}{2} - \frac{z^2}{4}\right)u = 0. \tag{B.10}$$

A linearly independent solution is $u = D_{-\nu-1}(iz)$ and, for ν not an integer, $D_\nu(-z)$. If $\nu = n = 0, 1, 2, \ldots,$ then

$$D_n(z) = 2^{-n/2}\exp(-z^2/4)H_n(2^{-1/2}z) \tag{B.11}$$

where

$$H_n(z) = (-1)^n \exp(z^2)\frac{d^n}{dz^n}\exp(-z^2) \tag{B.12}$$

is the Hermite polynomial of order n.

Differential recurrence relations:

$$\left[\frac{d}{dz} + \frac{z}{2}\right]D_\nu(z) = \nu D_{\nu-1}(z), \qquad \left[-\frac{d}{dz} + \frac{z}{2}\right]D_\nu(z) = D_{\nu+1}(z). \tag{B.13}$$

5. Bessel Functions

The Bessel function $J_\nu(z)$ is given by (B.9ii) or by

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(\nu+1 \middle| \frac{-z^2}{4}\right), \qquad |\arg z| < \pi, \tag{B.14}$$

where

$${}_0F_1(c|x) = \sum_{n=0}^{\infty}\frac{x^n}{(c)_n n!}, \tag{B.15}$$

convergent for all x. Here, $z^{-\nu}J_\nu(z)$ is an entire function of z.

Differential equation:

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{\nu^2}{z^2}\right)u = 0. \tag{B.16}$$

This equation has solutions $u_1 = J_\nu(z)$ and $u_2 = J_{-\nu}(z)$, linearly independent for ν not an integer n. However, $J_{-n}(z) = (-1)^n J_n(z)$ and for $\nu = n, J_n(z)$ is the only solution of (B.16) which is bounded near $z = 0$. Differen-

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tial recurrence formulas:

$$\left[-\frac{d}{dz}+\frac{\nu}{z}\right]J_\nu(z)=J_{\nu+1}(z), \qquad \left[\frac{d}{dz}+\frac{\nu}{z}\right]J_\nu(z)=J_{\nu-1}(z). \quad \text{(B.17)}$$

The functions in these classes (Sections 2–5) are either hypergeometric functions ${}_2F_1$ or various special and limiting cases of the ${}_2F_1$. However, the functions in Sections 6 and 7 are generalizations of the ${}_2F_1$, the first to differential equations of higher order and the second to functions of several variables.

6. Generalized Hypergeometric Functions

The functions ${}_pF_q$ are defined by the series

$${}_pF_q\left(\begin{matrix}a_1,a_2,\ldots,a_p\\ b_1,b_2,\ldots,b_q\end{matrix}\middle| z\right)={}_pF_q(a_i;b_j;z)$$

$$=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}. \quad \text{(B.18)}$$

Unless the parameters a_i, b_j are chosen such that the series terminates or becomes undefined, it can be shown that the series converges for all z if $p\leqslant q$, converges for $|z|<1$ if $p=q+1$, and diverges for all $z\neq 0$ if $p>q+1$.

Differential equation:

$$\left(z\frac{d}{dz}+a_1\right)\cdots\left(z\frac{d}{dz}+a_p\right)u-\frac{d}{dz}\left(z\frac{d}{dz}+b_1-1\right)\cdots\left(z\frac{d}{dz}+b_q-1\right)u=0. \quad \text{(B.19)}$$

This equation has the solution $u={}_pF_q(a_i;b_j;z)$ and, except for special choices of the parameters a_i, b_j, this is the only solution of (B.19) which is bounded in a neighborhood of $z=0$.

Differential recurrence formulas:

$$\left(z\frac{d}{dz}+a_1\right){}_pF_q(a_i;b_j;z)=a_1\,{}_pF_q\left[\begin{matrix}a_1+1,a_2,\ldots,a_p\\ b_j\end{matrix}\middle| z\right],$$

$$\left(z\frac{d}{dz}+b_1-1\right){}_pF_q(a_i;b_j;z)=(b_1-1)\,{}_pF_q\left(\begin{matrix}a_i\\ b_1-1,b_2,\ldots,b_q\end{matrix}\middle| z\right),$$

$$\frac{d}{dz}\,{}_pF_q(a_i;b_j;z)=\frac{a_1\cdots a_p}{b_1\cdots b_q}\,{}_pF_q(a_i+1;b_j+1;z). \quad \text{(B.20)}$$

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Symmetry relation: The ${}_pF_q(a_i;b_j;z)$ is a symmetric function of $a_1,\ldots,a_p$ and of $b_1,\ldots,b_q$.

7. The Lauricella Functions

The Lauricella functions are generalizations of the ${}_2F_1$ to n variables $z_1,\ldots,z_n$. They fall into four classes:

$$F_A[a;b_1,\ldots,b_n;c_1,\ldots,c_n;z_1,\ldots,z_n] = \sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}}\frac{z_1^{m_1}\cdots z_n^{m_n}}{m_1!\cdots m_n!}, \tag{B.21}$$

$$|z_1|+\cdots+|z_n|<1,$$

$$F_B[a_1,\ldots,a_n;b_1,\ldots,b_n;c;z_1,\ldots,z_n] = \sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(a_1)_{m_1}\cdots(a_n)_{m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}}\frac{z_1^{m_1}\cdots z_n^{m_n}}{m_1!\cdots m_n!}, \tag{B.22}$$

$$|z_1|<1,\ldots,|z_n|<1,$$

$$F_C[a;b;c_1,\ldots,c_n;z_1,\ldots,z_n] = \sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(a)_{m_1+\cdots+m_n}(b)_{m_1+\cdots+m_n}}{(c_1)_{m_1}\cdots(c_n)_{m_n}}\frac{z_1^{m_1}\cdots z_n^{m_n}}{m_1!\cdots m_n!}, \tag{B.23}$$

$$|z_1|^{1/2}+\cdots+|z_n|^{1/2}<1,$$

and

$$F_D[a;b_1,\ldots,b_n;c;z_1,\ldots,z_n] = \sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}}\frac{z_1^{m_1}\cdots z_n^{m_n}}{m_1!\cdots m_n!}, \tag{B.24}$$

$$|z_1|<1,\ldots,|z_n|<1.$$

The functions in the following sections cannot be obtained as special cases or limits of functions of hypergeometric type.

8. Mathieu Functions

Mathieu's differential equation is

$$\frac{d^2u}{dx^2}+(a-2q\cos 2x)u=0 \tag{B.25}$$

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where usually the variable x is real and q is a given real nonzero parameter. If we impose the periodic boundary condition $u(x)=u(x+2\pi)$ on the solutions of (B.25), we can cast this equation into the form of a regular Sturm–Liouville eigenvalue problem for the eigenvalues a. It follows from the general theory of such problems that there exist countably infinitely many such eigenvalues, all real, each of multiplicity one, and bounded below but increasing to $+\infty$. Moreover, due to the symmetry properties of (B.25) this equation has four types of periodic solutions (called Mathieu functions of the first kind, or just Mathieu functions):

$$\text{(i)}\quad ce_{2n}(x,q)=\sum_{m=0}^{\infty} A_{2m}^{(2n)}\cos 2mx,$$

$$\text{(ii)}\quad ce_{2n+1}(x,q)=\sum_{m=0}^{\infty} A_{2m+1}^{(2n+1)}\cos(2m+1)x,$$

$$\text{(iii)}\quad se_{2n+1}(x,q)=\sum_{m=0}^{\infty} B_{2m+1}^{(2n+1)}\sin(2m+1)x,$$

$$\text{(iv)}\quad se_{2n+2}(x,q)=\sum_{m=0}^{\infty} B_{2m+2}^{(2n+2)}\sin(2m+2)x,\qquad n=0,1,2,\ldots. \tag{B.26}$$

The coefficients A,B depend on q and recurrence relations for these coefficients can be easily obtained by substituting expressions (B.26) into (B.25) [7]. The eigenvalues a of $ce_{2n}, ce_{2n+1}, se_{2n+1}, se_{2n+2}$ are denoted $a_{2n}, a_{2n+1}, b_{2n+1}, b_{2n+2}$, respectively. The eigenvalues are just those values of a such that the functions (B.26) whose coefficients are determined by the recurrence relations belong to $L_2[-\pi,\pi]$; that is, such that the functions are square integrable. The coefficients can always be chosen to be real and the Mathieu functions are normalized so that

$$\int_{-\pi}^{\pi}\left[u(x)\right]^2 dx=\pi. \tag{B.27}$$

Furthermore, the normalization is such that

$$\lim_{q\to 0} ce_0(x,q)=2^{-1/2},\qquad \lim_{q\to 0} ce_n(x,q)=\cos nx,\qquad n\neq 0,$$
$$\lim_{q\to 0} se_n(x,q)=\sin nx. \tag{B.28}$$

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