APPENDIX A

Lie Groups and Algebras

We list here some of the basic facts concerning Lie groups and algebras that are needed in this book. Complete proofs and further details can be found in [85]. Since almost all Lie groups that arise in mathematical physics are groups of matrices, we shall confine our attention to local linear Lie groups.

Let W be an open, connected set containing $\mathbf{e}=(0,\ldots,0)$ in the space R^n of all real n-tuples $\mathbf{g}=(g_1,\ldots,g_n)$.

DEFINITION. An n-dimensional (*real*) *local linear Lie group* G is a set of $m\times m$ nonsingular complex matrices $A(\mathbf{g})=A(g_1,\ldots,g_n)$ defined for each $\mathbf{g}\in W$ such that

1. $A(\mathbf{e})=E_m$ (the identity matrix).
2. The matrix elements of $A(\mathbf{g})$ are analytic functions of the parameters $g_1,\ldots,g_n$ and the map $\mathbf{g}\to A(\mathbf{g})$ is one to one.
3. The n matrices $\partial A(\mathbf{g})/\partial g_j, j=1,\ldots,n$, are linearly independent for each $\mathbf{g}\in W$.
4. There exists a neighborhood W' of $\mathbf{e}$ in R^n, $W'\subset W$, with the property that for every pair of n-tuples $\mathbf{g},\mathbf{h}$ in W' there is an n-tuple $\mathbf{k}$ in W satisfying $A(\mathbf{g})A(\mathbf{h})=A(\mathbf{k})$ where the operation on the left is matrix multiplication.

A local Lie group can be considered as a neighborhood of the identity in a global Lie group. (For the theory of global Lie groups see [46,47].) If in the foregoing definition W and W' are neighborhoods of $\mathbf{e}$ in $\mathcal{C}^n$, then G is a *complex* local linear Lie group.

The parameters $\mathbf{g}=(g_1,\ldots,g_n)$ define *local coordinates* on G and it can be shown that group multiplication can be expressed in terms of local coordinates by $\mathbf{k}=\varphi(\mathbf{g},\mathbf{h})$ where φ is an analytic vector-valued function of its $2n$

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arguments for $\mathbf{g},\mathbf{h}$ sufficiently close to $\mathbf{e}$ and $\boldsymbol{\varphi}(\mathbf{e},\mathbf{g})=\boldsymbol{\varphi}(\mathbf{g},\mathbf{e})=\mathbf{g}$. Any local coordinate transformation $\mathbf{g}'=\mathbf{f}(\mathbf{g})$ leads to a new Lie group which we identify with G.

Let $\mathbf{g}(t)$ be an analytic curve in R^n such that $\mathbf{g}(0)=\mathbf{e}$. (Here t is a real parameter and $\mathbf{g}(t)$ is defined and analytic in t for $|t|<1$.) The *Lie algebra* $\mathcal{G}$ of G is the set of all $m\times m$ matrices $\mathcal{A}=(d/dt)A(\mathbf{g}(t))|_{t=0}$ where $\mathbf{g}$ ranges over all analytic curves through $\mathbf{e}$. It follows easily that every $\mathcal{A}\in\mathcal{G}$ is a linear combination of the n linearly independent matrices

$$\mathcal{C}_j=\left.\frac{\partial A(\mathbf{g})}{\partial g_j}\right|_{\mathbf{g}=\mathbf{e}}.$$

Indeed, $\mathcal{A}=\sum_{j=1}^{n}\alpha_j\mathcal{C}_j$ where $\alpha_j=(dg_j/dt)(t)|_{t=0}$. This shows that $\mathcal{G}$ is an n-dimensional real vector space under addition and scalar multiplication of matrices. The matrices $\mathcal{C}_j$ form a basis for $\mathcal{G}$.

Furthermore, the matrix *commutator* $[\mathcal{A},\mathcal{B}]=\mathcal{A}\mathcal{B}-\mathcal{B}\mathcal{A}$ belongs to $\mathcal{G}$ for any $\mathcal{A},\mathcal{B}\in\mathcal{G}$. In particular, $[\mathcal{C}_l,\mathcal{C}_s]=\sum_{j=1}^{n}c_j^{ls}\mathcal{C}_j$, $1\leqslant l,s\leqslant n$, where $c_j^{ls}=c_{j,ls}-c_{j,ls}$ and

$$c_{j,ls}=\left.\frac{\partial^2}{\partial g_l\,\partial h_s}\varphi_j(\mathbf{g},\mathbf{h})\right|_{\mathbf{g}=\mathbf{h}=\mathbf{e}},\qquad \boldsymbol{\varphi}=(\varphi_1,\ldots,\varphi_n).$$

The *matrix exponential* $\exp(\mathcal{A})$ of an $m\times m$ matrix $\mathcal{A}$ is the $m\times m$ matrix

$$\exp(\mathcal{A})=\sum_{p=0}^{\infty}(p!)^{-1}\mathcal{A}^p. \tag{A.1}$$

This series is convergent and analytic in the matrix elements of $\mathcal{A}$. Here $\exp(\mathcal{A})\exp(-\mathcal{A})=E_m$ and $\exp(\mathcal{A})\exp(\mathcal{B})=\exp(\mathcal{A}+\mathcal{B})$ for $m\times m$ matrices $\mathcal{A},\mathcal{B}$ with $\mathcal{A}\mathcal{B}=\mathcal{B}\mathcal{A}$.

Denote the matrix elements of $\mathcal{A}$ by $\mathcal{A}_{ij}$, $1\leqslant i,j\leqslant m$, and define the *norm* of $\mathcal{A}$ by $\|\mathcal{A}\|=\max_{i,j}|\mathcal{A}_{ij}|$. There exist positive numbers ε,δ such that (1) $\exp(\mathcal{A})\in G$ for each $\mathcal{A}\in\mathcal{G}$ with $\|\mathcal{A}\|<\varepsilon$ and (2) each $A\in G$ with $\|A-E_m\|<\delta$ can be expressed as $A=\exp(\mathcal{A})$ for a unique $\mathcal{A}\in\mathcal{G}$ with $\|\mathcal{A}\|<\varepsilon$. This is a one-to-one analytic mapping of a neighborhood of the zero matrix in $\mathcal{G}$ onto a neighborhood of E_m in G. Writing $A=\exp(\mathcal{A})$ where $\mathcal{A}=\sum_{j=1}^{n}\alpha_j\mathcal{C}_j$, we can use the *canonical coordinates* $\alpha_1,\ldots,\alpha_n$ to parametrize G.

Let U be an open connected set in $\mathbb{C}^p$. Any $\mathbf{z}\in U$ can be designated by its coordinates $\mathbf{z}=(z_1,\ldots,z_p)$, $z_j\in\mathbb{C}$. Let $\mathbf{Q}$ be a mapping that associates to each pair $(\mathbf{z},A)$, $\mathbf{z}\in U$, $A\in G$, an element $\mathbf{Q}(\mathbf{z},A)$ in $\mathbb{C}^p$. We write $\mathbf{Q}(z,A)=\mathbf{z}^A\in\mathbb{C}^p$.

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DEFINITION. The n-dimensional local linear Lie group G acts on the manifold U as a *local Lie transformation group* if $\mathbf{Q}$ satisfies the properties

1. $\mathbf{z}^A$ is analytic in the $p+n$ coordinates of $\mathbf{z}$ and A;
2. $\mathbf{z}^{E_m}=\mathbf{z}$ all $\mathbf{z}\in U$;
3. if $\mathbf{z}^A\in U$, then $(\mathbf{z}^A)^B=\mathbf{z}^{(AB)}$ for all $A,B\in G$ such that $AB\in G$.

Suppose G is a local Lie transformation group on U and let $\mathfrak{F}$ be the space of all functions $f(\mathbf{z})$ analytic in a neighborhood of a fixed $\mathbf{z}^0\in U$. (Here the neighborhood is allowed to vary with the function.) A *local multiplier* ν for this transformation group is a scalar-valued function $\nu(\mathbf{z},A)$ analytic in the $p+n$ coordinates $\mathbf{z}\in U, A\in G$ such that (1) $\nu(\mathbf{z},E_m)=1$ and (2) $\nu(\mathbf{z},AB)=\nu(\mathbf{z},A)\nu(\mathbf{z}^A,B)$ for $A,B,AB\in G$. Note that $\nu(\mathbf{z},A)\equiv 1$ is a (trivial) local multiplier. (For the general theory of local multipliers see [82, Chapter 8].)

A *local multiplier representation* $\mathbf{T}$ corresponding to $G,U,\mathfrak{F},\nu$ is a mapping $\mathbf{T}(A)$ of $\mathfrak{F}$ onto $\mathfrak{F}$ defined for $A\in G$ and $f\in\mathfrak{F}$ by

$$\left[\mathbf{T}(A)f\right](\mathbf{z})=\nu(\mathbf{z},A)f(\mathbf{z}^A). \tag{A.2}$$

Since ν is a local multiplier, it follows that

1. $\mathbf{T}(E_m)f=f$ for all $f\in\mathfrak{F}$;
2. $\mathbf{T}(AB)f=\mathbf{T}(A)[\mathbf{T}(B)f]$ for all $A,B\in G$ sufficiently close to E_m.

Let $A(\mathbf{g}(t))$ be a one-parameter curve in G with Lie algebra element

$$\mathcal{Q}=\frac{d}{dt}A(\mathbf{g}(t))\Big|_{t=0}=\sum_{j=1}^{n}\alpha_j\mathcal{C}_j \tag{A.3}$$

as defined earlier. Let $\mathbf{T}$ be a multiplier representation of G and $f\in\mathfrak{F}$.

DEFINITION. The *generalized Lie derivative* $D_{\mathcal{Q}}f$ of f is the analytic function

$$D_{\mathcal{Q}}f(\mathbf{z})=\frac{d}{dt}\left[\mathbf{T}(A\mathbf{g}(t))f\right](\mathbf{z})\Big|_{t=0}. \tag{A.4}$$

Direct computation yields

$$D_{\mathcal{Q}}=\sum_{i=1}^{p}\sum_{j=1}^{n}P_{ij}(\mathbf{z})\alpha_j\partial_{z_i}+\sum_{j=1}^{n}\alpha_jP_j(\mathbf{z}). \tag{A.5}$$

Here $D_{\mathcal{Q}}$ depends only on $\mathcal{Q}\in\mathcal{G}$, not on the possible curves $\mathbf{g}(t)$ that lead to $\mathcal{Q}$. The $P_{ij}(\mathbf{z})$ can be computed uniquely from $\mathbf{Q}(\mathbf{z},A)$, while the $P_j(\mathbf{z})$ follow from $\nu(\mathbf{z},A)$. In particular, if $\nu\equiv 1$, then $P_j\equiv 0$ and $D_{\mathcal{Q}}$ is an *ordinary* Lie derivative.

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The following theorems, which are used frequently in this book, are essentially due to Sophus Lie [82, 85].

THEOREM *A*.1. *The generalized Lie derivatives of a local multiplier representation form a Lie algebra under the operations of addition of derivatives and Lie bracket*

$$[D_{\mathcal{A}}, D_{\mathcal{B}}] = D_{\mathcal{A}}D_{\mathcal{B}} - D_{\mathcal{B}}D_{\mathcal{A}}. \tag{A.6}$$

This algebra is a homomorphic image of $\mathcal{G}$:

$$D_{(a\mathcal{A}+b\mathcal{B})} = aD_{\mathcal{A}} + bD_{\mathcal{B}}, \qquad D_{[\mathcal{A},\mathcal{B}]} = [D_{\mathcal{A}}, D_{\mathcal{B}}], \qquad \mathcal{A}, \mathcal{B} \in \mathcal{G},\ a, b \in R. \tag{A.7}$$

(The equalities in (A.6), (A.7) are meant in the sense that both sides of the equation yield the same result when applied to a fixed $f \in \mathcal{F}$.)

THEOREM *A*.2

$$[\mathbf{T}(\exp(t\mathcal{A}))f](\mathbf{z}) = \sum_{j=0}^{\infty} (j!)^{-1} t^j D_{\mathcal{A}}^j f(\mathbf{z}) = (\exp D_{\mathcal{A}}) f(\mathbf{z}), \qquad \mathcal{A} \in \mathcal{G}. \tag{A.8}$$

(*This result is valid for all* $t \in R$ *with* $|t|$ *sufficiently small.*)

THEOREM *A*.3. *Let*

$$D_j = \sum_{i=1}^{p} P_{ij}(\mathbf{z})\,\partial_{z_i} + P_j(\mathbf{z}), \qquad j = 1, \dots, n, \tag{A.9}$$

be n *linearly independent differential operators defined and analytic in an open set* $U \subseteq \mathcal{C}^p$. *If there exist real constants* c_{jk}^l *such that*

$$[D_j, D_k] = \sum_{l=1}^{n} c_{jk}^l D_l, \qquad 1 \leq j, k \leq n, \tag{A.10}$$

then the D_j *form a basis for a Lie algebra that is the algebra of generalized Lie derivatives for a local multiplier representation* $\mathbf{T}$ *of a local Lie group* G. *There is a basis* $\{\mathcal{C}_j\}$ *for the Lie algebra* $\mathcal{G}$ *of* G *such that*

$$[\mathcal{C}_j, \mathcal{C}_k] = \sum_{l=1}^{n} c_{jk}^l \mathcal{C}_l.$$

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The action of G is obtained by integration of the equations

$$\frac{dz_i(t)}{dt} = \sum_{j=1}^{n} P_{ij}(\mathbf{z}(t))\alpha_j, \qquad \frac{d}{dt}\ln \nu(\mathbf{z}^0, \exp t\mathcal{Q}) = \sum_{j=1}^{n} \alpha_j P_j(\mathbf{z}(t)) \quad \text{(A.11)}$$

where

$$\mathbf{z}(0) = \mathbf{z}^0, \qquad \nu(\mathbf{z}^0, E_m) = 1, \qquad \mathbf{z}(t) = \mathbf{z}^{0(\exp t\mathcal{Q})}, \qquad 1 \leqslant i \leqslant p, \quad \text{(A.12)}$$

and $\mathcal{Q}$ *is given by* (A.3).

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