APPENDIX C

Elliptic Functions

We list here the basic properties of elliptic functions that are needed in this book. For further details see [7,37,136a].

Elliptic functions depend on a complex variable z and a real parameter k (the *modulus*) which in this book will always satisfy $0 \leqslant k \leqslant 1$. The *complementary modulus* is $k' = (1-k^2)^{1/2}, 1 \geqslant k' \geqslant 0$. The *elliptic functions* $\operatorname{sn}(z,k), \operatorname{cn}(z,k), \operatorname{dn}(z,k)$, or briefly $\operatorname{sn} z, \operatorname{cn} z, \operatorname{dn} z$, are defined by

$$z = \int_0^{\operatorname{sn} z} \left[(1-t^2)(1-k^2t^2)\right]^{-1/2} dt = \int_{\operatorname{cn} z}^{1} \left[(1-t^2)(k'^2+k^2t^2)\right]^{-1/2} dt$$

$$= \int_{\operatorname{dn} z}^{1} \left[(1-t^2)(t^2-k'^2)\right]^{-1/2} dt. \tag{C.1}$$

The values of the integrals depend on the integration contours and this is reflected in the periodicity properties of elliptic functions.

As $k \to 0$ we have

$$\operatorname{sn}(z,k) \to \sin z, \qquad \operatorname{cn}(z,k) \to \cos z, \qquad \operatorname{dn}(z,k) \to 1,$$

and as $k \to 1$

$$\operatorname{sn}(z,k) \to \tanh z, \qquad \operatorname{cn}(z,k) \to \operatorname{sech} z, \qquad \operatorname{dn}(z,k) \to \operatorname{sech} z.$$

Periodicity:

$$\begin{aligned} \operatorname{sn}(z+2K) &= -\operatorname{sn} z, & \operatorname{sn}(z+2iK') &= \operatorname{sn} z, \\ \operatorname{cn}(z+2K) &= -\operatorname{cn} z, & \operatorname{cn}(z+2iK') &= -\operatorname{cn} z, \\ \operatorname{dn}(z+2K) &= \operatorname{dn} z, & \operatorname{dn}(z+2iK') &= -\operatorname{dn} z. \end{aligned} \tag{C.2}$$

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Here K, K' are defined by

$$K(k)=\int_0^{\pi/2}(1-k^2\sin^2\theta)^{-1/2}\,d\theta, \qquad K'=K(k'). \tag{C.3}$$

Special relations:

$$\begin{aligned}&\operatorname{sn}(-z)=-\operatorname{sn}(z), \quad \operatorname{cn}(-z)=\operatorname{cn}z, \quad \operatorname{dn}(-z)=\operatorname{dn}z,\\ &\operatorname{sn}^2z+\operatorname{cn}^2z=1, \quad k^2\operatorname{sn}^2z+\operatorname{dn}^2z=1.\end{aligned} \tag{C.4}$$

Special values:

$$\begin{aligned}&\operatorname{sn}0=0, \quad \operatorname{sn}K=1, \quad \operatorname{sn}(K+iK')=1/k,\\ &\operatorname{cn}0=1, \quad \operatorname{cn}K=0, \quad \operatorname{cn}(K+iK')=-ik'/k,\\ &\operatorname{dn}0=1, \quad \operatorname{dn}K=k', \quad \operatorname{dn}(K+iK')=0.\end{aligned} \tag{C.5}$$

The elliptic functions all have simple poles at $z=iK'$. As z increases from 0 to K, $\operatorname{sn}z$ increases from 0 to 1, $\operatorname{cn}z$ decreases from 1 to 0, and $\operatorname{dn}z$ decreases from 1 to k'. As z varies from K to $K+iK'$, $\operatorname{sn}z$ increases from 1 to k^{-1}, $\operatorname{cn}z$ is pure imaginary and varies from 0 to $-ik'/k$, and $\operatorname{dn}z$ decreases from k' to 0. As z varies from $K+iK'$ to iK', $\operatorname{sn}z$ increases from $1/k$ to $+\infty$, $\operatorname{cn}z$ is pure imaginary and varies from $-ik'/k$ to $-i\infty$, and $\operatorname{dn}z$ is pure imaginary and varies from 0 to $-i\infty$.

Derivatives:

$$\frac{d}{dz}\operatorname{sn}z=\operatorname{cn}z\operatorname{dn}z, \qquad \frac{d}{dz}\operatorname{cn}z=-\operatorname{sn}z\operatorname{dn}z, \qquad \frac{d}{dz}\operatorname{dn}z=-k^2\operatorname{sn}z\operatorname{cn}z. \tag{C.6}$$

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