EXACT AND QUASI - EXACT SOLVABILITY OF SECOND ORDER SUPERINTEGRABLE QUANTUM SYSTEMS.

E. G. Kalnins
Department of Mathematics, University of Waikato, Hamilton, New Zealand.

W. Miller, Jr.
School of Mathematics, University of Minnesota, Minneapolis, Minnesota, 55455, U.S.A.

G. S. Pogosyan
Yerevan State University, Yerevan, Armenia
and
Departamento de Matematicas, Universidad de Guadalajara, Guadalajara, Mexico

Abstract

Quasi-exactly solvable (QES) problems in quantum mechanics are eigenvalue problems for the Schrödinger operator where it is posssible to compute exactly a certain finite number of eigenvalues and eigenfunctions, even though exact algebraic expressions for the full set of eigenvalues do not exist. In the past, mathematical physicists have used hidden symmetry methods to attack these problems. We give a brief review of these methods and then show the increased insight into the structure of such problems provided by superintegrability theory and separation of variables.

1 Introduction

It is well known that N-dimensional nonrelativistic quantum systems described by the Hamiltonian

$$\mathcal{H} = -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_i^2} + V(x_1, x_2, ..., x_N) \tag{1}$$

are integrable if there exist N linearly independent and global differential operators $\mathcal{I}_\ell$, $\ell = 0, 1, ..N-1$ and $\mathcal{I}_0 = \mathcal{H}$, commuting with the Hamiltonian $\mathcal{H}$ and with each other

$$[\mathcal{I}_\ell, \mathcal{H}] = 0, \qquad [\mathcal{I}_\ell, \mathcal{I}_j] = 0, \qquad \ell, j = 1, 2, ...N-1. \tag{2}$$

An integrable system is called **superintegrable** (this term was introduced first time by S.Rauch-Wojciechowski in [1]) or **maximally integrable** if it is integrable and, also, possesses additional integrals $\mathcal{L}_k$, commuting with the Hamiltonian

$$[\mathcal{L}_k, \mathcal{H}] = 0, \qquad k = 1, 2, ...N - 1 \tag{3}$$

but not necessarily with each other.

Two examples of this kind have been well-known a long time, namely the Kepler-Coulomb problem and the isotropic harmonic oscillator. Another famous example is the many-body Calogero - Moser model. The existence of additional integrals of motion for these systems (second order superintegrability) leads to many interesting properties not shared by standard integrable systems [2, 3, 4, 5, 6]

1. In quantum mechanics, there is the phenomenon of *accidental degeneracy* when the energy eigenvalues are multiply degenerate.

2. This property is intimately related to the existence of a dynamical symmetry group (or algebra), a so-called *hidden symmetry group (algebra)*.

3. In classical mechanics the additional integrals of motion have the consequence that in the case of superintegrable systems in two dimensions and maximally superintegrable systems in three dimensions all finite trajectories are found to be periodic.

4. One of the most important properties of (second order) superintegrable systems is *multiseparability*, the separation of variables for the Hamilton-Jacobi and Schrödinger equations in more than one orthogonal coordinate system.

1.1 Quasi-Exactly Solvable Systems

The crucial example that stimulated the investigation of quasi-exactly solvable systems is the quantum system with the anisotropic potential

$$V(x) = \frac{1}{2}\omega^2 x^6 + 2\beta\omega^2 x^4 + (2\beta^2\omega^2 - 2\delta\omega - \lambda)x^2 + 2\frac{(\delta - \frac{1}{4})(\delta - \frac{3}{4})}{x^2}, \tag{4}$$

where ω, β, $\delta > 1/2$ and λ are the constants.

As noticed by many authors [7, 8], this system admits polynomial solutions (which obviously do not form a basis) only for special values of the constant $\lambda = \omega(2n+1)$, $(n = 0, 1, 2...)$

$$\Psi(x) \approx x^{2\delta - \frac{1}{2}}\, \mathrm{e}^{-\frac{\omega}{4}x^4 - \beta\omega x^2}\, P_n(x^2). \tag{5}$$

where

$$P_n(x^2) = \sum_{s=0}^{n} A_s x^2.$$

is a polynomial of degree $2n$.

There are two different approaches to the investigation of quasi-exactly solvable systems. In the algebraic approach formulated by Turbiner [9] quasi-exactly solvability is explained in terms of a "hidden symmetry algebra" $sl(2,R)$. More precisely this means following: the one-dimensional Hamiltonian $\mathcal{H} = \partial^2_{x_i} + V(x)$ after suitable changes of variable $z = \xi(x)$ and "gauge transformation" $H = e^{-\alpha(z)}\mathcal{H}e^{\alpha(z)}$ can be written in the form

$$H = \sum_{a,b=0,\pm} C_{ab} J_a J_b + \sum_{a=0,\pm} C_a J_a \tag{6}$$

where the first-order differential operators $\{J_\pm, J_0\}$ satisfy the commutation relations for the Lie algebra $sl(2,R)$.

The second approach, known as analytic, was formulated by Uschveridze [10, 11] and represents a one-dimensional reduction of the Niven-Stielties method for solving multiparameter spectral problems such as the generalized Lamé equation (or ellipsoidal equation). The solution in this method is determined by the zeros of polynomials $P_n(x^2)$. Indeed the wave function (5) can be rewritten in the form

$$\Psi(x) \approx x^{2\delta-\frac{1}{2}}\, \mathrm{e}^{-\frac{\omega}{4}x^4-\beta\omega x^2}\, \Pi_{i=0}^{n}(x^2-\xi_i), \tag{7}$$

where the numbers $(\xi_1, \xi_2, ...\xi_n)$ satisfy a system of n algebraic equations.

According to the oscillation theorem, the number of zeros in the physical interval $\xi_i \in [0,\infty)$ enumerates the ground state and first n - excitations, described in terms of all zeros (complete solutions of the systems of algebraic equations and including the non physical section $\xi_i \in (-\infty, 0])$ as

$$E = 4\delta \left[\beta\omega + \sum_{i=1}^{n} \frac{1}{\xi_i}\right]. \tag{8}$$

A few natural questions occur in these two approaches:

(i) why does the constant $\lambda = \omega(2n+1)$ in potential (4) have this special form,

(ii) what is the physical meaning of the negative zeros ξ_i, and finally,

(iii) why in the correct formula for the energy spectrum (8) do n zeros of the polynomial $P_n(x^2)$ appear?

Let us now consider the quantum motion in the plane for a charged particle with two fixed Coulomb centers with coordinates $(\pm D/2, 0)$ (the so-called plane two center problem)

$$V(x,y) = -\frac{\alpha_1}{\sqrt{y^2+(x+D/2)^2}} - \frac{\alpha_2}{\sqrt{y^2+(x-D/2)^2}} \tag{9}$$

This system is not superintegrable and separation of variables is possible only in two-dimensional elliptic coordinates

$$x = \frac{D}{2}\cosh\nu\cos\mu, \qquad y = \frac{D}{2}\sinh\nu\sin\mu.$$

Upon the substitution $\psi(\nu,\mu;D^2) = X(\nu;D^2)Y(\mu;D^2)$ with separation constant $A(D)$, the Schrödinger equation separates into a **system** of two ordinary differential equations

$$\frac{d^2X}{d\nu^2} + \left[\frac{D^2E}{2}\cosh^2\nu + D(\alpha_1+\alpha_2)\cosh\nu + A(D)\right]X = 0,$$

$$\frac{d^2Y}{d\mu^2} - \left[\frac{D^2E}{2}\cos^2\mu + D(\alpha_1-\alpha_2)\cos\mu + A(D)\right]Y = 0.$$

These equations belong to the class of **non-exactly solvable** problems. Polynomial solutions do not exist even for the case of discrete spectrum $E < 0$, and each of the wave functions $X(\nu;D^2)$ and $Y(\mu;D^2)$ is expressed as an infinite series with a three-term recurrence relation.

Let us now put $\alpha_2 = 0$. Then the two center potential transforms to the ordinary two-dimensional (2D) hydrogen atom problem, which is well-known as a superintegrable system with dynamical symmetry group $SO(3)$, and admits separation of variables in three systems of coordinates: polar, parabolic and elliptic [12]. In this case we can see that two separation equations in elliptic coordinates [13], namely

$$\frac{d^2X}{d\nu^2} + \left[\frac{D^2E}{2}\cosh^2\nu + D\alpha_1\cosh\nu + A(D)\right]X = 0, \tag{10}$$

$$\frac{d^2Y}{d\mu^2} - \left[\frac{D^2E}{2}\cos^2\mu + D\alpha_1\cos\mu + A(D)\right]Y = 0. \tag{11}$$

transform into each other by the change $\mu \leftrightarrow i\nu$. Thus separation of variables in elliptic coordinates for the 2D hydrogen atom gives **two functionally identical** one-dimensional Schrödinger type equations with two parameters: coupling constant E and energy $A(D)$ (correspondingly energy and separation constant for 2D hydrogen atom), but one defined on the real and the other on the imaginary axis.

In other words, instead of yielding a system of differential equations (10)-(11), the problem reduces to solving only one of the equations (10) or (11), for instance

$$\frac{d^2Z}{d\xi^2} - \left[\frac{D^2E}{2}\cos^2\xi + D\alpha_1\cos\xi + A(D)\right]Z = 0.$$

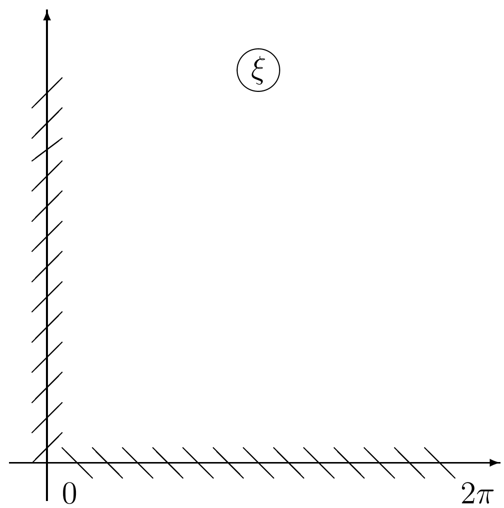
for which the "domain of definition" of variable ξ is the complex plane (see Fig.).

The requirement of finiteness for the wave functions in the complex plane permits **only** polynomial solutions. As result we obtain *simultaneous* quantization of the energy

$$E_n = -\frac{\alpha_1^2}{2(n+1/2)^2}, \qquad n = 0,1,2,... \tag{12}$$

and the elliptic separation constant $A_s(D)$ where $s = 0,1,2,...n$ (as a solution of an nth-degree algebraic equation). The polynomial solutions are defined with the help of a finite series with a three-term recurrence relation for the coefficients. They cannot be considered as exactly-solvable and can be investigated only numerically.

We note that each of equations (10) or (11) has the form of a one dimensional Schrödinger equation with parameter E and eigenvalue $A(D)$, and could separately be considered in the regions $\mu \in [0, 2\pi]$ or $\nu \in [0, \infty)$, correspondingly. Then for arbitrary values of constant E the solutions of eqs. (10) or (11) expressed via infinite series and only on the "energy surface" of the 2D hydrogen atom (12) split into polynomial and nonpolynomial sectors (each of these sectors is non complete) and for fixed number n, only *some* of the eigenvalue $A_s(D)$, $(s = 0, 1, 2 \ldots n)$ can be calculated from an nth-degree algebraic equation.



To understand the intimate connection between superintegrable and quasi-exactly solvable systems we consider one of the two dimensional superintegrable systems, namely the anisotropic oscillator.

2 Anisotropic oscillator

The potential

$$V(x, y) = \frac{1}{2}\omega^2(4x^2 + y^2) + k_1 x + \frac{k_2^2 - \frac{1}{4}}{2y^2}, \qquad k_2 > 0, \tag{13}$$

is known as the *singular anisotropic oscillator*. The Schrödinger equation separates in two coordinate systems: *Cartesian and parabolic.*

2.1. Cartesian bases

Separation of variables in Cartesian coordinates leads to two *independent* one-dimensional Schrödinger equations

$$\frac{d^2\psi_1}{dx^2} + (2\lambda_1 - 4\omega^2 x^2 - 2k_1 x)\psi_1 = 0. \tag{14}$$

$$\frac{d^2\psi_2}{dy^2} + \left(2\lambda_2 - \omega^2 y^2 - \frac{k_2^2 - \frac{1}{4}}{y^2}\right)\psi_2 = 0. \tag{15}$$

where

$$\Psi(x,y;k_1,\pm k_2)=\psi_1(x;k_1)\psi_2(y;\pm k_2) \tag{16}$$

and λ_1, λ_2 are *two Cartesian separation constants* with $\lambda_1+\lambda_2=E$.

Equation (15) represents the well-known linear singular oscillator system. It is an exactly-solvable problem and the complete set of orthonormalized eigenfunctions is

$$\psi_{n_2}(y;\pm k_2)=\sqrt{\frac{2\omega^{(1\pm k_2)}n_2!}{\Gamma(n_2\pm k_2+1)}}\, y^{\frac{1}{2}\pm k_2}\, e^{-\frac{1}{2}\omega y^2}\, L_{n_2}^{\pm k_2}(\omega y^2) \tag{17}$$

where $\lambda_2=\omega(2n_2+1\pm k_2)$. The positive sign at the k_2 occurs alone when $k_2>\frac{1}{2}$ and both the positive and the negative signs occurs if $0<k_2<\frac{1}{2}$.

The second equation easily transforms to the ordinary one-dimensional oscillator problem:

$$\psi_{n_1}(x;k_1)=\left(\frac{2\omega}{\pi}\right)^{1/4}\frac{e^{-\omega z^2}}{\sqrt{2^{n_1}n_1!}}H_{n_1}(\sqrt{2\omega}z), \qquad z=x+\frac{k_1}{4\omega^2}, \tag{18}$$

where $\lambda_1=\omega(2n_1+1)-\frac{k_1^2}{8\omega^2}$. Thus the complete energy spectrum is

$$E=\lambda_1+\lambda_2=\omega[2n+2\pm k_2]-\frac{k_1^2}{8\omega^2}, \qquad n=n_1+n_2=0,1,2,... \tag{19}$$

and the degree of degeneracy for fixed principal quantum number n is $(n+1)$.

2.2. Parabolic bases

In parabolic coordinates ξ and η

$$x=\frac{1}{2}(\xi^2-\eta^2)\ , \qquad y=\xi\eta\ , \qquad \xi\in\mathbf{R},\,\eta>0. \tag{20}$$

upon the substitution

$$\Psi(\xi,\eta)=X(\xi)Y(\eta)$$

and the introduction of the parabolic separation constant λ, the Schrödinger equation splits into two ordinary differential equations

$$\frac{d^2X}{d\xi^2}\ +\ \left(2E\xi^2-\omega^2\xi^6-k_1\xi^4-\frac{k_2^2-\frac{1}{4}}{\xi^2}\right)X=-\lambda X, \tag{21}$$

$$\frac{d^2Y}{d\eta^2}\ +\ \left(2E\eta^2-\omega^2\eta^6+k_1\eta^4-\frac{k_2^2-\frac{1}{4}}{\eta^2}\right)Y=+\lambda Y. \tag{22}$$

Equations (21) and (22) are transformed into one another by the change $\xi\longleftrightarrow i\eta$. The complete wave function is

$$\Psi(\xi,\eta;E,\lambda)=C(E,\lambda)Z(\xi;E,\lambda)Z(i\eta;E,\lambda) \tag{23}$$

where $C(E,\lambda)$ is the normalization constant and the wave function $Z(\mu;E,\lambda)$ is a solution of the equation

$$\left[-\frac{d^2}{d\mu^2}+\left(\omega^2\mu^6+k_1\mu^4-2E\mu^2+\frac{k_2^2-\frac{1}{4}}{\mu^2}\right)\right]Z(\mu;E,\lambda)=\lambda Z(\mu;E,\lambda). \tag{24}$$

Thus, at $\mu\in(-\infty,\infty)$ we have eq. (21) and at $\mu\in[0,i\infty)$ - the eq. (22) and in the complex μ domain the "physical" region is just the two lines Im $\mu=0$ and Re $\mu=0$, Im $\mu>0$.

Our task is to find the solutions of eq. (24) that are regular and decreasing as $\mu\to\pm\infty$ and $\mu\to i\infty$.

Requiring a solution in the form

$$Z(\mu;E,\lambda)=\exp\left(-\frac{\omega}{4}\mu^4-\frac{k_1}{4\omega}\mu^2\right)\mu^{\frac{1}{2}\pm k_2}\,\psi(\mu;E,\lambda), \tag{25}$$

and passing to a new variable $z=\mu^2$

$$z\frac{d^2\psi}{dz^2}+\left[(1\pm k_2)-\omega z\left(z+\frac{k_1}{2\omega^2}\right)\right]\frac{d\psi}{dz}+\left[\frac{1}{2}\tilde{E}z+\frac{1}{4}\tilde{\lambda}\right]\psi=0 \tag{26}$$

where

$$\tilde{E}=E+\frac{k_1^2}{8\omega^2}-\omega(2\pm k_2),\qquad \tilde{\lambda}=\lambda-\frac{k_1}{\omega}(1\pm k_2), \tag{27}$$

we express the wave function $\psi(z)$ as

$$\psi(z;E,\lambda)=\sum_{s=0}^{\infty}A_s(E,\lambda)z^s. \tag{28}$$

The substitution (28) in eq. (26) leads to the following three-term recurrence relation for the expansion coefficients $A_s\equiv A_s(E,\lambda)$,

$$\begin{aligned}(s+1)(s+1\pm k_2)A_{s+1} &+ \frac{1}{4}\Big[\lambda-\frac{k_1}{\omega}(2s+1\pm k_2)\Big]A_s\\ &+ \frac{1}{2}\Big[E+\frac{k_1}{8\omega^2}-\omega(2s\pm k_2)\Big]A_{s-1}=0,\end{aligned} \tag{29}$$

with the initial conditions $A_{-1}=0$ and $A_0=1$.

Let us now examine the asymptotics for a function $\psi(z)$ on the real axis $z\in(-\infty,\infty)$. At large s, from (29) we have

$$[s^2+s(2\pm k_2)]\frac{A_{s+1}}{A_s}\frac{A_s}{A_{s-1}}-\frac{k_1}{2\omega}s\frac{A_s}{A_{s-1}}-\omega s=0 \tag{30}$$

As $s^{-1}\ll 1$ the following expansions hold

$$\frac{A_{s+1}}{A_s}\sim c_0+\frac{c_1}{\sqrt{s}}+\text{Ø}\left(\frac{1}{s}\right). \tag{31}$$

Using (31) in (30) we obtain

$$c_0 = 0, \qquad c_1 = \pm\sqrt{\omega}, \qquad A_s \sim \frac{(\pm\sqrt{\omega})^s}{\sqrt{s!}} \tag{32}$$

and therefore the function $\psi(z)$ may have two asymptotic regimes

$$\psi(z) \sim \sum \frac{(\pm\sqrt{\omega}z)^s}{\sqrt{s!}}. \tag{33}$$

Then we have for $z > 0$ [the case of eq. (21)]

$$\sum \frac{(\sqrt{\omega}z)^s}{\sqrt{s!}} > \sqrt{\sum \frac{(\omega z^2)^s}{s!}} = \exp\left(\frac{\omega}{2}z^2\right) \tag{34}$$

and the same for $z < 0$ [the case of eq. (22)]

$$\sum \frac{(-\sqrt{\omega}z)^s}{\sqrt{s!}} = \sum \frac{(\sqrt{\omega}|z|)^s}{\sqrt{s!}} > \sqrt{\sum \frac{(\omega z^2)^s}{s!}} = \exp\left(\frac{\omega}{2}z^2\right). \tag{35}$$

Thus the function $Z(\mu)$ cannot converge **simultaneously** at large μ for real and imaginary μ and therefore the series should be truncated. Note that a more detailed mathematical analysis of the three-term recurrence relations (29) has been carried out in our recent article [14].

The condition for the series to be truncated results in the energy spectrum, giving the already known formula. The coefficients $A_s \equiv A_s^{nq}(k_1, \pm k_2)$ satisfy the following relation

$$(s+1)(s+1\pm k_2)A_{s+1} + \beta_s A_s + \omega(n+1-s)A_{s-1} = 0, \tag{36}$$

$$\beta_s = \frac{\lambda}{4} - \frac{k_1}{4\omega}(2s+1\pm k_2).$$

The three-term recurrence relation (36) represents a homogeneous system of $n+1$ - algebraic equations for $n+1$ - coefficients $\{A_0, A_1, A_2, ...A_n\}$. The requirement for the existence of a non-trivial solution leads to a vanishing of the determinant

$$D_n(\lambda) = \begin{vmatrix} \beta_0 & 1\pm k_2 & & \cdot & & & \\ \omega n & \beta_1 & 2(2\pm k_2) & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ & & & \cdot & 2\omega & \beta_{n-1} & n(n\pm k_2) \\ & & & \cdot & & \omega & \beta_n \end{vmatrix} = 0.$$

The roots of the corresponding algebraic equation give us the $(n+1)$ eigenvalues of the parabolic separation constant $\lambda_{nq}(k_1, \pm k_2)$ and can be enumerated with the help of the integer q, where $0 \leq q \leq n$. The degeneracy for the n - energy state, as in the Cartesian case, equals $(n+1)$.

The physical admissible solutions have the form

$$Mk_{nq}(z;k_1,\pm k_2) \equiv \psi(z,E,\lambda) = \sum_{s=0}^{n} A_s^{nq}(k_1,\pm k_2)\, z^s,$$

$$Z_{nq}(\mu;k_1,\pm k_2) = exp\left(-\frac{\omega}{4}\mu^4 - \frac{k_1}{4\omega}\mu^2\right)\, \mu^{\frac{1}{2}\pm k_2}\, Mk_{nq}(\mu^2;k_1,\pm k_2). \tag{37}$$

For $\mu = \xi$ the function $Z_{nq}(\mu;k_1,\pm k_2)$ gives the solution of equation (21), and for $\mu = i\eta$ the solution for equation (22). Thus the parabolic wave function (23) can be written in following way

$$\Psi_{nq}(\xi,\eta;k_1,\pm k_2) = C_{nq}(k_1,\pm k_2)\, Z_{nq}(\xi;k_1,\pm k_2)\, Z_{nq}(i\eta;k_1,\pm k_2).$$

2.1 Niven-Stilties approach

Let us express the solution of the Schrödinger equation for the anisotropic oscillator in the following form

$$\Psi(x,y) = e^{-\omega(x+\frac{k_1}{4\omega^2})^2 - \frac{1}{2}\omega y^2}\, y^{\frac{1}{2}\pm k_2}\, \Phi(x,y). \tag{38}$$

The function $\Phi(x,y)$ is polynomial in variables (x,y^2) in Cartesian coordinates and (ξ^2,η^2) in parabolic ones. It satisfies the equation

$$\mathcal{R}\Phi(x,y) = -2E\Phi(x,y), \tag{39}$$

where the operator $\mathcal{R}$ is

$$\mathcal{R} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left[\frac{(1\pm 2k_2)}{y} - 2\omega y\right]\frac{\partial}{\partial y} - 4\omega\left[x + \frac{k_1}{4\omega^2}\right]\frac{\partial}{\partial x} - \omega(2\pm k_2) + \frac{k_1^2}{8\omega^2}.$$

Taking into account that

$$Mk_{nq}(z;k_1,\pm k_2) = \sum_{s=0}^{n} A_s^{nq}(k_1,\pm k_2)\, z^s \cong \Pi_{\ell=1}^{n}(z-\alpha_\ell), \tag{40}$$

where α_ℓ, $(\ell = 1,2,...n)$ are zeros of polynomials $Mk_{nq}(z)$ on the real axis $-\infty < z < \infty$, and that in parabolic coordinates

$$\frac{y^2}{\alpha} + 2x - \alpha = \frac{(\xi^2-\alpha)(\eta^2+\alpha)}{\alpha}, \tag{41}$$

we can choose a solution of eq. (39) in the form

$$\Phi(x,y) \quad = \quad Mk_{nq_1}(\xi^2;k_1,\pm k_2)\, Mk_{nq_2}(-\eta^2;k_1,\pm k_2) \cong \Pi_{\ell=1}^{n}\left(\frac{y^2}{\alpha_\ell} + 2x - \alpha_\ell\right). \tag{42}$$

Then from (39) follows that zeros α_ℓ must satisfy the system of n - algebraic equations

$$\sum_{m\neq\ell}^{n} \frac{2}{\alpha_\ell - \alpha_m} + \frac{(1\pm k_2)}{\alpha_\ell} - \omega\alpha_\ell = \frac{k_1}{2\omega}, \qquad \ell = 1,2,....n, \tag{43}$$

and for the energy spectrum we again have formula (19).

The system of algebraic equations (43) contains the n - set of solutions (zeros) $(\alpha_1^{(q)}, \alpha_2^{(q)}, ...\alpha_n^{(q)})$, $q = 1, 2, ...n$ and all zeros are real. The positive zeros $\alpha_\ell > 0$ define the nodes of wave functions for equation (21), whereas negative zeros $\alpha_\ell < 0$ define the nodes of wave functions for equation (22).

The eigenvalues for the parabolic separation constant can be calculated in the same way via the operator equation

$$\Lambda\Phi(x, y) = \lambda\Phi(x, y).$$

A more elegant way is to use the differential equation (26) directly. Rewriting first the eq. (26) in the form

$$\begin{aligned} \Bigg\{4z\frac{d^2}{dz^2} &+ 4\left[(1 \pm k_2) - \omega z\left(z + \frac{k_1}{2\omega^2}\right)\right]\frac{d}{dz} \\ &+ \left[4n\omega z - \frac{k_1}{\omega}(1 \pm k_2)\right]\Bigg\} Mk_{nq}(z; k_1, \pm k_2) = \lambda Mk_{nq}(z; k_1, \pm k_2) \end{aligned} \tag{44}$$

and expressing the wave function $Mk_{nq}(z; k_1, \pm k_2)$ in the form of (40), we obtain

$$\lambda_{nq}(k_1, \pm k_2) = 4(1 \pm k_2)\left[\frac{k_1}{4\omega} + \sum_{\ell=1}^{n}\frac{1}{\alpha_\ell^{(q)}}\right]. \tag{45}$$

2.2 Connection with quasi-exactly solvable systems

Substitution of the formula for the energy spectrum in eq. (24) yields

$$\left[-\frac{d^2}{d\mu^2} + \left(\omega^2\mu^6 + k_1\mu^4 + \left[\frac{k_1^2}{4\omega^2} - \omega(4n + 4 \pm 2k_2)\right]\mu^2 + \frac{k_2^2 - \frac{1}{4}}{\mu^2}\right)\right] Z_n(\mu) = \lambda Z_n(\mu),$$

which on the real axis completely coincides for $k_1 = 4\beta\omega^2$ and $1 \pm k_2 = 2\delta$, with the one-dimensional spectral problem (4), and is called a quasi-exactly solvable system. Now it is easy to understand the origin of the occurence of quasi-exactly solvable systems. The requirement of convergence just in real space in the vicinity of the singular points $\mu = \pm\infty$ implies that there are polynomial solutions of the form (37) We also can shed light on the mystery of zeros of polynomial $P_n(x^2)$. Indeed, the substitution of the wave function

$$\Psi(x) \approx x^{2\delta - \frac{1}{2}}\, \mathrm{e}^{-\frac{\omega}{4}x^4 - \beta\omega x^2}\, P_n(x^2).$$

into the one-dimensional Schrödinger equation with potential (4) leads to the differential equation for polynomial $P_n(x^2)$ in the same form as equation (44) (in variable $x^2 = z$), but with the difference that the physical region of eq. (44) is whole real axis $z \in (-\infty, \infty)$, and therefore all zeros (for positive and negative x^2) of $P_n(x^2)$ correspond to the zeros of two-dimensional eigenfunction of singular anisotropic oscillator in parabolic coordinates.

The situation is repeated in the case of the potential $(k_1, k_2 > 0)$

$$V(x,y) = \frac{1}{2}\omega^2(x^2+y^2) + \frac{1}{2}\left(\frac{k_1^2-\frac{1}{4}}{x^2} + \frac{k_2^2-\frac{1}{4}}{y^2}\right). \tag{46}$$

The corresponding Schrödinger equation separates in three different orthogonal coordinate systems: **Cartesian, polar and elliptical coordinates**.

Separation of variables in two-dimensional elliptic coordinates leads to a Schrödinger type equation

$$\frac{d^2Z(\zeta)}{d\zeta^2} + \Big[\frac{D^4\omega^2}{64}\cos^4\zeta - \frac{D^2E}{4}\cos^2\zeta - \frac{k_1^2-\frac{1}{4}}{\cos^2\zeta} - \frac{k_2^2-\frac{1}{4}}{\sin^2\zeta} - \lambda(D^2)\Big]Z(\zeta) = 0, \tag{47}$$

in the complex plane and the requirement of convergence at the point $\zeta = 0, 2\pi$ and $\zeta = i\infty$ forces polynomial solutions and determines the energy spectrum $E_n = \omega(2n+2\pm k_1 \pm k_2)$, $(n = 0,1,2,...)$.

As a consequence trigonometric and hyperbolic quasi-exactly solvable systems [10] are generated in the form

$$\frac{d^2X}{d\nu^2} + \Big[\left(\frac{\alpha^2}{4} + \alpha(2n+2\pm k_1\pm k_2)\right)\cosh^2\nu - \frac{\alpha^2}{4}\cosh^4\nu - \frac{k_1^2-\frac{1}{4}}{\sinh^2\nu} + \frac{k_2^2-\frac{1}{4}}{\cosh^2\nu} + \lambda\Big]X = 0,$$

$$\frac{d^2Y}{d\mu^2} - \Big[\left(\frac{\alpha^2}{4} + \alpha(2n+2\pm k_1\pm k_2)\right)\cos^2\mu - \frac{\alpha^2}{4}\cos^4\mu + \frac{k_1^2-\frac{1}{4}}{\cos^2\mu} + \frac{k_2^2-\frac{1}{4}}{\sin^2\mu} + \lambda\Big]Y = 0,$$

where $\alpha = D^2\omega/2$.

Thus we have established that an integral part of the notion of quasi-exact solvability is the reduction of superintegrable systems to one dimensional problems.

We can express our observation in the form of the following hypothesis: **Quantum mechanical problems which are expressible as one-dimensional quasi-exactly solvable systems can be obtained via separation of variables from N-dimensional Schrödinger equations for superintegrable systems**.

This analogy prompts us to use the term quasi-exactly solvable for equations of type (24) or (47), defined in the complex plane and which are not exactly-solvable but admit polynomial solutions.

We suggest calling quantum mechanical systems **first-order quasi-exactly solvable** if the polynomial solutions of the one-parametric differential equation obtained fom the N -dimensional Schrödinger equation after separation of variables are defined by a recurrence relation which contain three terms or more and the discrete eigenvalues can be calculated as the solutions of algebraic equations. According to this definition, systems (24) and (47) are first order quasi-exactly solvable.

We propose also another definition of exact-solvability through the solutions of ordinary differential equation: a *quantum mechanical system is called exactly solvable if the solutions of the Schrödinger equation, can be expressed in terms of hypergeometric functions* $_mF_n$. More precisely this means that the coefficients in power series expansions of the solutions satisfy two-term recurrence relations, rather than recurrence relations of higher order.

It is obvious, that an N-dimensional Schrödinger equation is exactly solvable if it is separable in some coordinate system and each of the separated equations is exactly solvable. Obviously, a superintegrable system is exactly-solvable if is exactly solvable in at least one system of coordinates.

3 Magyari - Uschveridze example.

A critical further example is the tenth-order polynomial quasi-exactly solvable problem from an article of Magyari [7] and Ushveridze's book [11].

Consider the Schrödinger equation in the form $H\Psi = E\Psi$ with the Hamiltonian

$$H = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + 36k_1^2\left[2(x-iy)^2 - 4(x^2+y^2) - z^2\right]$$

$$+48k_1k_2\left[3(x-iy)^2 - (x+iy)\right] - 16(2k_2^2 + 3k_1k_3)(x+iy) - \frac{p(p-1)}{z^2}.$$

This is essentially the Euclidean space superintegrable system with nondegenerate potential

$$V = \alpha\left[z^2 - 2(x-iy)^3 + 4(x^2+y^2)\right) + \beta\left(2(x+iy) - 3(x-iy)2\right] + \gamma(x+iy) + \frac{\delta}{z^2},$$

where the six basis symmetry operators can be taken in the form

$$H = \partial_x^2 + \partial_y^2 + \partial_z^2 + V,$$

$$L_1 = (\partial_x - i\partial_y)2 + f_1, \quad L_2 = \partial_z^2 + f_2, \quad L_3 = \{\partial_z, J_2 + iJ_1\} + f_3,$$

$$L_4 = \frac{1}{2}\{J_3, \partial_x - i\partial_y\} - \frac{i}{4}(\partial_x + i\partial_y)^2 + f_4, \quad L_5 = (J_2 + iJ_1)^2 + 2i\{\partial_z, J_1\} + f_5,$$

where $\{A, B\} = AB + BA$ and the f_i $(i = 1, 2, ..., 5)$ are the function of potential V. There is a quadratic algebra generated by these symmetries.

One choice of separable coordinates in three dimensions is

$$z = iuvw,$$

$$x+iy=\frac{1}{2}(u^2v^2+u^2w^2+v^2w^2)-\frac{1}{4}(u^4+v^4+w^4),\quad x-iy=\frac{1}{2}(u^2+v^2+w^2).$$

The separation equations in these coordinates have the form

$$\begin{aligned}\Big[\frac{d^2}{d\lambda^2} &- 36k_1^2\lambda^{10}-48k_1k_2\lambda^8-8(2k_2^2+3k_1k_3)\lambda^6+\frac{p(1-p)}{\lambda^2}\\ &+ E\lambda^4+\ell_2\lambda^2+\ell_3\Big]\Lambda(\lambda)=0.\end{aligned}\tag{48}$$

where ℓ_2 and ℓ_3 are the separation constant. This is essentially (for $p=1,0$), the Magyari-Ushveridze one-dimensional quasi-exactly solvable problem.

In searching for finite solutions of the Schrödinger equation we write

$$\Psi(u,v,w)=\exp\Big[k_1(u^6+v^6+w^6)+k_2(u^4+v^4+w^4)+k_3(u^2+v^2+w^2)\Big]\,(uvw)^p\Phi(u,v,w)$$

where the function Φ has polynomial form

$$\Phi(u,v,w)=\prod_{j=1}^{n}(u^2-\theta_j)(v^2-\theta_j)(w^2-\theta_j).$$

The zeros of the polynomials satisfy the relations

$$\frac{2n+1}{2\theta_i}-12k_1\theta_i^2-4k_2\theta_i-k_3+\sum_{j\neq i}\frac{1}{\theta_i-\theta_j}=0.$$

Solving these equations we see that the eigenvalues E,ℓ_2,ℓ_3 have the form

$$E=-(30+24n+12p)k_1-16k_2k_3,$$

$$\ell_2=-4k_3^2-(12+16r)k_2-24k_1\sum_{j=1}^{r}\theta_j,$$

$$\ell_3=-(2+8r+4p)k_3-16k_2\sum_{j=1}^{r}\theta_j-24k_1\sum_{j=1}^{r}\theta_j^2.$$

The last equations, in fact, give us also the requirements that the one-dimensional differential equation (48) with tenth-order anharmonicity admits polynomial solutions, or is quasi-exactly solvable a la Turbiner and Ushveridze.

References

[1] S.R.Wojciechowski. *Superintegrability of the Calogero-Moser System*. Phys. Lett., **A 95** (1983) 279.

[2] N.W.Evans. *Superintegrability in Classical Mechanics*; Phys.Rev. **A 41** (1990) 5666;

[3] C.Grosche, G.S.Pogosyan, A.N.Sissakian. *Path Integral Discussion for Smorodinsky - Winternitz Potentials: I. The Two - and Three Dimensional Euclidean Space.* Fortschritte der Physik, **43**, 453-521, 1995.

[4] E.G.Kalnins, W.Miller Jr. and G.S.Pogosyan. *Superintegrability and associated polynomial solutions. Euclidean space and sphere in two-dimensions.* J.Math.Phys. **37**, 6439-6467, 1996

[5] E.G.Kalnins, G.Williams, W.Miller Jr. and G.S.Pogosyan. *Superintegrability in the three-dimensional Euclidean space.* J.Math.Phys., **40**, 708-725, 1999.

[6] E.G.Kalnins, W.Miller Jr., G.S.Pogosyan and G.C.Williams. *On superintegrable symmetry-breaking potentials in N-dimensional Euclidean space.* J.Phys. **A35**(22), 4755-4773, 2002.

[7] E.Magyari. *Exact Quantum-Mechanical Solutions for Anharmonic Oscillators.* Phys. Lett., **81A**, 116-118, 1981.

[8] A.V.Turbiner and A.G.Ushveridze. *Spectral singularities and quasi-exactly solvable quantum problem.* Phys. Lett., **A126**, 181-183, 1987.

[9] A.V.Turbiner. *Quasi-Exactly-Solvable Problems and* $sl(2)$ *Algebra.* Comm. Math. Phys., **118**, 467-474, 1988.

[10] A.G.Ushveridze. *Quasi-exactly solvable models in quantum mechanics.* Sov. J.Part. Nucl., **20**, 504-528, 1989.

[11] A.G.Ushveridze, *Quasi-Exactly solvable models in quantum mechanics* Institute of Physics, Bristol, 1993.

[12] L.G.Mardoyan, G.S.Pogosyan, A.N.Sissakian and V.M.Ter-Antonyan. *Hidden symmetry, Separation of Variables and Interbasis Expansions in the Two-Dimensional Hydrogen Atom.* J.Phys., **A18**, 455-466, 1985.

[13] L.G.Mardoyan, G.S.Pogosyan, A.N.Sissakian and V.M.Ter-Antonyan. Two-Dimensional Hydrogen Atom: I. Elliptic Bases; *Theor.Math.Phys.* **61**, (1984), 1021;

[14] E.G.Kalnins, W.Miller Jr., and G.S.Pogosyan. *Exact and quasi-exact solvability of second order superintegrable quantum systems. I Euclidean space preliminaries.* J.Math.Phys., **47**, 033502, 2006.