Chapter 1

Fourier Series

Jean Baptiste Joseph Fourier (1768-1830) was a French mathematician, physicist and engineer, and the founder of Fourier analysis. In 1822 he made the claim, seemingly preposterous at the time, that any function of t, continuous or discontinuous, could be represented as a linear combination of functions $\sin nt$. This was a dramatic distinction from Taylor series. While not strictly true in fact, this claim was true in spirit and it led to the modern theory of Fourier analysis with wide applications to science and engineering.

1.1 Definitions, Real and complex Fourier series

We have observed that the functions $e_n(t) = e^{int}/\sqrt{2\pi}$, $n = 0, \pm 1, \pm 2, \cdots$ form an ON set in the Hilbert space $L^2[0, 2\pi]$ of square-integrable functions on the interval $[0, 2\pi]$. In fact we shall show that these functions form an ON basis. Here the inner product is

$$(u, v) = \int_0^{2\pi} u(t)\overline{v}(t)\, dt, \qquad u, v \in L^2[0, 2\pi].$$

We will study this ON set and the completeness and convergence of expansions in the basis, both pointwise and in the norm. Before we get started, it is convenient to assume that $L^2[0, 2\pi]$ consists of square-integrable functions on the unit circle, rather than on an interval of the real line. Thus we will replace every function $f(t)$ on the interval $[0, 2\pi]$ by a function $f^*(t)$ such that $f^*(0) = f^*(2\pi)$ and $f^*(t) = f(t)$ for $0 \le t < 2\pi$. Then we will extend

f^* to all $-\infty < t < \infty$ by requiring periodicity: $f^*(t+2\pi) = f^*(t)$. This will not affect the values of any integrals over the interval $[0, 2\pi]$, though it may change the value of f at one point. Thus, from now on our functions will be assumed $2\pi -$ *periodic*. One reason for this assumption is the

Lemma 1 *Suppose F is $2\pi -$ periodic and integrable. Then for any real number a*

$$\int_a^{2\pi+a} F(t)dt = \int_0^{2\pi} F(t)dt.$$

Proof Each side of the identity is just the integral of F over one period. For an analytic proof, note that

$$\int_0^{2\pi} F(t)dt = \int_0^a F(t)dt + \int_a^{2\pi} F(t)dt = \int_0^a F(t+2\pi)dt + \int_a^{2\pi} F(t)dt$$

$$= \int_{2\pi}^{2\pi+a} F(t)dt + \int_a^{2\pi} F(t)dt = \int_a^{2\pi} F(t)dt + \int_{2\pi}^{2\pi+a} F(t)dt$$

$$= \int_a^{2\pi+a} F(t)dt.$$

□

Thus we can transfer all our integrals to any interval of length 2π without altering the results.

Exercise 1 *Let $G(a) = \int_a^{2\pi+a} F(t)dt$ and give a new proof of of Lemma 1 based on a computation of the derivative $G'(a)$.*

For students who don't have a background in complex variable theory we will define the complex exponential in terms of real sines and cosines, and derive some of its basic properties directly. Let $z = x + iy$ be a complex number, where x and y are real. (Here and in all that follows, $i = \sqrt{-1}$.) Then $\bar{z} = x - iy$.

Definition 1 $e^z = \exp(x)(\cos y + i \sin y)$

Lemma 2 *Properties of the complex exponential:*

- $e^{z_1} e^{z_2} = e^{z_1+z_2}$
- $|e^z| = \exp(x)$

- $\overline{e^z} = e^{\overline{z}} = \exp(x)(\cos y - i \sin y)$.

Exercise 2 *Verify lemma 2. You will need the addition formula for sines and cosines.*

Simple consequences for the basis functions $e_n(t) = e^{int}/\sqrt{2\pi},\ n = 0, \pm 1, \pm 2, \cdots$ where t is real, are given by

Lemma 3 *Properties of e^{int}:*

- $e^{in(t+2\pi)} = e^{int}$
- $|e^{int}| = 1$
- $\overline{e^{int}} = e^{-int}$
- $e^{imt}e^{int} = e^{i(m+n)t}$
- $e^0 = 1$
- $\frac{d}{dt}e^{int} = ine^{int}$.

Lemma 4 $(e_n, e_m) = \delta_{nm}$.

Proof If $n \neq m$ then

$$(e_n, e_m) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)t}dt = \frac{1}{2\pi}\frac{e^{i(n-m)t}}{i(n-m)}\ |_0^{2\pi} = 0.$$

If $n = m$ then $(e_n, e_m) = \frac{1}{2\pi}\int_0^{2\pi} 1\ dt = 1$. □

Since $\{e_n\}$ is an ON set, we can project any $f \in L^2[0, 2\pi]$ on the closed subspace generated by this set to get the Fourier expansion

$$f(t) \sim \sum_{n=-\infty}^{\infty} (f, e_n)e_n(t),$$

or

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{int}, \qquad c_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt. \tag{1.1.1}$$

This is the *complex version* of Fourier series. (For now the $\sim$ just denotes that the right-hand side is the Fourier series of the left-hand side. In what sense the Fourier series represents the function is a matter to be resolved.) From our study of Hilbert spaces we already know that Bessel's inequality holds: $(f,f) \geq \sum_{n=-\infty}^{\infty} |(f,e_n)|^2$ or

$$\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 dt \geq \sum_{n=-\infty}^{\infty} |c_n|^2. \qquad (1.1.2)$$

An immediate consequence is the Riemann-Lebesgue Lemma.

Lemma 5 *(Riemann-Lebesgue, weak form)* $\lim_{|n|\to\infty} \int_0^{2\pi} f(t)e^{-int}dt = 0.$

Thus, as $|n|$ gets large the Fourier coefficients go to 0.

If f is a real-valued function then $\bar{c}_n = c_{-n}$ for all n. If we set

$$c_n = \frac{a_n - ib_n}{2}, \qquad n = 0, 1, 2, \cdots$$

$$c_{-n} = \frac{a_n + ib_n}{2}, \qquad n = 1, 2, \cdots$$

and rearrange terms, we get the *real version* of Fourier series:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt), \qquad (1.1.3)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\ dt \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\ dt$$

with Bessel inequality

$$\frac{1}{\pi}\int_0^{2\pi} |f(t)|^2 dt \geq \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2),$$

and Riemann-Lebesgue Lemma

$$\lim_{n\to\infty}\int_0^{2\pi} f(t)\cos(nt)dt = \lim_{n\to\infty}\int_0^{2\pi} f(t)\sin(nt)dt = 0.$$

Remark: The set $\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos nt, \frac{1}{\sqrt{\pi}}\sin nt\}$ for $n = 1, 2, \cdots$ is also ON in $L^2[0, 2\pi]$, as is easy to check, so (1.1.3) is the correct Fourier expansion in this basis for complex functions $f(t)$, as well as real functions.

Later we will prove the following basic results:

Theorem 1 *Parseval's equality. Let $f \in L^2[0, 2\pi]$. Then*

$$(f, f) = \sum_{n=-\infty}^{\infty} |(f, e_n)|^2.$$

In terms of the complex and real versions of Fourier series this reads

$$\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{1.1.4}$$

or

$$\frac{1}{\pi} \int_0^{2\pi} |f(t)|^2 dt = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2).$$

Let $f \in L^2[0, 2\pi]$ and remember that we are assuming that all such functions satisfy $f(t + 2\pi) = f(t)$. We say that f is *piecewise continuous* on $[0, 2\pi]$ if it is continuous except for a finite number of discontinuities. Furthermore, at each t the limits $f(t + 0) = \lim_{h \to 0, h>0} f(t + h)$ and $f(t - 0) = \lim_{h \to 0, h>0} f(t - h)$ exist. NOTE: At a point t of continuity of f we have $f(t+0) = f(t-0)$, whereas at a point of discontinuity $f(t+0) \neq f(t-0)$ and $f(t + 0) - f(t - 0)$ is the magnitude of the jump discontinuity.

Theorem 2 *Suppose*

- *$f(t)$ is periodic with period 2π.*
- *$f(t)$ is piecewise continuous on $[0, 2\pi]$.*
- *$f'(t)$ is piecewise continuous on $[0, 2\pi]$.*

Then the Fourier series of $f(t)$ converges to $\frac{f(t+0)+f(t-0)}{2}$ at each point t.

1.2 Examples

We will use the real version of Fourier series for these examples. The transformation to the complex version is elementary.

1. Let

$$f(t) = \begin{cases} 0, & t = 0 \\ \frac{\pi - t}{2}, & 0 < t < 2\pi \\ 0, & t = 2\pi. \end{cases}$$

and $f(t+2\pi) = f(t)$. We have $a_0 = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi-t}{2}\, dt = 0$. and for $n \geq 1$,

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - t}{2}\cos nt\ dt = \frac{\frac{\pi-t}{2}\sin nt}{n\pi}\Big|_0^{2\pi} + \frac{1}{2\pi n}\int_0^{2\pi} \sin nt\ dt = 0,$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - t}{2}\sin nt\ dt = -\frac{\frac{\pi-t}{2}\cos nt}{n\pi}\Big|_0^{2\pi} - \frac{1}{2\pi n}\int_0^{2\pi} \cos nt\ dt = \frac{1}{n}.$$

Therefore,

$$\frac{\pi - t}{2} = \sum_{n=1}^{\infty} \frac{\sin nt}{n}, \qquad 0 < t < 2\pi.$$

By setting $t = \pi/2$ in this expansion we get an alternating series for $\pi/4$:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots .$$

Parseval's identity gives

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

2. Let

$$f(t) = \begin{cases} \frac{1}{2}, & t = 0 \\ 1, & 0 < t < \pi \\ \frac{1}{2}, & t = \pi \\ 0 & \pi < t < 2\pi. \end{cases}$$

and $f(t+2\pi) = f(t)$ (a step function). We have $a_0 = \frac{1}{\pi}\int_0^{\pi} dt = 1$, and for $n \geq 1$,

$$a_n = \frac{1}{\pi}\int_0^{\pi} \cos nt\ dt = \frac{\sin nt}{n\pi}\Big|_0^{\pi} = 0,$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin nt\ dt = -\frac{\cos nt}{n\pi}\Big|_0^{\pi} = \frac{(-1)^{n+1}+1}{n\pi} = \begin{cases} \frac{2}{\pi n}, & n \text{ odd} \\ 0, & n \text{ even}. \end{cases}$$

Therefore,

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{j=1}^{\infty} \frac{\sin(2j-1)t}{2j-1}.$$

For $0 < t < \pi$ this gives

$$\frac{\pi}{4} = \sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots ,$$

and for $\pi < t < 2\pi$ it gives

$$-\frac{\pi}{4} = \sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots .$$

Parseval's equality becomes

$$\frac{\pi^2}{8} = \sum_{j=1}^{\infty} \frac{1}{(2j-1)^2}.$$

1.3 Fourier series on intervals of varying length, Fourier series for odd and even functions

Although it is convenient to base Fourier series on an interval of length 2π there is no necessity to do so. Suppose we wish to look at functions $f(x)$ in $L^2[\alpha, \beta]$. We simply make the change of variables

$$t = \frac{2\pi(x-\alpha)}{\beta-\alpha}$$

in our previous formulas. Every function $f(x) \in L^2[\alpha, \beta]$ is uniquely associated with a function $\hat{f}(t) \in L^2[0, 2\pi]$ by the formula $f(x) = \hat{f}(\frac{2\pi(x-\alpha)}{\beta-\alpha})$. The set $\{\frac{1}{\sqrt{\beta-\alpha}}, \sqrt{\frac{2}{\beta-\alpha}} \cos \frac{2\pi n(x-\alpha)}{\beta-\alpha}, \sqrt{\frac{2}{\beta-\alpha}} \sin \frac{2\pi n(x-\alpha)}{\beta-\alpha}\}$ for $n = 1, 2, \cdots$ is an ON basis for $L^2[\alpha, \beta]$, The real Fourier expansion is

$$\begin{aligned} f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{2\pi n(x-\alpha)}{\beta-\alpha} + b_n \sin \frac{2\pi n(x-\alpha)}{\beta-\alpha}), \qquad (1.3.5) \\ a_n &= \frac{2}{\beta-\alpha} \int_{\alpha}^{\beta} f(x) \cos \frac{2\pi n(x-\alpha)}{\beta-\alpha}\, dx, \quad b_n = \frac{2}{\beta-\alpha} \int_{\alpha}^{\beta} f(x) \sin \frac{2\pi n(x-\alpha)}{\beta-\alpha}\, dx \end{aligned}$$

with Parseval equality

$$\frac{2}{\beta-\alpha} \int_{\alpha}^{\beta} |f(x)|^2 dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2).$$

For our next variant of Fourier series it is convenient to consider the interval $[-\pi, \pi]$ and the Hilbert space $L^2[-\pi, \pi]$. This makes no difference in the formulas, since all elements of the space are 2π-periodic. Now suppose $f(t)$ is defined and square integrable on the interval $[0, \pi]$. We define $F(t) \in L^2[-\pi, \pi]$ by

$$F(t) = \begin{cases} f(t) & \text{on } [0, \pi] \\ f(-t) & \text{for } -\pi < t < 0 \end{cases}$$

The function F has been constructed so that it is *even*, i.e., $F(-t) = F(t)$. For an even functions the coefficients $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t) \sin nt \, dt = 0$ so

$$F(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$$

on $[-\pi, \pi]$ or

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt, \quad \text{for } 0 \le t \le \pi \tag{1.3.6}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t) \cos nt \, dt = \frac{2}{\pi}\int_0^{\pi} f(t) \cos nt \, dt.$$

Here, (1.3.6) is called the *Fourier cosine series* of f.

We can also extend the function $f(t)$ from the interval $[0, \pi]$ to an odd function on the interval $[-\pi, \pi]$. We define $G(t) \in L^2[-\pi, \pi]$ by

$$G(t) = \begin{cases} f(t) & \text{on } (0, \pi] \\ 0 & \text{for } t = 0 \\ -f(-t) & \text{for } -\pi < t < 0. \end{cases}$$

The function G has been constructed so that it is *odd*, i.e., $G(-t) = -G(t)$. For an odd function the coefficients $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} G(t) \cos nt \, dt = 0$ so

$$G(t) \sim \sum_{n=1}^{\infty} b_n \sin nt$$

on $[-\pi, \pi]$ or

$$f(t) \sim \sum_{n=1}^{\infty} b_n \sin nt, \quad \text{for } o < t \le \pi, \tag{1.3.7}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} G(t) \sin nt \, dt = \frac{2}{\pi}\int_0^{\pi} f(t) \sin nt \, dt.$$

Here, (1.3.7) is called the *Fourier sine series* of f.

Example 1 *Let* $f(t) = t, \quad 0 \le t \le \pi$.

1. *Fourier Sine series.*

$$b_n = \frac{2}{\pi}\int_0^\pi t \sin nt\ dt = \frac{-2t\cos nt}{n\pi}\ |_0^\pi + \frac{2}{n\pi}\int_0^\pi \cos nt\ dt = \frac{2(-1)^{n+1}}{n}.$$

Therefore,

$$t = \sum_{n=1}^\infty \frac{2(-1)^{n+1}}{n}\sin nt, \qquad 0 < t < \pi.$$

2. *Fourier Cosine series.*

$$a_n = \frac{2}{\pi}\int_0^\pi t \cos nt\ dt = \frac{2t\sin nt}{n\pi}\ |_0^\pi - \frac{2}{n\pi}\int_0^\pi \sin nt\ dt = \frac{2[(-1)^n - 1}{n^2\pi},$$

for $n \ge 1$ *and* $a_0 = \frac{2}{\pi}\int_0^\pi t\ dt = \pi$, *so*

$$t = \frac{\pi}{2} - \frac{4}{\pi}\sum_{j=1}^\infty \frac{\cos(2j-1)t}{(2j-1)^2}, \qquad 0 < t < \pi.$$

1.4 Convergence results

In this section we will prove Theorem 2 on pointwise convergence. Let f be a complex valued function such that

- $f(t)$ is periodic with period 2π.
- $f(t)$ is piecewise continuous on $[0, 2\pi]$.
- $f'(t)$ is piecewise continuous on $[0, 2\pi]$.

Expanding f in a Fourier series (real form) we have

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^\infty (a_n \cos nt + b_n \sin nt) = S(t), \qquad (1.4.8)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\ dt, \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\ dt.$$

For a fixed t we want to understand the conditions under which the Fourier series converges to a number $S(t)$, and the relationship between this number and f. To be more precise, let

$$S_k(t) = \frac{a_0}{2} + \sum_{n=1}^{k}(a_n \cos nt + b_n \sin nt)$$

be the k-th partial sum of the Fourier series. This is a finite sum, a *trigonometric polynomial*, so it is well defined for all $t \in R$. Now we have

$$S(t) = \lim_{k\to\infty} S_k(t),$$

if the limit exists. To better understand the properties of $S_k(t)$ in the limit, we will recast this finite sum as a single integral. Substituting the expressions for the Fourier coefficients a_n, b_n into the finite sum we find

$$S_k(t) = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx + \frac{1}{\pi}\sum_{n=1}^{k}\left(\int_0^{2\pi} f(x)\cos nx\ dx \cos nt + \int_0^{2\pi} f(x)\sin nx\ dx \sin nt\right),$$

so

$$\begin{aligned} S_k(t) &= \frac{1}{\pi}\int_0^{2\pi}\left[\frac{1}{2} + \sum_{n=1}^{k}(\cos nx \cos nt + \sin nx \sin nt\right] f(x)dx \\ &= \frac{1}{\pi}\int_0^{2\pi}\left[\frac{1}{2} + \sum_{n=1}^{k}\cos[n(t-x)]\right] f(x)dx \\ &= \frac{1}{\pi}\int_0^{2\pi} D_k(t-x)f(x)dx. \end{aligned} \tag{1.4.9}$$

We can find a simpler form for the kernel $D_k(t) = \frac{1}{2} + \sum_{n=1}^{k}\cos nt = -\frac{1}{2} + \sum_{m=0}^{k}\cos mt$. The last cosine sum is the real part of the geometric series

$$\sum_{m=0}^{k}(e^{it})^m = \frac{(e^{it})^{k+1} - 1}{e^{it} - 1}$$

so

$$-\frac{1}{2} + \sum_{m=0}^{k}\cos mt = -\frac{1}{2} + \mathrm{Re}\ \frac{(e^{it})^{k+1} - 1}{e^{it} - 1}$$

$$= \text{Re}\ \frac{(e^{it})^{k+1} - \frac{1}{2}e^{it} - \frac{1}{2}}{e^{it} - 1} = \text{Re}\ \frac{\frac{1}{2}(e^{-iu/2}e^{iu/2} - e^{i(k+1/2)u}}{e^{-iu/2} - e^{iu/2}}$$

$$= \text{Re}\ \frac{\sin(k+1/2)u + i(\cos u/2 - \cos(k+1/2)u)}{2\sin u/2}.$$

Thus,

$$D_k(t) = \frac{\sin(k+\frac{1}{2})t}{2\sin\frac{t}{2}}. \tag{1.4.10}$$

Note that D_k has the properties:

- $D_k(t) = D_k(t + 2\pi)$
- $D_k(-t) = D_k(t)$
- $D_k(t)$ is defined and differentiable for all t and $D_k(0) = k + \frac{1}{2}$.

From these properties it follows that the integrand of (1.4.9) is a 2π-periodic function of x, so that we can take the integral over any full 2π-period:

$$S_k(t) = \frac{1}{\pi}\int_{a-\pi}^{a+\pi} D_k(t-x)f(x)dx,$$

for any real number a. Let us set $a = t$ and fix a δ such that $0 < \delta < \pi$. (Think of δ as a very small positive number.) We break up the integral as follows:

$$S_k(t) = \frac{1}{\pi}\left(\int_{t-\pi}^{t-\delta} + \int_{t+\delta}^{t+\pi}\right) D_k(t-x)f(x)dx + \frac{1}{\pi}\int_{t-\delta}^{t+\delta} D_k(t-x)f(x)dx.$$

For fixed t we can write $D_k(t-x)$ in the form

$$D_k(t-x) = \frac{f_1(x,t)\cos k(t-x) + f_2(x,t)\sin k(t-x)}{\sin[\frac{1}{2}(t-x)]}$$

$$= g_1(x,t)\cos k(t-x) + g_2(x,t)\sin k(t-x).$$

In the interval $[t-\pi, t-\delta]$ the functions g_1, g_2 are bounded. Thus the functions

$$G_\ell(x,t) = \begin{cases} g_\ell(x,t) & \text{for } x \in [t-\pi, t-\delta] \\ 0 & \text{elsewhere} \end{cases}, \qquad \ell = 1, 2$$

are elements of $L^2[-\pi,\pi]$ (and its 2π-periodic extension). Thus, by the Riemann-Lebesgue Lemma, applied to the ON basis determied by the orthogonal functions $\cos k(t-x), \sin k(t-x)$ for fixed t, the first integral goes to 0 as $k \to \infty$. A similar argument shows that the integral over the interval $[t+\delta, t+\pi]$ goes to 0 as $k \to \infty$. [This argument doesn't hold for the interval $[t-\delta, t+\delta]$ because the term $\sin[\frac{1}{2}(t-x)]$ vanishes in the interval, so that the G_ℓ are not square integrable.] Thus,

$$\lim_{k\to\infty} S_k(t) = \lim_{k\to\infty} \frac{1}{\pi} \int_{t-\delta}^{t+\delta} D_k(t-x) f(x) dx, \tag{1.4.11}$$

where,

$$D_k(t) = \frac{\sin(k+\frac{1}{2})t}{2\sin\frac{t}{2}}.$$

Theorem 3 *Localization Theorem. The sum $S(t)$ of the Fourier series of f at t is completely determined by the behavior of f in an arbitrarily small interval $(t-\delta, t+\delta)$ about t.*

This is a remarkable fact! Although the Fourier coefficients contain information about all of the values of f over the interval $[0, 2\pi)$, only the local behavior of f affects the convergence at a specific point t.

Now we are ready to prove a basic pointwise convergence result. It makes use of another simple property of the kernal funtion $D_k(t)$:

Lemma 6

$$\int_0^{2\pi} D_k(x) dx = \pi$$

Proof;

$$\int_0^{2\pi} D_k(x) dx = \int_0^{2\pi} (\frac{1}{2} + \sum_{n=1}^{k} \cos nx) dx = \pi.$$

□

Theorem 4 *Suppose*

- *$f(t)$ is periodic with period 2π.*
- *$f(t)$ is piecewise continuous on $[0, 2\pi]$.*
- *$f'(t)$ is piecewise continuous on $[0, 2\pi]$.*

Then the Fourier series of $f(t)$ converges to $\frac{f(t+0)+f(t-0)}{2}$ at each point t.

Proof: It will be convenient to modify f, if necessary, so that

$$f(t) = \frac{f(t+0)+f(t-0)}{2}$$

at each point t. This condition affects the definition of f only at a finite number of points of discontinuity. It doesn't change any integrals and the values of the Fourier coefficients. Then, using Lemma 6 we can write

$$S_k(t) - f(t) = \frac{1}{\pi}\int_0^{2\pi} D_k(t-x)[f(x)-f(t)]dx$$

$$= \frac{1}{\pi}\int_0^{\pi} D_k(x)[f(t+x)+f(t-x)-2f(t)]dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{[f(t+x)+f(t-x)-2f(t)]}{2\sin\frac{x}{2}}\sin(k+\frac{1}{2})x\ dx$$

$$= \frac{1}{\pi}\int_0^{\pi} [H_1(t,x)\sin kx + H_2(t,x)\cos kx]dx$$

From the assumptions, H_1, H_2 are square integrable in x. Indeed, we can use L'Hospital's rule and the assumptions that f and f' are piecewise continuous to show that the limit

$$\lim_{x\to 0+} \frac{[f(t+x)+f(t-x)-2f(t)]}{2\sin\frac{x}{2}}$$

exists. Thus H_1, H_2 are bounded for $x \to 0+$. Then, by the Riemann-Lebesgue Lemma, the last expression goes to 0 as $k \to \infty$:

$$\lim_{k\to\infty}[S_k(t) - f(t)] = 0.$$

□

Exercise 3 *Suppose f is piecewise continuous and 2π-periodic. For any point t define the right-hand derivative $f'_R(t)$ and the left-hand derivative $f'_L(t)$ of f by*

$$f'_R(t) = \lim_{u\to t+}\frac{f(u)-f(t+0)}{u-t}, \quad f'_L(t) = \lim_{u\to t-}\frac{f(u)-f(t-0)}{u-t},$$

respectively. Show that in the proof of Theorem 4 we can drop the requirement that f' is piecewise continuous and the conclusion of the theorem will still hold at any point t such that both $f'_R(t)$ and $f'_L(t)$ exist.

Exercise 4 *Show that if f and f' are piecewise continuous then for any point t we have $f'(t+0) = f'_R(t)$ and $f'(t-0) = f'_L(t)$. Hint: By the mean value theorem of calculus, for $u > t$ and u sufficiently close to t there is a point c such that $t < c < u$ and*

$$\frac{f(u) - f(t+0)}{u-t} = f'(c).$$

Exercise 5 *Let*

$$f(t) = \begin{cases} t^2 \sin(\frac{1}{t}) & \text{for } t \neq 0 \\ 0 & \text{for } t = 0. \end{cases}$$

Show that f is continuous for all t and that $f'_R(0) = f'_L(0) = 0$. Show that $f'(t+0$ and $f'(t-0)$ do not exist for $t = 0$. Hence, argue that f' is not a piecewise continuous function. This shows that the result of Exercise 3 is a true strengthening of Theorem 4.

Exercise 6 *Let*

$$f(t) = \begin{cases} \frac{2\sin\frac{t}{2}}{t} & \text{if } 0 < |t| \le \pi, \\ 1 & \text{if } t = 0. \end{cases}$$

Extend f to be a 2π-periodic function on the entire real line. Verify that f satisfies the hypotheses of Theorem 4 and is continuous at $t = 0$. Apply the Localization Theorem (1.4.11) to f at $t = 0$ to give a new evaluation of the improper integral $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

1.4.1 Uniform pointwise convergence

We have shown that for functions f with the properties:

- $f(t)$ is periodic with period 2π,
- $f(t)$ is piecewise continuous on $[0, 2\pi]$,
- $f'(t)$ is piecewise continuous on $[0, 2\pi]$,

then at each point t the partial sums of the Fourier series of f,

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = S(t), \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\; dt$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\; dt,$$

converge to $\frac{f(t+0)+f(t-0)}{2}$:

$$S_k(t) = \frac{a_0}{2} + \sum_{n=1}^{k}(a_n \cos nt + b_n \sin nt), \qquad \lim_{k\to\infty} S_k(t) = \frac{f(t+0)+f(t-0)}{2}.$$

(If we require that f satisfies $f(t) = \frac{f(t+0)+f(t-0)}{2}$ at each point then the series will converge to f everywhere. In this section we will make this requirement.) Now we want to examine the rate of convergence.

We know that for every $\epsilon > 0$ we can find an integer $N(\epsilon, t)$ such that $|S_k(t) - f(t)| < \epsilon$ for every $k > N(\epsilon, t)|$. Then the finite sum trigonometric polynomial $S_k(t)$ will approximate $f(t)$ with an error $< \epsilon$. However, in general N depends on the point t; we have to recompute it for each t. What we would prefer is *uniform convergence*. The Fourier series of f will converge to f *uniformly* if for every $\epsilon > 0$ we can find an integer $N(\epsilon)$ such that $|S_k(t) - f(t)| < \epsilon$ for every $k > N(\epsilon)$ and *for all* t. Then the finite sum trigonometric polynomial $S_k(t)$ will approximate $f(t)$ everywhere with an error $< \epsilon$.

We cannot achieve uniform convergence for all functions f in the class above. The partial sums are continuous functions of t, Recall from calculus that if a sequence of continuous functions converges uniformly, the limit function is also continuous. Thus for any function f with discontinuities, we cannot have uniform convergence of the Fourier series.

If f is continuous, however, then we do have uniform convergence.

Theorem 5 *Assume f has the properties:*

- *$f(t)$ is periodic with period 2π,*
- *$f(t)$ is continuous on $[0, 2\pi]$,*
- *$f'(t)$ is piecewise continuous on $[0, 2\pi]$.*

Then the Fourier series of f converges uniformly.

Proof: : Consider the Fourier series of both f and f':

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt), \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\, dt$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\, dt,$$

$$f'(t) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos nt + B_n \sin nt), \qquad A_n = \frac{1}{\pi}\int_0^{2\pi} f'(t)\cos nt\ dt$$
$$B_n = \frac{1}{\pi}\int_0^{2\pi} f'(t)\sin nt\ dt,$$

Now

$$A_n = \frac{1}{\pi}\int_0^{2\pi} f'(t)\cos nt\ dt = \frac{1}{\pi}f(t)\cos nt|_0^{2\pi} + \frac{n}{\pi}\int_0^{2\pi} f(t)\sin nt\ dt$$
$$= \begin{cases} nb_n, & n \geq 1 \\ 0, & n = 0 \end{cases}$$

(We have used the fact that $f(0) = f(2\pi)$.) Similarly,

$$B_n = \frac{1}{\pi}\int_0^{2\pi} f'(t)\sin nt\ dt = \frac{1}{\pi}f(t)\sin nt|_0^{2\pi} - \frac{n}{\pi}\int_0^{2\pi} f(t)\cos nt\ dt = -na_n.$$

Using Bessel's inequality for f' we have

$$\sum_{n=1}^{\infty}(|A_n|^2 + |B_n|^2) \leq \frac{1}{\pi}\int_0^{2\pi} |f'(t)|^2 dt < \infty,$$

hence

$$\sum_{n=1}^{\infty} n^2(|a_n|^2 + |b_n|^2) \leq \infty.$$

Now

$$\begin{matrix}\sum_{n=1}^m |a_n| \\ \sum_{n=1}^m |b_n|\end{matrix} \leq \sum_{n=1}^{m}\sqrt{|a_n|^2 + |b_n|^2} = \sum_{n=1}^{m}\frac{1}{n}\sqrt{|A_n|^2 + |B_n|^2}$$
$$\leq (\sum_{n=1}^{m}\frac{1}{n^2})(\sum_{n=1}^{m}(|A_n|^2 + |B_n|^2))$$

which converges as $m \to \infty$. (We have used the Schwarz inequality for the last step.) Hence $\sum_{n=1}^{\infty}|a_n| < \infty$, $\sum_{n=1}^{\infty}|b_n| < \infty$. Now

$$|\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)| \leq |\frac{a_0}{2}| + \sum_{n=1}^{\infty}(|a_n \cos nt| + |b_n \sin nt|) \leq$$
$$|\frac{a_0}{2}| + \sum_{n=1}^{\infty}(|a_n| + |b_n|) < \infty,$$

so the series converges uniformly and absolutely. □

Corollary 1 *Parseval's Theorem. For f satisfying the hypotheses of the preceding theorem*

$$\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2) = \frac{1}{\pi}\int_0^{2\pi} |f(t)|^2 dt.$$

Proof: The Fourier series of f converges uniformly, so for any $\epsilon > 0$ there is an integer $N(\epsilon)$ such that $|S_k(t) - f(t)| < \epsilon$ for every $k > N(\epsilon)$ and *for all* t. Thus

$$\int_0^{2\pi} |S_k(t) - f(t)|^2\, dt = ||S_k - f||^2 = ||f||^2 - \pi(\frac{|a_0|^2}{2} + \sum_{n=1}^{k}(|a_n|^2 + |b_n|^2) < 2\pi\epsilon^2$$

for $k > N(\epsilon)$. $\Box$

Remark 1: Parseval's Theorem actually holds for any $f \in L^2[0, 2\pi]$, as we shall show later.

Remark 2: As the proof of the preceding theorem illustrates, differentiability of a function improves convergence of its Fourier series. The more derivatives the faster the convergence. There are famous examples to show that continuity alone is *not* sufficient for convergence.

1.5 More on pointwise convergence, Gibbs phenomena

Let's return to our Example 1 of Fourier series:

$$h(t) = \begin{cases} 0, & t = 0 \\ \frac{\pi - t}{2}, & 0 < t < 2\pi \\ 0, & t = 2\pi. \end{cases}$$

and $h(t + 2\pi) = h(t)$. In this case, $a_n = 0$ for all n and $b_n = \frac{1}{n}$. Therefore,

$$\frac{\pi - t}{2} = \sum_{n=1}^{\infty} \frac{\sin nt}{n}, \qquad 0 < t < 2\pi.$$

The function h has a discontinuity at $t = 0$ so the convergence of this series can't be uniform. Let's examine this case carefully. What happens to the partial sums near the discontinuity?

Here, $S_k(t) = \sum_{n=1}^{k} \frac{\sin nt}{n}$ so

$$S_k'(t) = \sum_{n=1}^{k} \cos nt = D_k(t) - \frac{1}{2} = \frac{\sin(k+\frac{1}{2})t}{2\sin\frac{t}{2}} - \frac{1}{2} = \frac{\sin\frac{kt}{2}\cos\frac{(k+1)t}{2}}{\sin\frac{t}{2}}.$$

Thus, since $S_k(0) = 0$ we have

$$S_k(t) = \int_0^t S_k'(x)dx = \int_0^t \left(\frac{\sin\frac{kx}{2}\cos\frac{(k+1)x}{2}}{2\sin\frac{x}{2}}\right) dx.$$

Note that $S_k'(0) > 0$ so that S_k starts out at 0 for $t = 0$ and then increases. Looking at the derivative of S_k we see that the first maximum is at the critical point $t_k = \frac{\pi}{k+1}$ (the first zero of $\cos\frac{(k+1)x}{2}$ as x increases from 0). Here, $h(t_k) = \frac{\pi - t_k}{2}$. The error is

$$S_k(t_k) - h(t_k) = \int_0^{t_k} \frac{\sin(k+\frac{1}{2})x}{2\sin\frac{x}{2}}dx - \frac{\pi}{2}$$

$$= \int_0^{t_k} \frac{\sin(k+\frac{1}{2})x}{x}dx + \int_0^{t_k} \left(\frac{1}{2\sin\frac{x}{2}} - \frac{1}{x}\right)\sin(k+\frac{1}{2})x\ dx - \frac{\pi}{2}$$

$$= I(t_k) + J(t_k) - \frac{\pi}{2}$$

where

$$I(t_k) = \int_0^{t_k} \frac{\sin(k+\frac{1}{2})x}{x}dx = \int_0^{(k+\frac{1}{2})t_k} \frac{\sin u}{u}du \to \int_0^{\pi} \frac{\sin u}{u}du \approx 1.851397052$$

(according to MAPLE). Also

$$J(t_k) = \int_0^{t_k} \left(\frac{1}{2\sin\frac{x}{2}} - \frac{1}{x}\right)\sin(k+\frac{1}{2})x\ dx.$$

Note that the integrand is bounded near $x = 0$, (indeed it goes to 0 as $x \to 0$) and the interval of integration goes to 0 as $k \to \infty$. Hence we have $J(t_k) \to 0$ as $k \to \infty$. We conclude that

$$\lim_{k\to\infty}(S_k(t_k) - h(t_k)) \approx 1.851397052 - \frac{\pi}{2} \approx .280600725$$

To sum up,$\lim_{k\to\infty} S_k(t_k) \approx 1.851397052$ whereas $\lim_{k\to\infty} h(t_k) = \frac{\pi}{2} \approx 1.570796327$. The partial sum is overshooting the correct value by about 17.86359697%! This is called the *Gibbs Phenomenon*.

To understand it we need to look more carefully at the convergence properties of the partial sums $S_k(t) = \sum_{n=1}^{k} \frac{\sin nt}{n}$ for all t.

First some preparatory calculations. Consider the geometric series $E_k(t) = \sum_{n=1}^{k} e^{int} = \frac{e^{it}(1-e^{ikt})}{1-e^{it}}$.

Lemma 7 *For* $0 < t < 2\pi$,

$$|E_k(t)| \le \frac{2}{|1-e^{it}|} = \frac{1}{\sin\frac{t}{2}}.$$

Note that $S_k(t)$ is the imaginary part of the complex series $\sum_{n=1}^{k} \frac{e^{int}}{n}$.

Lemma 8 *Let* $0 < \alpha < \beta < 2\pi$. *The series* $\sum_{n=1}^{\infty} \frac{e^{int}}{n}$ *converges uniformly for all* t *in the interval* $[\alpha, \beta]$.

Proof: (tricky)

$$\sum_{n=j}^{k} \frac{e^{int}}{n} = \sum_{n=j}^{k} \frac{E_n(t)-E_{n-1}(t)}{n} = \sum_{n=j}^{k} \frac{E_n(t)}{n} - \sum_{n=j}^{k} \frac{E_n(t)}{n+1} - \frac{E_{j-1}(t)}{j} + \frac{E_k(t)}{K+1}$$

and

$$|\sum_{n=j}^{k} \frac{e^{int}}{n}| \le \frac{1}{\sin\frac{t}{2}} \left(\sum_{n=j}^{k} (\frac{1}{n} - \frac{1}{n+1}) + \frac{1}{j} + \frac{1}{k+1} \right) = \frac{2}{j\sin\frac{t}{2}}.$$

This implies by the Cauchy Criterion that $\sum_{n=j}^{k} \frac{e^{int}}{n}$ converges uniformly on $[\alpha, \beta]$. □

This shows that the Fourier series for $h(t)$ converges uniformly on any closed interval that doesn't contain the discontinuities at $t = 2\pi\ell$, $\ell = 0, \pm1, \pm2, \cdots$. Next we will show that the partial sums $S_k(t)$ are bounded for *all* t and all k. Thus, even though there is an overshoot near the discontinuities, the overshoot is strictly bounded.

From the lemma on uniform convergence above we already know that the partial sums are bounded on any closed interval not containing a discontinuity. Also, $S_k(0) = 0$ and $S_k(-t) = -S_k(t)$, so it suffices to consider the interval $0 < t < \frac{\pi}{2}$.

We will use the facts that $\frac{2}{\pi} \leq \frac{\sin t}{t} \leq 1$ for $0 < t \leq \frac{\pi}{2}$. The right-hand inequality is a basic calculus fact and the left-hand one is obtained by solving the calculus problem of minimizing $\frac{\sin t}{t}$ over the interval $0 < t < \frac{\pi}{2}$. Note that

$$|\sum_{n=1}^{k} \frac{\sin nt}{n}| \leq |\sum_{1 \leq n < 1/t} \frac{t \sin nt}{nt}| + |\sum_{1/t \leq n \leq k} \frac{\sin nt}{n}|.$$

Using the calculus inequalities and the lemma, we have

$$|\sum_{n=1}^{k} \frac{\sin nt}{n}| \leq \sum_{1 \leq n < 1/t} t + \frac{2}{\frac{1}{t} \sin \frac{t}{2}} \leq t \cdot \frac{1}{t} + \frac{2}{\frac{1}{t} \frac{t}{2} \frac{2}{\pi}} = 1 + 2\pi.$$

Thus the partial sums are uniformly bounded for all t and all k.

We conclude that the Fourier series for $h(t)$ converges uniformly to $h(t)$ in any closed interval not including a discontinuity. Furthermore the partial sums of the Fourier series are uniformly bounded. At each discontinuity $t_N = 2\pi N$ of h the partial sums S_k overshoot $h(t_N + 0)$ by about 17.9% (approaching from the right) as $k \to \infty$ and undershoot $h(t_N - 0)$ by the same amount.

All of the work that we have put into this single example will pay off, because the facts that have emerged are of broad validity. Indeed we can consider any function f satisfying our usual conditions as the sum of a continuous function for which the convergence is uniform everywhere and a finite number of translated and scaled copies of $h(t)$.

Theorem 6 *Let f be a complex valued function such that*

- *$f(t)$ is periodic with period 2π.*
- *$f(t)$ is piecewise continuous on $[0, 2\pi]$.*
- *$f'(t)$ is piecewise continuous on $[0, 2\pi]$.*
- *$f(t) = \frac{f(t+0)+f(t-0)}{2}$ at each point t.*

Then

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

pointwise. The convergence of the series is uniform on every closed interval in which f is continuous.

Proof: let $x_1, x_2, \cdots, x_\ell$ be the points of discontinuity of f in $[0, 2\pi)$ Set $s(x_j) = f(x_j + 0) - f(x_j - 0)$. Then the function

$$g(t) = f(t) - \sum_{j=1}^{\ell} \frac{s(x_j)}{\pi} h(t - x_j)$$

is everywhere continuous and also satisfies all of the hypotheses of the theorem. Indeed, at the discontinuity x_j of f we have

$$g(x_j - 0) = f(x_j - 0) - \frac{2s(x_j)}{\pi} h(-0) = f(x_j - 0) - \frac{f(x_j + 0) - f(x_j - 0))}{\pi} (\frac{-\pi}{2})$$

$$= \frac{f(x_j - 0) + f(x_j + 0)}{2} = f(x_j).$$

Similarly, $g(x_j + 0) = f(x_j)$. Therefore $g(t)$ can be expanded in a Fourier series that converges absolutely and uniformly. However, $\sum_{j=1}^{\ell} \frac{s(x_j)}{\pi} h(t - x_j)$ can be expanded in a Fourier series that converges pointwise and uniformly in every closed interval that doesn't include a discontinuity. But

$$f(t) = g(t) + \sum_{j=1}^{\ell} \frac{s(x_j)}{\pi} h(t - x_j),$$

and the conclusion follows. □

Corollary 2 *Parseval's Theorem. For f satisfying the hypotheses of the preceding theorem*

$$\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2) = \frac{1}{\pi} \int_0^{2\pi} |f(t)|^2 dt.$$

Proof; As in the proof of the theorem, let $x_1, x_2, \cdots, x_\ell$ be the points of discontinuity of f in $[0, 2\pi)$ and set $s(x_j) = f(x_j + 0) - f(x_j - 0)$. Choose $a \geq 0$ such that the discontinuities are contained in the interior of $I = (-a, 2\pi - a)$. From our earlier results we know that the partial sums of the Fourier series of h are uniformly bounded with bound $M > 0$. Choose $P = \sup_{t \in [0, 2\pi]} |f(t)|$. Then $|S_k(t) - f(t)|^2 \leq (M + P)^2$ for all t and all k. Given $\epsilon > 0$ choose non-overlapping open intervals $I_1, I_2, \cdots, I_\ell \subset I$ such that $x_j \in I_j$ and $(\sum_{j=1}^{\ell} |I_j|)(M + P)^2 < \frac{\epsilon}{2}$. Here, $|I_j|$ is the length of the interval I_j. Now the Fourier series of f converges uniformly on the closed set $A = [-a, 2\pi - a] - I_1 \cup I_2 \cup \cdots \cup I_\ell$. Choose an integer $N(\epsilon)$ such that $|S_k(t) - f(t)|^2 < \frac{\epsilon}{4\pi}$ for all $t \in A, k \geq N(\epsilon)$. Then

$$\int_0^{2\pi} |S_k(t) - f(t)|^2 dt = \int_{-a}^{2\pi - a} |S_k(t) - f(t)|^2 dt =$$

$$\int_A |S_k(t) - f(t)|^2 dt + \int_{I_1 \cup I_2 \cup \cdots \cup I_\ell} |S_k(t) - f(t)|^2 dt$$

$$< 2\pi \frac{\epsilon}{4\pi} + (\sum_{j=1}^{\ell} |I_j|)(M + P)^2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\lim_{k\to\infty} ||S_k - f|| = 0$ and the partial sums converge to f in the mean.

Furthermore,

$$\epsilon > \int_0^{2\pi} |S_k(t) - f(t)|^2 \, dt = ||S_k - f||^2 = ||f||^2 - \pi(\frac{|a_0|^2}{2} + \sum_{n=1}^{k}(|a_n|^2 + |b_n|^2)$$

for $k > N(\epsilon)$. □

1.6 Mean convergence, Parseval's equality, integration and differentiation of Fourier series

The convergence theorem and the version of the Parseval identity proved in the previous section apply to step functions on $[0, 2\pi]$. However, we already know that the space of step functions on $[0, 2\pi]$ is dense in $L^2[0, 2\pi]$. Since every step function is the limit in the norm of the partial sums of its Fourier series, this means that the space of all finite linear combinations of the functions $\{e^{int}\}$ is dense in $L^2[0, 2\pi]$. Hence $\{e^{int}/\sqrt{2\pi}\}$ is an ON basis for $L^2[0, 2\pi]$ and we have

Theorem 7 *Parseval's Equality (strong form) [Plancherel Theorem]. If $f \in L^2[0, 2\pi]$ then*

$$\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2) = \frac{1}{\pi} \int_0^{2\pi} |f(t)|^2 dt,$$

where a_n, b_n are the Fourier coefficients of f.

Integration of a Fourier series term-by-term yields a series with improved convergence.

Theorem 8 *Let f be a complex valued function such that*

- *$f(t)$ is periodic with period 2π.*
- *$f(t)$ is piecewise continuous on $[0, 2\pi]$.*
- *$f'(t)$ is piecewise continuous on $[0, 2\pi]$.*

Let

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

be the Fourier series of f. Then

$$\int_0^t f(x)dx = \frac{a_0}{2}t + \sum_{n=1}^{\infty} \frac{1}{n}\left[a_n \sin nt - b_n(\cos nt - 1)\right].$$

where the convergence is uniform on the interval $[0, 2\pi]$.

Proof: Let $F(t) = \int_0^t f(x)dx - \frac{a_0}{2}t$. Then

- $F(2\pi) = \int_0^{2\pi} f(x)dx - \frac{a_0}{2}(2\pi) = 0 = F(0)$.
- $F(t)$ is continuous on $[0, 2\pi]$.
- $F'(t) = f(t) - \frac{a_0}{2}$ is piecewise continuous on $[0, 2\pi]$.

Thus the Fourier series of F converges to F uniformly and absolutely on $[0, 2\pi]$:

$$F(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos nt + B_n \sin nt).$$

Now

$$A_n = \frac{1}{\pi}\int_0^{2\pi} F(t)\cos nt\; dt = \left.\frac{F(t)\sin nt}{n\pi}\right|_0^{2\pi} - \frac{1}{n\pi}\int_0^{2\pi}(f(t) - \frac{a_0}{2})\sin nt\; dt$$

$$= -\frac{b_n}{n}, \qquad n \neq 0,$$

and

$$B_n = \frac{1}{\pi}\int_0^{2\pi} F(t)\sin nt\ dt = -\frac{F(t)\cos nt}{n\pi}\Big|_0^{2\pi} + \frac{1}{n\pi}\int_0^{2\pi}(f(t) - \frac{a_0}{2})\cos nt\ dt$$

$$= \frac{a_n}{n}.$$

Therefore,

$$F(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(-\frac{b_n}{n}\cos nt + \frac{a_n}{n}\sin nt),$$

$$F(2\pi) = 0 = \frac{A_0}{2} - \sum_{n=1}^{\infty}\frac{b_n}{n}.$$

Solving for A_0 we find

$$F(t) = \int_0^t f(x)dx - \frac{a_0}{2}t = \sum_{n=1}^{\infty}\frac{1}{n}\left[a_n \sin nt - b_n(\cos nt - 1)\right].$$

□

Example 2 *Let*

$$f(t) = \begin{cases} \frac{\pi - t}{2} & 0 < t < 2\pi \\ 0 & t = 0, 2\pi. \end{cases}$$

Then

$$\frac{\pi - t}{2} \sim \sum_{n=1}^{\infty}\frac{\sin nt}{n}.$$

Integrating term-by term we find

$$\frac{2\pi t - t^2}{4} = -\sum_{n=1}^{\infty}\frac{1}{n^2}(\cos nt - 1), \quad 0 \le t \le 2\pi.$$

Differentiation of Fourier series, however, makes them less smooth and may not be allowed. For example, differentiating the Fourier series

$$\frac{\pi - t}{2} \sim \sum_{n=1}^{\infty}\frac{\sin nt}{n},$$

formally term-by term we get

$$-\frac{1}{2} \sim \sum_{n=1}^{\infty} \cos nt,$$

which doesn't converge on $[0, 2\pi]$. In fact it can't possible be a Fourier series for an element of $L^2[0, 2\pi]$. (Why?)

If f is sufficiently smooth and periodic it *is* OK to differentiate term-by-term to get a new Fourier series.

Theorem 9 *Let f be a complex valued function such that*

- *$f(t)$ is periodic with period 2π.*
- *$f(t)$ is continuous on $[0, 2\pi]$.*
- *$f'(t)$ is piecewise continuous on $[0, 2\pi]$.*
- *$f''(t)$ is piecewise continuous on $[0, 2\pi]$.*

Let

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

be the Fourier series of f. Then at each point $t \in [0, 2\pi]$ where $f''(t)$ exists we have

$$f'(t) = \sum_{n=1}^{\infty} n \left[-a_n \sin nt + b_n \cos nt\right].$$

Proof: By the Fourier convergence theorem the Fourier series of f' converges to $\frac{f'(t_0+0)+f'(t_0-0)}{2}$ at each point t_0. If $f''(t_0)$ exists at the point then the Fourier series converges to $f'(t_0)$, where

$$f'(t) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos nt + B_n \sin nt).$$

Now

$$A_n = \frac{1}{\pi} \int_0^{2\pi} f'(t) \cos nt \; dt = \left.\frac{f(t) \cos nt}{\pi}\right|_0^{2\pi} + \frac{n}{\pi} \int_0^{2\pi} f(t) \sin nt \; dt$$

$$= nb_n,$$

$A_0 = \frac{1}{\pi}\int_0^{2\pi} f'(t)dt = \frac{1}{\pi}(f(2\pi) - f(0)) = 0$ (where, if necessary, we adjust the interval of length 2π so that f' is continuous at the endpoints) and

$$B_n = \frac{1}{\pi}\int_0^{2\pi} f'(t)\sin nt\ dt = \left.\frac{f(t)\sin nt}{\pi}\right|_0^{2\pi} - \frac{n}{\pi}\int_0^{2\pi} f(t)\cos nt\ dt$$
$$= -na_n.$$

Therefore,

$$f'(t) \sim \sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt).$$

□

Note the importance of the requirement in the theorem that f is continuous everywhere and periodic, so that the boundary terms vanish in the integration by parts formulas for A_n and B_n. Thus it is OK to differentiate the Fourier series

$$f(t) = \frac{2\pi t - t^2}{4} - \frac{\pi^2}{6} = -\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nt, \quad 0 \le t \le 2\pi$$

term-by term, where $f(0) = f(2\pi)$, to get

$$f'(t) = \frac{\pi - t}{2} \sim \sum_{n=1}^{\infty}\frac{\sin nt}{n}.$$

However, even though $f'(t)$ is infinitely differentiable for $0 < t < 2\pi$ we have $f'(0) \neq f'(2\pi)$, so we cannot differentiate the series again.

1.7 Arithmetic summability and Fejér's theorem

We know that the kth partial sum of the Fourier series of a square integrable function f:

$$S_k(t) = \frac{a_0}{2} + \sum_{n=1}^{k}(a_n \cos nt + b_n \sin nt)$$

is the trigonometric polynomial of order k that best approximates f in the Hilbert space sense. However, the limit of the partial sums

$$S(t) = \lim_{k\to\infty} S_k(t),$$

doesn't necessarily converge pointwise. We have proved pointwise convergence for piecewise smooth functions, but if, say, all we know is that f is continuous then pointwise convergence is much harder to establish. Indeed there are examples of continuous functions whose Fourier series diverges at uncountably many points. Furthermore we have seen that at points of discontinuity the Gibbs phenomenon occurs and the partial sums overshoot the function values. In this section we will look at another way to recapture $f(t)$ from its Fourier coefficients, by Cesàro sums (arithmetic means). This method is surprisingly simple, gives uniform convergence for continuous functions $f(t)$ and avoids most of the Gibbs phenomena difficulties.

The basic idea is to use the arithmetic means of the partial sums to approximate f. Recall that the kth partial sum of $f(t)$ is defined by

$$S_k(t) = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx \quad +$$

$$\frac{1}{\pi}\sum_{n=1}^{k}\left(\int_0^{2\pi} f(x)\cos nx\ dx\cos nt + \int_0^{2\pi} f(x)\sin nx\ dx\sin nt\right),$$

so

$$\begin{aligned} S_k(t) &= \frac{1}{\pi}\int_0^{2\pi}\left[\frac{1}{2}+\sum_{n=1}^{k}(\cos nx\cos nt+\sin nx\sin nt\right] f(x)dx \\ &= \frac{1}{\pi}\int_0^{2\pi}\left[\frac{1}{2}+\sum_{n=1}^{k}\cos[n(t-x)]\right] f(x)dx \\ &= \frac{1}{\pi}\int_0^{2\pi} D_k(t-x)f(x)dx. \end{aligned}$$

where the kernel $D_k(t) = \frac{1}{2}+\sum_{n=1}^{k}\cos nt = -\frac{1}{2}+\sum_{m=0}^{k}\cos mt$. Further,

$$D_k(t) = \frac{\cos kt - \cos(k+1)t}{4\sin^2\frac{t}{2}} = \frac{\sin(k+\frac{1}{2})t}{2\sin\frac{t}{2}}.$$

Rather than use the partial sums $S_k(t)$ to approximate $f(t)$ we use the arithmetic means $\sigma_k(t)$ of these partial sums:

$$\sigma_k(t) = \frac{S_0(t)+S_1(t)+\cdots+S_{k-1}(t)}{k}, \qquad k=1,2,\cdots. \tag{1.7.12}$$

Then we have

$$\sigma_k(t) = \frac{1}{k\pi}\sum_{j=0}^{k-1}\int_0^{2\pi} D_j(t-x)f(x)dx = \int_0^{2\pi}\left[\frac{1}{k\pi}\sum_{j=0}^{k-1} D_j(t-x)\right]f(x)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} F_k(t-x)f(x)dx \qquad (1.7.13)$$

where

$$F_k(t) = \frac{1}{k}\sum_{j=0}^{k-1} D_j(t) = \frac{1}{k}\sum_{j=0}^{k-1}\frac{\sin(j+\frac{1}{2})t}{2\sin\frac{t}{2}}.$$

Lemma 9

$$F_k(t) = \frac{1}{k}\left(\frac{\sin kt/2}{\sin t/2}\right)^2.$$

Proof: Using the geometric series, we have

$$\sum_{j=0}^{k-1} e^{i(j+\frac{1}{2})t} = e^{i\frac{t}{2}}\frac{e^{ikt}-1}{e^{it}-1} = e^{i\frac{kt}{2}}\frac{\sin\frac{kt}{2}}{\sin\frac{t}{2}}.$$

Taking the imaginary part of this identity we find

$$F_k(t) = \frac{1}{k\sin\frac{t}{2}}\sum_{j=0}^{k-1}\sin(j+\frac{1}{2})t = \frac{1}{k}\left(\frac{\sin kt/2}{\sin t/2}\right)^2.$$

□

Note that F has the properties:

- $F_k(t) = F_k(t+2\pi)$
- $F_k(-t) = F_k(t)$
- $F_k(t)$ is defined and differentiable for all t and $F_k(0) = k$
- $F_k(t) \geq 0$.

From these properties it follows that the integrand of (1.7.13) is a 2π-periodic function of x, so that we can take the integral over any full 2π-period. Finally, we can change variables and divide up the integral, in analogy with our study of the Fourier kernel $D_k(t)$, and obtain the following simple expression for the arithmetic means:

Lemma 10

$$\sigma_k(t) = \frac{2}{k\pi}\int_0^{\pi/2} \frac{f(t+2x)+f(t-2x)}{2}\left(\frac{\sin kx}{\sin x}\right)^2 dx.$$

Exercise 7 *Derive Lemma 10 from expression(1.7.13) and Lemma 9.*

Lemma 11

$$\frac{2}{k\pi}\int_0^{\pi/2}\left(\frac{\sin kx}{\sin x}\right)^2 dx = 1.$$

Proof: Let $f(t) \equiv 1$ for all t. Then $\sigma_k(t) \equiv 1$ for all k and t. Substituting into the expression from lemma 10 we obtain the result. □

Theorem 10 *(Fejér) Suppose $f(t) \in L^1[0, 2\pi]$, periodic with period 2π and let*

$$\sigma(t) = \lim_{x\to 0_+} \frac{f(t+x)+f(t-x)}{2} = \frac{f(t+0)+f(t-0)}{2}$$

whenever the limit exists. For any t such that $\sigma(t)$ is defined we have

$$\lim_{k\to\infty}\sigma_k(t) = \sigma(t) = \frac{f(t+0)+f(t-0)}{2}.$$

Proof: From lemmas 10 and 11 we have

$$\sigma_k(t) - \sigma(t) = \frac{2}{k\pi}\int_0^{\pi/2}\left[\frac{f(t+2x)+f(t-2x)}{2} - \sigma(t)\right]\left(\frac{\sin kx}{\sin x}\right)^2 dx.$$

For any t for which $\sigma(t)$ is defined, let $G_t(x) = \frac{f(t+2x)+f(t-2x)}{2} - \sigma(t)$. Then $G_t(x) \to 0$ as $t \to 0$ through positive values. Thus, given $\epsilon > 0$ there is a $\delta < \pi/2$ such that $|G_t(x)| < \epsilon/2$ whenever $0 < x \leq \delta$. We have

$$\sigma_k(t) - \sigma(t) = \frac{2}{k\pi}\int_0^{\delta} G_t(x)\left(\frac{\sin kx}{\sin x}\right)^2 dx + \frac{2}{k\pi}\int_{\delta}^{\pi/2} G_t(x)\left(\frac{\sin kx}{\sin x}\right)^2 dx.$$

Now

$$\left|\frac{2}{k\pi}\int_0^{\delta} G_t(x)\left(\frac{\sin kx}{\sin x}\right)^2 dx\right| \leq \frac{\epsilon}{k\pi}\int_0^{\pi/2}\left(\frac{\sin kx}{\sin x}\right)^2 dx = \frac{\epsilon}{2}$$

and

$$\left|\frac{2}{k\pi}\int_{\delta}^{\pi/2} G_t(x)\left(\frac{\sin kx}{\sin x}\right)^2 dx\right| \le \frac{2}{k\pi\sin^2\delta}\int_{\delta}^{\pi/2}|G_t(x)|dx \le \frac{2I}{k\pi\sin^2\delta},$$

where $I = \int_0^{\pi/2}|G_t(x)|dx$. This last integral exists because F is in L^1. Now choose K so large that $2I/(N\pi\sin^2\delta) < \epsilon/2$. Then if $k \ge K$ we have

$$|\sigma_k(t)-\sigma(t)| \le \left|\frac{2}{k\pi}\int_0^{\delta} G_t(x)\left(\frac{\sin kx}{\sin x}\right)^2 dx\right| + \left|\frac{2}{k\pi}\int_{\delta}^{\pi/2} G_t(x)\left(\frac{\sin kx}{\sin x}\right)^2 dx\right| < \epsilon.$$

□

Corollary 3 *Suppose $f(t)$ satisfies the hypotheses of the theorem and also is continuous on the closed interval $[a,b]$. Then the sequence of arithmetic means $\sigma_k(t)$ converges uniformly to $f(t)$ on $[a,b]$.*

Proof: If f is continuous on the closed bounded interval $[a,b]$ then it is uniformly continuous on that interval and the function G_t is bounded on $[a,b]$ with upper bound M, independent of t. Furthermore one can determine the δ in the preceding theorem so that $|G_t(x)| < \epsilon/2$ whenever $0 < x \le \delta$ and uniformly for all $t \in [a,b]$. Thus we can conclude that $\sigma_k \to \sigma$, uniformly on $[a,b]$. Since f is continuous on $[a,b]$ we have $\sigma(t) = f(t)$ for all $t \in [a,b]$. □

Corollary 4 *(Weierstrass approximation theorem) Suppose $f(t)$ is real and continuous on the closed interval $[a,b]$. Then for any $\epsilon > 0$ there exists a polynomial $p(t)$ such that*

$$|f(t) - p(t)| < \epsilon$$

for every $t \in [a,b]$.

Sketch of Proof: Using the methods of Section 1.3 we can find a linear transformation to map $[a,b]$ one-to-one on a closed subinterval $[a',b']$ of $[0,2\pi]$, such that $0 < a' < b' < 2\pi$. This transformation will take polynomials in t to polynomials. Thus, without loss of generality, we can assume $0 < a < b < 2\pi$. Let $g(t) = f(t)$ for $a \le t \le b$ and define $g(t)$ outside that interval so that it is continuous at $T = a, b$ and is periodic with period 2π. Then from the first corollary to Fejér's theorem, given an $\epsilon > 0$ there is an integer N and arithmetic sum

$$\sigma(t) = \frac{A_0}{2} + \sum_{j=1}^{N}(A_j \cos jt + B_j \sin jt)$$

such that $|f(t) - \sigma(t)| = |g(t) - \sigma(t)| < \frac{\epsilon}{2}$ for $a \le t \le b$. Now $\sigma(t)$ is a trigonometric polynomial and it determines a poser series expansion in t about the origin that converges uniformly on every finite interval. The partial sums of this power series determine a series of polynomials $\{p_n(t)\}$ of order n such that $p_n \to \sigma$ uniformly on $[a, b]$, Thus there is an M such that $|\sigma(t) - p_M(t)| < \frac{\epsilon}{2}$ for all $t \in [a, b]$. Thus

$$|f(t) - p_M(t)| \le |f(t) - \sigma(t)| + |\sigma(t) - p_M(t)| < \epsilon$$

for all $t \in [a, b]$. □

This important result implies not only that a continuous function on a bounded interval can be approximated uniformly by a polynomial function but also (since the convergence is uniform) that continuous functions on bounded domains can be approximated with arbitrary accuracy in the L^2 norm on that domain. Indeed the space of polynomials is dense in that Hilbert space.

Another important offshoot of approximation by arithmetic sums is that the Gibbs phenomenon doesn't occur. This follows easily from the next result.

Lemma 12 *Suppose the 2π-periodic function $f(t) \in L^2[-\pi, \pi]$ is bounded, with $M = \sup_{t\in[-\pi,\pi]} |f(t)|$. Then $|\sigma_n(t)| \le M$ for all t.*

Proof: From (1.7.13) and (10) we have

$$|\sigma_k(t)| \le \frac{1}{2k\pi}\int_0^{2\pi} |f(t+x)| \left(\frac{\sin kx/2}{\sin x/2}\right)^2 dx \le \frac{M}{2k\pi}\int_0^{2\pi} \left(\frac{\sin kx/2}{\sin x/2}\right)^2 dx = M.$$

Q.E.D.

Now consider the example which has been our prototype for the Gibbs phenomenon:

$$h(t) = \begin{cases} 0, & t = 0 \\ \frac{\pi - t}{2}, & 0 < t < 2\pi \\ 0, & t = 2\pi. \end{cases}$$

and $h(t+2\pi) = h(t)$. Here the ordinary Fourier series gives

$$\frac{\pi - t}{2} = \sum_{n=1}^{\infty} \frac{\sin nt}{n}, \qquad 0 < t < 2\pi.$$

and this series exhibits the Gibbs phenomenon near the simple discontinuities at integer multiples of 2π. Furthermore the supremum of $|h(t)|$ is $\pi/2$ and it approaches the values $\pm\pi/2$ near the discontinuities. However, the lemma shows that $|\sigma(t)| < \pi/2$ for all n and t. Thus the arithmetic sums never overshoot or undershoot as t approaches the discontinuities, so there is no Gibbs phenomenon in the arithmetic series for this example.

In fact, the example is universal; there is no Gibbs phenomenon for arithmetic sums. To see this, we can mimic the proof of Theorem 6. This then shows that the arithmetic sums for all piecewise smooth functions converge uniformly except in arbitrarily small neighborhoods of the discontinuities of these functions. In the neighborhood of each discontinuity the arithmetic sums behave exactly as does the series for $h(t)$. Thus there is no overshooting or undershooting.

Remark 1 *The pointwise convergence criteria for the arithmetic means are much more general (and the proofs of the theorems are simpler) than for the case of ordinary Fourier series. Further, they provide a means of getting around the most serious problems caused by the Gibbs phenomenon. The technical reason for this is that the kernel function $F_k(t)$ is nonnegative. Why don't we drop ordinary Fourier series and just use the arithmetic means? There are a number of reasons, one being that the arithmetic means $\sigma_k(t)$ are not the best L^2 approximations for order k, whereas the $S_k(t)$ are the best L^2 approximations. There is no Parseval theorem for arithmetic means. Further, once the approximation $S_k(t)$ is computed for ordinary Fourier series, in order to get the next level of approximation one needs only to compute two more constants:*

$$S_{k+1}(t) = S_k(t) + a_{k+1}\cos(k+1)t + b_{k+1}\sin(k+1)t.$$

However, for the arithmetic means, in order to update $\sigma_k(t)$ to $\sigma_{k+1}(t)$ one must recompute ALL of the expansion coefficients. This is a serious practical difficulty.

1.8 Additional Exercises

Exercise 8 *(1) Let X be the Eucldean space $\mathbb{C}^n$ of all $n \geq 1$ tuples $(x_1, .., x_n)$ of complex numbers. For $x, y \in \mathbb{X}$, define*

$$(x.y) = \sum_{j=1}^{n} x_j \overline{y_j}$$

where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. Show that $(; .;)$ satisfies the properties of positivity, homogeneity and linearity and with symmetry, replaced by compex symmetry $(x, y) = \overline{(x, y)}$. Hence $(; .;)$ defines a compex inner product on X.

(2) Define $||x|| = \sqrt{(x, x)}$ for $x \in X$ and verify directly that (1) $||; ||$ is a norm on X, (2) that the Cauchy-Schwartz inequality

$$|(x, y)| \leq ||x|| ||y||$$

holds on X and (3) that the parallelogram law

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2$$

holds on X.

Exercise 9 *Let X b the space $R[-\pi, \pi]$ with inner product*

$$(f, g) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx.$$

Show that the sequence of functions

$$\left\{ \frac{1}{\sqrt{2}}, \cos(x), \sin(x), \cos(2x), \sin(2x), ... \right\}$$

is an orthonormal sequence in X.

Exercise 10 *Let $X = R[-1, 1]$ with inner product*

$$(f, g) = \int_{-1}^{1} f(x)g(x)dx.$$

Find polynomials P_0, P_1, P_2 of degree $0, 1, 2$ respectively such that for $j, k = 0, 1, 2$

$$(P_j, P_k) = \delta_{j,k} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

Hint: First find $P_0(x) = c$ satisfying $(P_0, P_0) = 1$. Next, find $P_1(x) = a + bx$ satisfying

$$(P_1, P_0) = 0,\ (P_1, P_1) = 1.$$

Next, find $P_2(x) = d + ex + fx^2$ satisfying

$$(P_2, P_0) = (P_2, P_1) = 0,\ (P_2, P_2) = 1.$$

This procedure can be continued indefinitely and is called the **Gram-Schmidt** *process.*

Exercise 11 *Let $f(x) = x$, $x \in [-\pi, \pi]$.*

(a) Show that $f(x)$ has the Fourier series

$$\sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n-1} \sin(nx).$$

(b) Let $\alpha > 0$. Show that $f(x) = \exp(\alpha x)$, $x \in [-\pi, \pi]$ has the Fourier series

$$\left(\frac{e^{\alpha\pi} - e^{-\alpha\pi}}{\pi}\right) \left(\frac{1}{2\alpha} + \sum_{k=1}^{\infty} (-1)^k \frac{\alpha \cos(kx) - k \sin(kx)}{\alpha^2 + k^2}\right).$$

(c) Let α be any real number other than an integer. Let $f(x) = \cos(\alpha x)$, $x \in [-\pi, \pi]$. Show that f has a Fourier series

$$\frac{\sin(\alpha\pi)}{\alpha\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\frac{\sin(\alpha + n)\pi}{\alpha + n} + + \frac{\sin(\alpha - n)\pi}{\alpha - n}\right] \cos(nx).$$

(d) Find the Fourier series of $f(x) = -\alpha \sin(\alpha x)$, $x \in [-\pi, \pi]$. Do you notice any relationship to that in (c)?

(e) Find the Fourier series of $f(x) = |x|$, $x \in [-\pi, \pi]$.

(f) *Find the Fourier series of*

$$f(x) = \operatorname{sign}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Do you notice any relationship to that in (e)?

Exercise 12 *Show that if $f \in R[-\pi, \pi]$ is odd, then all its Fourier cosine coefficients $a_n = 0$, $n \geq 0$ so f has a Fourier series $\sum_{n=1}^{\infty} b_n \sin(nx)$. Likewise, if f is even, show that all its Fourier sine coeffcients $b_n = 0$, $n \geq 0$ and f has a Fourier series $a_0/2 + \sum_{n=1}^{\infty} a_n \cos(nx)$.*

Exercise 13 *By substituting special values of x in convergent Fourier series, we can often deduce interesting series expansions for various numbers, or just find the sum of important series.*

(a) *Use the Fourier series for $f(x) = x^2$ to show that*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

(b) *Prove that the Fourier series for $f(x) = x$ converges in $(-\pi, \pi)$ and hence that*

$$\sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} = \frac{\pi}{4}.$$

(c) *Show that the Fourier series of $f(x) = \exp(\alpha x)$, $x \in [-\pi, \pi)$ converges in $[-\pi, \pi)$ to $\exp(\alpha x)$ and at $x = \pi$ to $\frac{\exp(\alpha\pi)+\exp(-\alpha\pi)}{2}$. Hence show that*

$$\frac{\alpha\pi}{\tanh(\alpha\pi)} = 1 + \sum_{k=1}^{\infty} \frac{2\alpha^2}{k^2+\alpha^2}.$$

(d) *Show that*

$$\frac{\pi}{4} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n^2-1}.$$

(e) *Show that*

$$\sum_{l=0}^{\infty} \frac{1}{(2l+1)^2} = \frac{\pi^2}{8}.$$

Exercise 14 *Let $f \in R[-\pi,\pi]$ be 2π periodic and let σ_n, $n = 0, 1, 3, ...$ denote its Fejer means. Show that*

$$\max\{|\sigma_n(x)| : x \in [-\pi,\pi]\} \leq sup\{|f(x)| : x \in [-\pi,\pi]\}.$$

Exercise 15 *The following exercise concerns an extension of Parsevals formula to $R[-\pi,\pi]$. Consider $R[-\pi,\pi]$ with inner product*

$$(f,g) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx,\ f,g \in R[-\pi,\pi]$$

and $||f|| = \sqrt{(f,f)}$. Fix $f \in R[-\pi,\pi]$ and $\varepsilon > 0$.

(a) *Because $f \in R[-\pi,\pi]$, choose a partition*

$$-\pi = x_0 < x_1 < x_2 < ... < x_n = \pi$$

such that if

$$M_j := sup\{f(x) : x \in [x_{j-1}, x_j]\},\ m_j := inf\{f(x) : x \in [x_{j-1}, x_j]\},\ j = 1, ..., n$$

then

$$\sum_{j=1}^{n}(M_j - m_j)(x_j - x_{j-1}) < \varepsilon.$$

Now define $g(x) := M_j$, $x \in [x_{j-1}, x_j)$, $1 \leq j \leq n$ and $g(b)$ arbitrarily. Show that

$$\int_{-\pi}^{\pi} |f(x) - g(x)|dx < \varepsilon.$$

(b) *Show that there is a continuous function $h : [-\pi,\pi] \to \mathbb{R}$ such that*

$$\int_{-\pi}^{\pi} |g(x) - h(x))|dx < \varepsilon$$

and

$$|h(x)| \leq M = \max\{|M_1|, ..., |M_n|\},\ x \in [-\pi,\pi].$$

Hint: Sketch the graph of g and modify g near each x_j.

(c) *Deduce that*

$$\int_{-\pi}^{\pi} |f(x) - h(x))|dx < 2\varepsilon$$

and

$$||f - h||^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x) - h(x))|^2 dx < \frac{4\varepsilon}{\pi} M*$$

where $M^* = \sup\{|f(x)| : x \in [a, b)\}$.

(d) *Deduce that given any* $f \in R[-\pi, \pi]$ *and* $\delta > 0$, *we can find a continuous* $h : [-\pi, \pi] \to \mathbb{R}$ *such that* $||f - h|| < \delta/2$ *and hence, that there exists a trigonometric polynomial* t *such that*

$$||f - t|| < \delta.$$

(e) *Let* S_n *be the sequence of partial sums of the Fourier series of* f. *Show that*

$$\lim_{n\to\infty} ||f - S_n|| = 0.$$

(f) *If* (a_n) *and* (b_n) *are the Fourier series coefficients of* f, *deduce that*

$$a_0^2/2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx.$$

This is **Parseval's identity** *for* $R[-\pi, \pi]$.

Exercise 16 *By applying Parsevals identity to suitable Fourier series:*

(a) *Show that*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(b) *Show that*

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(c) *Evaluate*

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^2 + n^2},\ \alpha > 0.$$

(d) Show that

$$\sum_{l=0}^{\infty} \frac{1}{(2l+1)^4} = \frac{\pi^4}{96}.$$

Exercise 17 *Is*

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^{1/2}}$$

the Fourier series of some $f \in R[-\pi, \pi]$? The same question for the series

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{\log(n+2)}.$$

Exercise 18 *Show that*

$$\Pi_{n=2}^{\infty} \left(1 - \frac{2}{n(n+1)}\right) = 1/3.$$

Exercise 19 *Show that*

$$\Pi_{n=1}^{\infty} \left(1 + \frac{6}{(n+1)(2n+9)}\right) = 21/8.$$

Exercise 20 *Show that*

$$\Pi_{n=0}^{\infty}(1 + x^{2^n}) = \frac{1}{1-x}, \ |x| < 1.$$

Hint:(Multiply the partial products by $1 - x$.)

Exercise 21 *Let $a, b \in \mathbb{C}$ and suppose that they are not negative integers. Show that*

$$\Pi_{n=1}^{\infty} \left(1 + \frac{a-b-1}{(n+a)(n+b)}\right) = \frac{a}{b+1}.$$

Hint: Note that $\frac{(b+1)}{(a-1)} = ab + a - b - 1$, $(b+1) + (a-1) = a + b$.

Exercise 22 *Is $\Pi_{n=0}^{\infty}(1 + n^{-1})$ convergent?*

Exercise 23 *Prove the inequality*

$$0 \leq e^u - 1 \leq 3u, \ u \in (0, 1)$$

and deduce that $\Pi_{n=1}^{\infty} \left(1 + \frac{1}{n}[e^{1/n} - 1]\right)$ converges.

Exercise 24 *The* **Euler Mascheroni Constant**. *Let*

$$H_n = \sum_{k=1}^{n} \frac{1}{k},\ n \geq 1.$$

Show that $1/k \geq 1/x$, $x \in [k, k+1]$ *and hence that*

$$1/k \geq \int_k^{k+1} 1/x dx.$$

Deduce that

$$H_n \geq \int_1^{n+1} 1/x dx = \log(n+1)$$

and that

$$[H_n - \log(n+1)] - [H_n - \log n] = \frac{1}{n+1} + log\left(1 - \frac{1}{n+1}\right)$$
$$= -\sum_{k=2}^{\infty} \left(\frac{1}{n+1}\right)^k /k < 0.$$

Deduce that $H_n - \log n$ *decreases as* n *increases and thus has a non-negative limit* γ*, called the* **Euler-Mascheroni** *constant and with value approximately* 0.5772... *not known to be either rational or irrational. Finally show that*

$$\Pi_{k=1}^n (1 + 1/k) e^{-1/k} = \Pi_{k=1}^n (1 + 1/k) \Pi_{k=1}^n e^{-1/k} = \frac{n+1}{n} \exp(-(H_n - \log n)).$$

Deduce that

$$\Pi_{k=1}^{\infty} (1 + 1/k) e^{-1/k} = e^{-\gamma}.$$

Many functions have nice infinite product expansions which can be derived from first principles. The following examples illustrate some of these nice expansions.

Exercise 25 *The aim of this exercise, is to establish Euler's reflection formula:*

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)},\ \mathrm{Re}(x) > 0.$$

Step 1*: Set $y = 1 - x$, $0 < x < 1$ in the integral*

$$\int_0^\infty \frac{s^{x-1}}{(1+s)^{x+y}} ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

to obtain

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{t^{x-1}}{1+t} dt.$$

Step 2*: Let now C consist of 2 circles about the origin of radii R and ε respectively which are joined alog the negative real axis from $_R$ to $-\varepsilon$. Show that*

$$\int_C \frac{z^{x-1}}{1-z} dz = -2i\pi$$

where z^{x-1} takes its principle value.

Now let us write,

$$-2i\pi = \int_{-\pi}^{\pi} \frac{iR^x \exp(ix\theta)}{1 - R\exp(i\theta)} d\theta + \int_R^\varepsilon \frac{t^{x-1}\exp(ix\pi)}{1+t} dt$$
$$+ \int_{-\pi}^{\pi} \frac{i\varepsilon^x \exp(ix\theta)}{1 - \varepsilon\exp(i\theta)} d\theta + \int_\varepsilon^R \frac{t^{x-1}\exp(-ix\pi)}{1+t} dt.$$

Let $R \to \infty$ and $\varepsilon \to 0^+$ to deduce the result for $0 < x < 1$ and for $\mathrm{Re} x\, 0$ by continuation.

Exercise 26 *Using the previous exercise, derive the following expansions for all arguments the RHS is meaningful.*

(1)

$$\sin(x) = x\Pi_{j=1}^\infty \left(1 - (x/j\pi)^2\right).$$

(2)

$$cos(x) = \Pi_{j=1}^\infty \left(1 - \left(\frac{x}{(j-1/2)\pi}\right)^2\right).$$

(3) Eulers product formula for the Gamma function.

$$x\Gamma(x) = \Pi_{n=1}^\infty \left[(1+1/n)^x(1+x/n)^{-1}\right].$$

(4) Weierstrass's product formula for the Gamma function

$$(\Gamma(x))^{-1} = xe^{x\gamma}\Pi_{k=1}^{\infty}\left[(1+x/k)e^{-x/k}\right].$$

Here, γ is the **Euler-Mascheroni** *constant.*

Exercise 27 *Define*

$$H(z) = \Pi_{n=0}^{\infty}\left(\frac{1+azq^n}{1-zq^n}\right),\ |q|<1,\ |z|<1,\ |a|<1.$$

(i) Show that

$$H(z) = \Pi_{n=0}^{\infty}\left(1+\frac{(a+1)zq^n}{1-zq^n}\right)$$

converges absolutely.

(ii) Show that

$$(1-z)H(z) = (1+az)H(zq).$$

(iii) Write

$$H(z) = \sum_{n=0}^{\infty} A_n z^n,\ A_0 = 1.$$

Use (ii) to show that

$$(1-q^n)A_n = (1+aq^{n-1})A_{n-1},\ n\geq 1$$

Deduce that

$$A_n = \frac{(1+a)(1+aq)...(1+aq^{n-1})}{(1-q)(1-q^2)...(1-q^n)},\ n\geq 1.$$

So for $|z|<1$,

$$1+\sum_{n=1}^{\infty}\frac{(1+a)(1+aq)...(1+aq^{n-1})}{(1-q)(1-q^2)...(1-q^n)}z^n = \Pi_{n=0}^{\infty}\left(\frac{1+azq^n}{1-zq^n}\right).$$

(iv) Deduce that if $0<|b|<|a|$ and $|zb|<1$,

$$1+\sum_{n=1}^{\infty}\left(\frac{(b+a)(b+aq)...(b+aq^{n-1})}{(1-q)(1-q^2)...(1-q^n)}\right)z^n = \Pi_{n=0}^{\infty}\left(\frac{1+azq^n}{1-bzq^n}\right).$$

Exercise 28 *Let $D(n)$ denote the number of partitions of n into distinct parts (that is, n=sum of positive integers, all different). For example, $D(5) = 2$ as $5 = 4 + 1, 5 = 3 + 2$. Show formally that*

$$\sum_{n=0}^{\infty} D(n)q^n = \Pi_{j=1}^{\infty}(1 + q^j).$$