STRUCTURE THEORY FOR EXTENDED KEPLER–COULOMB 3D QUANTUM SUPERINTEGRABLE SYSTEMS

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A quantum superintegrable system is an integrable n-dimensional Hamiltonian system with potential: $H = \Delta_n + V$ that admits $2n-1$ algebraically independent partial differential operators commuting with the Hamiltonian, the maximum number possible. The system is of order ℓ if the maximum order of the symmetry operators, other than H, is ℓ. Typically, the algebra generated by the symmetry operators and their commutators has been proven to close polynomially. However the degenerate 3-parameter potential for the 3D extended Kepler-Coulomb system (2nd order superintegrable) appeared to be an exception as Kalnins et al. (2007) showed that it didn't close polynomially. The 3D 4-parameter potential for the extended Kepler-Coulomb system is not even 2nd order superintegrable. However, Verrier and Evans (2008) showed it was 4th order superintegrable, Tanoudis and Daskaloyannis (2011) showed it closed polynomially. We consider an infinite class of quantum extended Kepler-Coulomb systems that we show to be superintegrable of arbitrarily high order, compute the structure algebras and demonstrate that algebraic closure is the norm, whereas polynomial closure requires extra symmetry. This is a report on joint work with Ernie Kalnins (University of Waikato) and Jonathan Kress (University of new South Wales).

Keywords: Superintegrable systems; Schrödinger equation.

1. Definitions

A quantum system is defined by a Schrödinger operator $H = \Delta + V(\mathbf{x})$ where $\Delta = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i}(\sqrt{g} g^{ij})\partial_{x_j}$ is the Laplace-Beltrami operator on an n-dimensional Riemannian manifold, in local coordinates x_j. The system is *superintegrable* of order ℓ if it admits $2n-1$ algebraically independent globally defined differential symmetry operators (the maximal number possible) $S_j, \quad 1 \le j \le 2n-1, \quad n \ge 2$, with $S_1 = H$ and $[H, S_j] \equiv HS_j - S_jH = 0$, such that ℓ is the maximum order of the generating symmetries (other than

H) as a differential operator. Systems associated with Lie algebras ($\ell = 1$) and separation of variables ($\ell = 2$) are the simplest and best studied. An integrable system has n algebraically independent commuting symmetry operators whereas a superintegrable system has $2n-1$ independent symmetry operators which cannot all commute and this nonabelian structure is critical for finding the spectral resolution of H by algebraic methods alone. The importance of these systems is that they can be solved exactly.

2. The 4-parameter Kepler system

We shall start with the 4-parameter extended Kepler Coulomb system

$$H = \partial_x^2 + \partial_y^2 + \partial_z^2 + \frac{\alpha}{r} + \frac{\beta}{x^2} + \frac{\gamma}{y^2} + \frac{\delta}{z^2}, \qquad (1)$$

where x, y, z are Cartesian coordinates and $r = \sqrt{x^2+y^2+z^2}$. H is 4th order superintegrable. We use the idea introduced in[1,2] and extend the system by passing to spherical coordinates and replacing each angular coordinate θ_i by $k_i\theta_i$ where k_i is a fixed rational number. The extended Kepler-Coulomb operator is $H\Psi = E\Psi$, where $[L_2, L_3] = [L_j, H] = 0$ and

$$H = \partial_r^2 + \frac{2}{r}\partial_r + \frac{\alpha}{r} + \frac{1-k_1^2}{4r^2} + \frac{L_2}{r^2}, \; L_3 = \partial_{\theta_2}^2 + \frac{\beta}{\cos^2(k_2\theta_2)} + \frac{\gamma}{\sin^2(k_2\theta_2)},$$

$$L_2 = \partial_{\theta_1}^2 + k_1\cot(k_1\theta_1)\partial_{\theta_1} + \frac{L_3}{\sin^2(k_1\theta_1)} + \frac{\delta}{\cos^2(k_1\theta_1)}.$$

Here the 2nd order operators L_2, L_3 are just those that determine multiplicative separation of the Schrödinger equation. The scalar potential in spherical coordinates is now taken as

$$\tilde{V} = \frac{\alpha}{r} + \frac{1-k_1^2}{4r^2} + \frac{1}{r^2}\left(\frac{\beta}{\sin^2(k_1\theta_1)\cos^2(k_2\theta_2)} + \frac{\gamma}{\sin^2(k_1\theta_1)\sin^2(k_2\theta_2)} + \frac{\delta}{\cos^2(k_1\theta_2)}\right).$$

It differs from the classical potential V by the term $\frac{1-k_1^2}{4r^2}$ which corresponds to $R/8$ where R is the scalar curvature of the manifold, just the correction term needed for the conformally invariant Laplacian. It is also needed for superintegrability. Note that for $k_1 \neq 1$ the space isn't flat.

Our basic tasks are 1) to demonstrate that this generalized 4-parameter Kepler-Coulomb problem is superintegrable for all rational k_1, k_2 by constructing the two remaining basic generating symmetries explicitly, and 2) to investigate the closure of the symmetry algebra generated by the basic symmetries and compute the structure equations. The fundamental difficulty is the construction and manipulation of symmetry operators of arbitrarily high order.

The separated solutions for the eigenvalue equations $H\Psi = E\Psi$, $L_3\Psi = -\mu^2\Psi$, $L_2\Psi = \lambda\Psi$ are $\Xi_{p,m,n} = R_p^{k_1\rho}(r)\Phi_m^{(c,b)}(\cos(2k_2\theta_2))\frac{\Psi_n^{(\mu/k_1,d)}(\cos k_1\theta_1)}{\sqrt{\sin(k_1\theta_1)}}$,
$\Phi_m^{(c,b)} = \sin^{c+1/2}(k_2\theta_2)\cos^{b+1/2}(k_2\theta_2)P_m^{(c,b)}(\cos(2k_2\theta_2))$,
$\Psi_n^{(\mu/k_1,d)} = \sin^{\mu/k_1+1/2}(k_1\theta_1)\cos^{d+1/2}(k_1\theta_1)P_n^{(\mu/k_1,d)}(\cos(2k_1\theta_1))$,
$R_p^{k_1\rho} = \frac{e^{-\sqrt{E}r}r^{(k_1\rho+1)/2}}{2\sqrt{E}r}L_p^{k_1\rho}(2\sqrt{E}r)$.
Here $\beta = k_2^2(\frac14 - b^2)$, $\gamma = k_2^2(\frac14 - c^2)$, $\delta = k_2^2(\frac14 - d^2)$, $\mu = k_2(2m+b+c+1)$, $\beta = k_2^2(\frac14 - b^2)$, $\lambda = \frac{k_1^2}{4}(1-\rho^2)$; $P_n^{(\mu/k_1,d)}$, $P_m^{(c,b)}$ are Jacobi functions, $L_p^{k_1\rho}$ are associated Laguerre functions; $\rho = 2(2n + \frac{\mu}{k_1} + d + 1)$ and the energy is $E = \alpha^2/(2[p + 2k_1 n + 2k_2 m] + 2k_1[d+1] + 2k_2[b+c+1] + 1)^2$.

A basic insight here is that formal eigenspaces of the Hamiltonian are invariant under action of any symmetry operator, so the operator must induce recurrence relations for the basis of separated eigenfunctions. Our strategy is to use the known recurrence relations for hypergeometric functions to reverse this process and determine a symmetry operator from the recurrence relations. We look for recurrence operators that change the eigenvalues of L_2, L_3 but preserve E. For $k_1 = p_1/q_1, k_2 = p_2/q_2$ we use the mappings 1) : $p \to p + 2p_1, \quad m \to m, \quad n \to n - q_1$, 2) : $p \to p - 2p_1, \quad m \to m, \quad n \to n + q_1$, 3) : $p \to p, \quad m \to m - p_1 q_2, \quad n \to n + q_1 p_2$, 4) : $p \to p, \quad m \to m + p_1 q_2, \quad n \to n - q_1 p_2$. We first look at recurrences for associated Laguerre functions. To effect the r-dependent transformations 1) and 2) we use $Y(1)^{\pm p}$: $(\tau = -\frac{4\alpha}{2p+k_1\rho+1})$

$$Y(1)_-^p R_p^{k_1\rho} = [2(k_1\rho+1)\partial_r + (2\alpha + \frac{1-k_1^2\rho^2}{r})]R_p^{k_1\rho} = \tau R_{p-1}^{k_1\rho+2},$$

$$Y(1)_+^p R_p^{k_1\rho} = [2(-k_1\rho+1)\partial_r + (2\alpha + \frac{1-k_1^2\rho^2}{r})]R_p^{k_1\rho} = \tau(p+1)(p+k_1\rho)R_{p+1}^{k_1\rho-2}.$$

Similarly for Jacobi functions with $z = \cos(2k_1\theta_1)$, we can find first order parameter–dependent recursion operators $Z(1)_-^n$, $Z(1)_+^n$ such that

$$Z(1)_-^n \frac{\Psi_n^{\mu/k_1,d}}{\sqrt{\sin(k_1\theta_1)}} = -2(\frac{\mu}{k_1} + n)(d+n)\frac{\Psi_{n-1}^{\mu/k_1,d}}{\sqrt{\sin(k_1\theta_1)}},$$

$$Z(1)_+^n \frac{\Psi_n^{\mu/k_1,d}}{\sqrt{\sin(k_1\theta_1)}} = -2(\frac{\mu}{k_1} + d + n + 1)(n+1)\frac{\Psi_{n+1}^{\mu/k_1,d}}{\sqrt{\sin(k_1\theta_1)}}.$$

We now form the operators

$$J^\pm = \left(Y(1)_\pm^{p\pm 2p_1-1}Y(1)_\pm^{p\pm 2p_1-2}\cdots Y(1)_+^{p\pm 1}Y(1)_\pm^p\right)\left(Z(1)_\mp^{n\mp(q_1-1)}\cdots Z(1)_\mp^n\right),$$

We can observe that J^+ and J^- switch places under $\rho \to -\rho$ so $J_2 = J^+ + J^-$, $J_1 = (J^- - J^+)/\rho$ are even functions of both ρ and μ and can be interpreted as differential operators. Similarly we can find raising and lowering recurrences satisfying properties 3) and 4), acting on basis functions $\Psi_n^{\mu/k_1,d}$, $\Phi_m^{(c,b)}$ to construct ladder operators K^+, K^-, even functions of ρ^2 that switch places under the reflection $\mu \to -\mu$. Thus $K_2 = K^+ + K^-$, $K_1 = (K^- - K^+)/\mu$ are even functions in both ρ and μ so are differential operators. This proves superintegrability.

The action of J^- on a generalized eigenbasis is given in terms of the rising factorial $(\alpha)_n$ by:

$$J^-\Xi_{p,m,n} = \frac{(2)^{4p_1+q_1}(-1)^{q_1}\alpha^{2p_1}(n+1)_{q_1}(\mu/k_1+d+n+1)_{q_1}}{(2p+k_1\rho+1)^{2p_1}}\Xi_{p-2p_1,m,n+q_1}$$

with similar formulas for $J^+, K^\pm$. Recalling the actions of H, L_2, L_3 on this eigenbasis we have $J^+J^-\Xi_{p,m,n} = 4^{4p_1+q_1}E^{2p_1}(\frac{\rho/2-\frac{\mu}{k_1}-d+1}{2})_{q_1}\times (\frac{\rho/2+\frac{\mu}{k_1}+d+1}{2})_{q_1}(\frac{\rho/2+\frac{\mu}{k_1}-d+1}{2})_{q_1}(\frac{\rho/2-\frac{\mu}{k_1}+d+1}{2})_{q_1}(\frac{k_1\rho-\frac{\alpha}{\sqrt{E}}+1}{2})_{2p_1}(\frac{k_1\rho+\frac{\alpha}{\sqrt{E}}+1}{2})_{2p_1}$ $\times\Xi_{p,m,n}$ and similarly for $J^-J^+\Xi_{p,m,n}$, $K^+K^-\Xi_{p,m,n}$, $K^-K^+\Xi_{p,m,n}$. Thus, each of J^+J^-, J^-J^+ is a polynomial in μ^2 and E and since these operators switch places under the reflection $\rho \to -\rho$, $P_1 = J^+J^- + J^-J^+$, $P_2 = \frac{J^+J^- - J^-J^+}{\rho}$ are polynomials in H, L_2, L_3. Also K^+K^-, K^-K^+ are polynomials in ρ^2 and in μ and switch places under the reflection $\mu \to -\mu$ so $P_3 = K^+K^- + K^-K^+$, $P_4 = \frac{K^+K^- - K^-K^+}{\mu}$ are polynomials in L_2, L_3.

To determine the structure relations it is sufficient to establish them on the generalized eigenbases. The results can then be shown valid independent of basis,[3] For example $[J_1, J_2] = -2q_1J_1^2 - 2P_2(H, L_2, L_3)$, $[K_1, K_2] = -2p_1p_2K_1^2 - 2P_4(L_2, L_3)$, and the unsymmetrized structure relations $J_1^2(\frac{1}{4} - k_1^{-2}L_2) = J_2^2 - 2P_1(H, L_2, L_3) - 2q_1J_1J_2$, $K_1^2 = -K_2^2 + 2P_3(L_2, L_3) + 2p_1p_2K_1K_2$. Eventually, we find for $h, \ell = 1, 2$,

$$[K_\ell, J_h]Q = \{J_1, K_1\}P_{11}^{h\ell} + \{J_1, K_2\}P_{12}^{h\ell} + \{J_2, K_1\}P_{21}^{h\ell} + \{J_2, K_2\}P_{22}^{h\ell},$$

where Q, $P_{jk}^{h\ell}$ are explicit polynomials in L_2, L_3 and $\{A, B\} \equiv AB - BA$. Here, Q is defined by $B(d,\mu)B(d,-\mu)B(-d,\mu)B(-d,-\mu)$ where $B(d,\mu) = (-\frac{d}{2} - \frac{\mu}{k_1} - 2q_1p_2 + \frac{1}{2})_{q_1} + (-\frac{d}{2} - \frac{\mu}{k_1} + \frac{1}{2})_{q_1}$ on a generalized eigenbasis. These identities can be symmetrized, but they show clearly that double commutators of algebra generators cannot be expressed as polynomials in the generators. Rather the double commutators satisfy polynomial equations in the generators. In this sense the algebra closes, but it closes algebraically.

However, there is another problem: the generators $\{H, L_2, L_3, J_1, K_1\}$ are not of minimal order. In particular, for the case $k_1 = k_2 = 1$ we have found a system of generators of orders $(2,2,2,3,5)$ whereas it is known that a generating set of orders $(2,2,2,2,4)$ exists. It is easy to see that J_1 is of order 5 and J_2 is of order 6, whereas K_1 is of order 3 and K_2 is of order 4. We know that there must be other lower order symmetries J_0, K_0 for this case, such that

$$[L_2, J_0] = J_1, \quad [L_3, J_0] = 0, \; L_3, K_0] = K_1, \quad [L_2, K_0] = 0.$$

In[3] we show how to obtain these symmetries from the raising and lowering operators $J^{\pm}, K^{\pm}$, for all rational k_1, k_2. Now we consider the symmetry algebra generated by H, L_2, L_3, K_0, J_0. We can find algebraic relations between $[J_1, K_0]$ and the generators, so that the symmetry algebra closes algebraically. However, it doesn't close polynomially.

3. The caged Kepler problem: $k_1 = k_2 = 1$

In the case $k_1 = k_2 = 1$, (1), we are in Euclidean space and our system has additional symmetry. Note that in this case any permutation of the ordered pairs $(x,\beta), (y,\gamma), (z,\delta)$ leaves the Hamiltonian unchanged. This leads to additional structure in the symmetry algebra. This system was shown to be superintegrable by Verrier and Evans,[4] The basic symmetries are:

$$L_3 = I_{xy} = (x\partial_y - y\partial_x)^2 + \frac{\beta(x^2+y^2)}{x^2} + \frac{\gamma(x^2+y^2)}{y^2}.$$

By permutation symmetry, I_{xz}, I_{yz} are also symmetry operators, and $L_2 = I_{xy} + I_{xz} + I_{yz} - (\beta+\gamma+\delta)$. The constant of the motion K_0 is 2nd order: $K_0 = -\frac{1}{32}(I_{yz} - I_{xz})$, and J_0 is 4th order: $J_0 = -4M_3^2 - \frac{1}{2}\times$

$$\{(\{x,\partial_x\}+\{y,\partial_y\}+\{z,\partial_z\})^2, \frac{\delta}{z^2}\}+2H\left(I_{xz}+I_{yz}-(\beta+\gamma+\frac{3}{4})\right)+5\frac{\delta}{z^2}+\frac{\alpha^2}{2},$$

$$M_3 = \frac{1}{2}\{(y\partial_z - z\partial_y),\partial_y\} - \frac{1}{2}\{(z\partial_x - x\partial_z),\partial_x\} - z\left(\frac{\alpha}{2r}+\frac{\beta}{x^2}+\frac{\gamma}{y^2}+\frac{\delta}{z^2}\right).$$

The symmetries H, L_2, L_3, J_0, K_0 form a generating (algebraic) basis for the symmetry operators. Under the transposition $(x,\beta) \leftrightarrow (z,\delta)$ the operators H, L_2, L_3, K_0 transform to linear combinations of themselves. However, J_0 is mapped to a new symmetry J_0' linearly independent of the original basis symmetries. Tanoudis and Daskaloyannis[5] show that the algebra generated by the 6 algebraically dependent symmetries H, L_2, L_3, J_0, K_0 and J_0' closes polynomially, in the sense that all double commutators of the generators are again expressible as polynomials in the generators. However more is true. *All* formally self-adjoint symmetry operators, such as $\{K_1, J_1\}$, K_1^2, J_1^2, $[K_1, J_1]$

are expressible as symmetrized polynomials in the generators. Further we can deduce an explicit polynomial relation among the 6 generators, an operator identity of order 12, quadratic in J_0': $0 \equiv Q_5 + \{Q, J^2\} + 32\{K_0, J_0\}^2 - 16\{L_2+L_3-\delta-1, \{J, \{K_0, J_0\}\}\} + \{Q_1, K_0^2\} + \{Q_2, J_0^2\} + \{Q_3, K_0\} + \{Q_4, J_0\}$, where $J = -8(J_0 + 2J_0' + \frac{1}{2}H - \frac{\alpha^2}{2})$, and Q_1, Q_2, Q_3, Q_4, Q_5 are explicit polynomials in H, L_2, L_3 of orders 2,4,5,4,6, respectively.

4. Conclusions and Outlook

The recurrence relation method developed by Kalnins, Kress and Miller[6] for proving superintegrability and determining the structure equations for families of 2D quantum superintegrable systems can be extended to the 3D case. The construction appears to be quite general and not restricted to Kepler-Coulomb analogs. For these higher order superintegrable systems it appears that algebraic closure is the norm. For polynomial closure, extra symmetry is needed. We have not proved in all cases that there do not exist other generators of lower order but, if they exist, they must also be obtainable in terms of recurrence relations of hypergeometric functions. A crucial role is played by the raising and lowering operators. They are not defined independent of eigenbasis and are not even symmetries, but all symmetries are built from them.

References

1. F. Tremblay, A.Turbiner and P. Winternitz. An infinite family of solvable and integrable quantum systems in the plane. J.Phys.A: Math. Theor., **42**, (2009) 242001.
2. F. Tremblay, A. Turbiner and P. Winternitz. Periodic orbits for an infinite family of classical superintegrable systems. J. Phys. A: Math. Theor. **43** (2010) 015202.
3. E. G. Kalnins, J. M. Kress and W. Miller, Jr. The 3 and 4-parameter extended Kepler-Coulomb quantum systems in 3 dimensions (submitted)
4. N. W. Evans and P. E. Verrier. Superintegrability of the caged anisotropic oscillator. *J. Math. Phys.* **49**, 092902, 10 pages, arXiv:0808.2146, (2008).
5. Y Tanoudis and C. Daskaloyannis. Algebraic calculation of the energy eigenvalues for the non degenerate three dimensional Kepler Coulomb potential. arXiv:11020397v1, (2011)
6. E. G. Kalnins, J. M. Kress and W. Miller Jr. A recurrence relation approach to higher order quantum superintegrability, SIGMA 7 (2011), 031, 24 pages arXiv:1011.6548v3 [math-ph]