TOPICS IN HARMONIC ANALYSIS WITH APPLICATIONS TO RADAR AND SONAR

WILLARD MILLER JR.*

ABSTRACT. This minicourse is an introduction to basic concepts and tools in group representation theory, both commutative and noncommutative, that are fundamental for the analysis of radar and sonar imaging. Several symmetry groups of physical interest will be studied (circle, line, rotation, $ax + b$, Heisenberg, etc.) together with their associated transforms and representation theories (DFT, Fourier transform, expansions in spherical harmonics, wavelets, etc.). Through the unifying concepts of group representation theory, familiar tools for commutative groups, such as the Fourier transform on the line, extend to transforms for the noncommutative groups which arise in radar-sonar.

The insight and results obtained will be related directly to objects of interest in radar-sonar, such as the ambiguity function. The material will be presented with many examples and should be easily comprehensible by engineers and physicists, as well as mathematicians.

*School of Mathematics and IMA, University of Minnesota. The research contribution of this paper was supported in part by the National Science Foundation under grant DMS 88–23054

Typeset by $\mathcal{AMS}$-TeX

TABLE OF CONTENTS

1. Introduction 5
2. The Doppler effect 6
2.1 Wideband and narrow-band echos 6
2.2 Ambiguity functions 8
2.3 Exercises 10
3. A group theory primer 12
3.1 Definitions and examples 12
3.2 Group representations 17
3.3 Shur's lemmas 22
3.4 Orthogonality relations for finite group representations 23
3.5 Representations of Abelian groups 28
3.6 Exercises 29
4. Representation theory for infinite groups 30
4.1 Linear Lie groups 30
4.2 Invariant measures on Lie groups 31
4.3 Orthogonality relations for compact Lie groups 36
4.4 The Peter-Weyl theorem 39
4.5 The rotation group and spherical harmonics 41
4.6 Fourier transforms and their relation to Fourier series 46
4.7 Exercises 48
5. Representations of the Heisenberg group 49
5.1 Induced representations of H_R 49
5.2 The Schrödinger representation 50
5.3 Square integrable representations 51
5.4 Orthogonality of radar cross-ambiguity functions 54
5.5 The Heisenberg commutation relations 55
5.6 The Bargmann-Segal Hilbert space 59
5.7 The lattice representation of H_R 64
5.8 Functions of positive type 67
5.9 Exercises 71

6. Representations of the affine group 73
6.1 Induced irreducible representations of G_A 73
6.2 The wideband cross-ambiguity functions 75
6.3 Decomposition of the regular representation 77
6.4 Exercises 81
7. Weyl-Heisenberg frames 82
7.1 Windowed Fourier transforms 82
7.2 The Weil-Brezin-Zak transform 84
7.3 Windowed transforms and ambiguity functions 86
7.4 Frames 87
7.5 Frames of W-H type 89
7.6 Exercises 92
8. Affine frames and wavelets 93
8.1 Wavelets 93
8.2 Affine frames 93
8.3 Exercises 96
9. The Schrödinger group 97
9.1 Automorphisms of H_R 97
9.2 The metaplectic representation 100
9.3 Theta functions and the lattice Hilbert space 104
9.4 Exercises 108
References 109

§1. INTRODUCTION

These notes are intended as an introduction to those basic concepts and tools in group representation theory, both commutative and noncommutative, that are fundamental for the analysis of radar and sonar imaging. Several symmetry groups of physical interest will be studied (circle, line, rotation, Heisenberg, affine, etc.) together with their associated transforms and representation theories (DFT, Fourier transforms, expansions in spherical harmonics, Weyl-Heisenberg frames, wavelets, etc.) Through the unifying concepts of group representation theory, familiar tools for commutative groups, such as the Fourier transform on the line, extend to transforms for the noncommutative groups which arise in radar and sonar.

The insights and results obtained will be related directly to objects of interest in radar-sonar, in particular, the ambiguity and cross-ambiguity functions. (We will not, however, take up the study of tomography, even though this field has group-theoretic roots.) The material is presented with many examples and should be easily comprehensible by engineers and physicists, as well as mathematicians.

The main emphasis in these notes is on the matrix elements of irreducible representations of the Heisenberg and affine groups, i.e., the narrow and wide band ambiguity and cross-ambiguity functions of radar and sonar. In Chapter 2 we introduce the ambiguity functions in connection with the Doppler effect. Chapters 3 and 4 constitute a minicourse in the representation theory of groups. (Much of the material in these chapters is adapted from the author's textbook [M5] .) Chapters 5 and 6 specialize these ideas to the Heisenberg and affine groups. Chapters 7 and 8 are devoted to frames associated with the Heisenberg group. (Weyl-Heisenberg) and with the affine group (wavelets). We conclude with a chapter touching on the Schrödinger group and the metaplectic formula.

It is assumed that the reader is proficient in linear algebra and advanced calculus, and some concepts in functional analysis (including the basic properties of countable Hilbert spaces) are used frequently. (References such as [AG1], [K2], [K8], [NS] and [RN] contain all the necessary background information.) The theory presented here is largely algebraic and (sometimes) formal so as not to obscure the clarity of the ideas and to keep the notes short. However, the needed rigor can be supplied. (The knowledgeable reader can invoke Fubini's theorem when we interchange the order of integration, the Lebesgue dominated convergence theorem when we pass to a limit under the integral sign, etc.)

Finally, the author (who is not an expert on radar or sonar) wishes to thank the experts whose writings form the core of these notes, e.g., [AT1], [AT3], [D4], [G1], [HW], [N3], [S2], [W5].

§2. THE DOPPLER EFFECT

2.1 Wideband and narrow-band echos. We begin by reviewing the Doppler effect as it relates to radar and sonar. Consider a stationary transmitter/detector and a moving (point) target located at a distance $R(t)$ from the transmitter at time t. We assume that the distance from the transmitter to the target changes linearly with time,

$$R(t) = r + vt. \tag{2.1}$$

Now suppose that the transmitter emits an electromagnetic pulse $s(t)$. The pulse is transmitted at a constant speed c in the ambient medium (e.g., air or water) impinges the target, and is reflected back to the transmitter/receiver where it is detected as the echo $e_{r,v}(t)$. (We assume $c > |v|$.) Let $\Delta_{r,v}(t)$ be the time delay experienced by the pulse which is received as an echo at time t. (Thus this pulse is emitted at time $t - \Delta$ and travels a distance $c\Delta$ before it is received as an echo at time t.) It follows that the pulse impinges the target at time $t - \Delta/2$ and we have the identity

$$c\Delta = 2R(t - \Delta/2) = 2r + 2(t - \Delta/2)v.$$

Thus

$$\Delta_{r,v}(t) = \frac{2v}{c+v}t + \frac{2r}{c+v}. \tag{2.2}$$

The echo $e_{r,v}(t)$ is proportional to the signal at time $t - \Delta_{r,v}(t)$:

$$e_{r,v}(t) = -\sqrt{\frac{c-v}{c+v}}\, s\left(t\left[\frac{c-v}{c+v}\right] - \frac{2r}{c+v}\right). \tag{2.3}$$

Here the -1 factor in the amplitude of the echo is based on the boundary conditions for the normal component of the electric field at the surface of a (conducting) target, [J1]. The factor $\sqrt{\frac{c-v}{c+v}}$ is needed if we require that the energy of the pulse is conserved:

$$\int_{-\infty}^{\infty} |s(t)|^2 dt = \int_{-\infty}^{\infty} |e(t)|^2 dt.$$

Changing notation, we write the echo as

$$e_{x,y}(t) = -\sqrt{y}\, s(y[t + x]) \tag{2.4}$$

where $y = (c - v)/(c + v)$ and $x = -2r/(c - v)$. Note that the transformation $(x, y) \to (r, v)$ is one-to-one. The result (2.3) is the exact **(wideband)** solution for the given assumptions.

It is very common to approximate the wideband solution under the assumption that $|v| \ll c$ and t is small (of the order of r/c) over the period of observation. (See [CB] and [S7] for careful analyses of the relationship between the wideband solution and the narrow-band approximation.) One writes the signal in the form

$$s(t) = a(t)e^{2\pi i \omega_0 t}$$

where $a(t)$ is assumed to be a slowly varying complex function of t (the **envelope** of the waveform) with respect to the exponential factor.

Then we have

$$e_{x,y}(t) = -\sqrt{y}a(y[t+x])e^{2\pi i(y[t+x])\omega_0}$$

or, since $y \approx 1-2\beta$, $\quad yx \approx -(2r/c)(1-\beta)$ where $\beta = v/c$, we have

$$\begin{aligned} e_{x,y}(t) &\approx -a(t-2r/c)e^{2\pi i\omega_0[t(1-2\beta)-2(r/c)(1-\beta)]} \\ &\approx -s(t-2r/c)e^{-4\pi i\beta\omega_0[t-2r/c]}. \end{aligned} \tag{2.5}$$

This is called the **narrow-band approximation** of the Doppler effect. (Generally speaking, the narrow-band approximation is usually adequate for radar applications but is less appropriate for sonar.)

To see the significance of this approximation, consider the Fourier transform of the signal and the echo:

$$S(\omega) = \int_{-\infty}^{\infty} s(t)e^{-2\pi i\omega t}dt, \tag{2.6}$$

where, [DM2],

$$s(t) = \int_{-\infty}^{\infty} S(\omega)e^{2\pi i\omega t}d\omega. \tag{2.7}$$

Then

$$S(\omega) \to E(\omega) = -S(\omega + 2\beta\omega_0)e^{-4\pi i\omega r/c} \tag{2.8}$$

in the narrow-band approximation, whereas the exact result is

$$S(\omega) \to E(\omega) = -\frac{1}{\sqrt{y}}S\left(\frac{\omega}{y}\right)e^{2\pi i\omega x}. \tag{2.9}$$

Clearly (2.8) is a good approximation to (2.9) if the support of $S(\omega)$ lies in a narrow band around ω_0. In the narrow-band approximation the signal is delayed by the time interval $2r/c$ and the frequency changes by $2\omega_0 v/c$.

2.2 Ambiguity functions. To determine the position and velocity of a target from a narrow-band echo (2.5) one computes the inner product (cross-correlation) of the echo with a test signal

$$\begin{aligned} s_{r',v'}(t) &= -s\left(t-\frac{2r'}{c}\right)e^{-4\pi i\omega_0 v'\left[t-\frac{2r'}{c}\right]/c}, \\ \langle e_{r,v}, s_{r',v'}\rangle &= \int_{-\infty}^{\infty} e_{r,v}(t)\overline{s}_{r',v'}(t)dt \\ &= e^{\frac{4\pi i\omega_0}{c}\left[\frac{2r'v'}{c}+\frac{2r}{c}v\right]}\int_{-\infty}^{\infty} s(t-2r/c)\overline{s}\left(t-\frac{2r'}{c}\right)e^{\frac{4\pi i\omega_0}{c}[(v'-v)t]}dt \end{aligned} \tag{2.10}$$

Considering $|\langle e_{r,v}, s_{r',v'}\rangle|$ as a function of r', v', one tries to maximize this function for a signal $s(t) \in L_2(R)$, the square integrable functions on the real line. Assume that s is normalized to have unit energy: $||s||^2 = \langle s, s, \rangle = 1$. Then by the Schwarz inequality we have

$$|\langle e_{r,v}, s_{r',v'}\rangle| \leq ||e_{r,v}|| \cdot ||s_{r',v'}|| = 1.$$

Furthermore the maximum is assumed for $e_{r,v} \equiv e^{i\alpha} s_{r',v'}$ where α is a real constant. Hence, the maximum is assumed if and only if $r' = r, v' = v$.

We can simplify the notation by remarking that

$$\begin{aligned} I(u, w) &= |\langle e_{r,v}, s_{r',v'}\rangle|^2 \\ &= |A_s(u_0 - u, w_0 - w)|^2 \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} A_s(u, w) &= \int_{-\infty}^{\infty} s\left(t - \frac{u}{2}\right) \overline{s}\left(t + \frac{u}{2}\right) e^{4\pi i t w} dt \\ &= \int_{-\infty}^{\infty} S\left(\omega - w\right) \overline{S}\left(\omega + w\right) e^{-2\pi i \omega u} d\omega, \\ u_0 &= \frac{2r}{c}, u = \frac{2r'}{c}, w_0 = \frac{\omega_0 v}{c}, w = \frac{\omega_0 v'}{c}. \end{aligned} \tag{2.12}$$

Here, $A_s(u, w)$ is known as the **radar (self-) ambiguity function**.

To determine the velocity from the ambiguity function one typically chooses a single-frequency signal of the form $s(t) = \chi_T(t) e^{2\pi i \omega_o t}$ where $T \gg \omega_0^{-1}$ and

$$\chi_T(t) = \begin{cases} 1 \text{ if } -T \leq t \leq T \\ 0 \text{ otherwise }. \end{cases}$$

It can be shown in this case that as a function of v' the ambiguity function has a sharp peak about $v' = v$, whereas it is relatively insensitive to changes in r'. To estimate the range one typically chooses $s(t) = G_\sigma(t) e^{2\pi i \omega_0 t}$ where $G_\sigma(t)$ is a very peaked Gaussian wave function, centered at $t = 0$ and with standard deviation $G \ll 1$. This gives an ambiguity function with a sharp peak about $r' = r$, but which is relatively insensitive to changes in v'.

Now we consider the wideband case and generalize to allow a distribution of moving targets with density function $D(x, y)$, where the support of this function is contained in the set $\{(x, y) : \; y > 0\}$. Then from (2.4) the echo from the signal $s(t)$ is

$$e(t) = \int_0^{\infty} \int_{-\infty}^{\infty} \sqrt{y} s(y[t + x]) D(x, y) dx dy. \tag{2.13}$$

Correlating the echo with a "test" echo $e'(t)$ generated from the signal $s(t)$ and the "test" density function $D'(x, y)$ we obtain

$$\langle e, e' \rangle = \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} A(x, y, \tilde{x}, \tilde{y}) D(x, y) \overline{D'(\tilde{x}, \tilde{y})} dx dy d\tilde{x} d\tilde{y} \tag{2.14}$$

where

$$A(x, y, \tilde{x}, \tilde{y}) = \int_{-\infty}^{\infty} \sqrt{y\tilde{y}}\, s(y[t+x])\overline{s}(\tilde{y}[t+\tilde{x}])dt \tag{2.15}$$

is the **radar (self-) ambiguity function** in the wideband case. (A very similar construction of a moving target distribution can be carried out in the narrow-band case.) Note that if $D'(\tilde{x}, \tilde{y}) = \delta(\tilde{x} - x_1, \tilde{y} - y_1)$ and $D(x, y) = \delta(x - x_0, y - y_0)$ then

$$\langle e, e' \rangle (x_0, y_0, x_1, y_1) = A(x_0, y_0, x_1, y_1). \tag{2.16}$$

As with the narrow-band ambiguity function, if we normalize the signal s to have unit energy, then by the Schwarz inequality

$$|A(x, y, \tilde{x}, \tilde{y})| \leq ||s|| \cdot ||s|| = 1 \tag{2.17}$$

and the maximum is assumed for $y = \tilde{y}, x = \tilde{x}$. Note also that

$$A(x, y, \tilde{x}, \tilde{y}) = A\left((x - \tilde{x})\tilde{y}, \frac{y}{\tilde{y}}, 0, 1\right).$$

Thus to study the ambiguity function one can restrict to the case

$$A(x, y) = A(x, y, 0, 1) = \sqrt{y} \int_{-\infty}^{\infty} s(y[t+x])\overline{s}(t)dt. \tag{2.18}$$

Suppose $\{s_n\}$ is a basis for $L_2(R)$, the standard space of square integrable functions on the real line. Then we can consider the cross-correlation of s_m with the echo e_n from s_n:

$$A_{nm}(x, y) = \langle e_n, s_m \rangle = \sqrt{y} \int_{-\infty}^{\infty} s_n(y[t+x])\overline{s}_m(t)dt. \tag{2.19}$$

We call A_{nm} the **cross-ambiguity function** of s_n and s_m. Similarly the **(narrow-band)** cross-ambiguity function is

$$\psi_{nm}(u, w) = \int_{-\infty}^{\infty} s_n\left(t - \frac{u}{2}\right)\overline{s}_m\left(t + \frac{u}{2}\right)e^{4\pi i t w}dt.$$

The ambiguity and cross-ambiguity functions are of fundamental importance in the theory of radar/sonar. The use of the ambiguity function to estimate the range and velocity of point targets and, more generally, of the cross-ambiguity function to estimate target distribution functions in both the wideband and narrow band cases is basic to the theory.

In these lectures we shall summarize and elucidate this theory by exploiting the intimate relationship between the cross-ambiguity functions defined above and the theory of group representations. In particular the wide-band cross-ambiguity functions can be interpreted as matrix elements of unitary irreducible representations of the two-dimensional affine group; the narrow-band cross-ambiguity functions are matrix elements of unitary irreducible representations of the three-dimensional Heisenberg group. The basic properties of these functions thus emerge as consequences of analysis on affine and Heisenberg groups.

2.3 Exercises.

2.1 Consider the Gaussian pulse

$$s(t) = \left(\frac{2}{\pi T^2}\right)^{\frac{1}{4}} e^{-t^2/T^2 + 2\pi i \omega_0 t},$$

normalized to have unit energy. Verify that the ambiguity function is given by

$$A_s(u, w) = e^{-\frac{1}{2}\left(\frac{u^2}{T^2} + 4\pi^2 w^2 T^2\right)} e^{-2\pi i \omega_0 u}.$$

Describe the level curves $|A_s(u, w)| = k$ in the $u - w$ plane. Discuss the effect of varying the pulse length T on the problem of estimating the range and velocity of the target.

2.2 Show that the area enclosed by a level curve $|A_s(u, w)| = k$ in Exercise 2.1 is independent of T.

2.3 Consider the normalized frequency modulated pulse

$$s(t) = \left(\frac{2}{\pi T^2}\right)^{\frac{1}{4}} e^{-t^2/T^2 + 4\pi i(\omega_0 t + \gamma t^2)}.$$

The ambiguity function is given by

$$A_s(u, w) = e^{-\frac{1}{2}\left[(1+16\pi^2\gamma^2 T^4)\frac{u^2}{T^2} - 8\pi^2 \gamma T^2 wu + 4\pi^2 w^2 T^2\right]} e^{-2\pi i \omega_0 u}.$$

Describe the level curves $|A_s(u, w)| = k$ in the $u - w$ plane. Discuss the effect of varying the pulse length T and the "compression ratio" $m = \sqrt{1 + 16\pi^2\gamma^2 T^4}$ on the problem of estimating the range and velocity of the target.

2.4 Consider the rectangular pulse with unit energy

$$s(t) = \frac{1}{\sqrt{2T}} \chi_T(t) e^{2\pi i \omega_o t}$$

where

$$\chi_T(t) = \begin{cases} 1 \text{ if } -T \leq t \leq T \\ 0 \text{ otherwise }. \end{cases}$$

Show that the ambiguity function is

$$A_s(u, w) = \begin{cases} \sin[(1 - \frac{|u|}{2T})(4\pi w T)]/4\pi w T \text{ if } |u| \leq 2T \\ 0 \text{ if } |u| > 2T. \end{cases}$$

(This isn't easy.) Describe the level curves $|A_s(u, w)| = k$ in the $u - w$ plane. Show that for $k = 1 - c^2$ with c very close to zero, the level curves can be approximated by $\frac{|u|}{2T} + \frac{8}{3}\pi^2 w^2 T^2 = c^2$.

§3. A GROUP THEORY PRIMER

3.1 Definitions and examples. A group is an abstract mathematical entity which expresses the intuitive concept of symmetry.

Definition. A **group** G is a set of objects $\{g, h, k, \cdots\}$ (not necessarily countable) together with a binary operation which associates to any ordered pair of elements g, h in G a third element gh. the binary operation (called **group multiplication**) is subject to the following requirements:

(1) There exists an element e in G called the **identity element** such that $ge = eg = g$ for all $g \in G$.
(2) For every $g \in G$ there exists in G an **inverse element** g^{-1} such that $gg^{-1} = g^{-1}g = e$.
(3) Associative law. The identity $(gh)k = g(hk)$ is satisfied for all $g, h, k \in G$.

Any set together with a binary operation which satisfies conditions (1) - (3) is called a group. If $gh = hg$ we say that the elements g and h commute. If all elements of G commute then G is a **commutative** or **abelian** group. If G has a finite number $n(G)$ of elements it has **finite order**; otherwise G has **infinite order**.

A **subgroup** H of G is a subset which is itself a group under the group multiplication defined in G. The subgroups G and $\{e\}$ are called **improper** subgroups of G; all other subgroups are **proper**. It can be shown that the identity element e is unique. Also, every element g of G has a unique inverse g^{-1}.

Examples of groups.

(1) The real numbers R with addition as the group product. The product of $x_1, x_2 \in R$ is their sum $x_1 + x_2$. The identity is 0 and the inverse of x is $-x$. Here, R is an infinite abelian group. Among the proper subgroups of R are the integers and the even integers.
(2) The nonzero real numbers in R with multiplication of real numbers as the (commutative) group product. The identity is 1 and the inverse of x is $1/x$. The positive real numbers form a proper subgroup.
(3) The set of matrices

$$R' = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in R \right\}$$

with matrix multiplication as the group product. The identity element is the identity matrix. Here,

$$\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_1 + x_2 \\ 0 & 1 \end{pmatrix}$$

so the one-to-one mapping

$$x \longleftrightarrow r(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in R'$$

relating the group element x in R to r in R' takes products to products. (A one-to-one mapping from a group G onto a group G' which takes products to products is called a **group isomorphism**. We can identify the two groups in the sense that they have the same multiplication table.) Thus R and R' are isomorphic groups.

More generally we define a **homomorphism** $\mu : G \to G'$ as a mapping from the group G into a group G' which transforms products into products. Thus to every $g \in G$ there is associated $\mu(g) \in G'$ such that $\mu(g_1 g_2) = \mu(g_1)\mu(g_2)$ for all $g_1, g_2 \in G$. (It follows that $\mu(e) = e'$ where e' is the identity element in G', and $\mu(g^{-1}) = \mu(g)^{-1}$.) If μ is one-to-one and onto (i.e. if $\mu(G) = G'$) then μ is an isomorphism of G and G'.

(4) The **symmetric group** S_n. Let n be a positive integer. A **permutation** s of n objects (say the set $X = \{1, 2, \cdots, n\}$) is a one-to-one mapping of X onto itself. We can write such a permutation as

$$s = \begin{pmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix} \tag{3.1}$$

where 1 is mapped into p_1, 2 into p_2, $\cdots$, n into p_n. The numbers $p_1, \cdots, p_n$ are a reordering of $1, 2, \cdots, n$, and no two of the p_j are the same. The order in which the columns of (3.1) are written is unimportant. The **inverse** permutation s^{-1} is given by

$$s^{-1} = \begin{pmatrix} p_1 & p_2 & \cdots & p_n \\ 1 & 2 & \cdots & n \end{pmatrix}$$

and the **product** of two permutations s and

$$t = \begin{pmatrix} q_1 & q_2 & \cdots & q_n \\ 1 & 2 & \cdots & n \end{pmatrix}$$

is the permutation

$$st = \begin{pmatrix} q_1 & q_2 & \cdots & q_n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}.$$

(Here we read the product from right to left: t maps q_i to i and s maps i to p_i, so st maps q_i to p_i.) The **identity** permutation is

$$e = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}.$$

It is straightforward to show that the permutations of n objects form a group S_n of finite order $n(S_n) = n!$. For $n > 2$, S_n is not commutative.

(5) The **real general linear group** $GL(n, R)$. The group elements A are nonsingular $n \times n$ matrices with real coefficients:

$$GL(n, R) = \{A = (A_{ij}), 1 \le i, j \le n : A_{ij} \in R \text{ and } \det A \neq 0\}.$$

Group multiplication is ordinary matrix multiplication. The identity element is the identity matrix $E = (\delta_{ij})$ where δ_{ij} is the Kronecker delta. The inverse of an element A is its matrix inverse. This group is infinite and for $n \geq 2$ it is non-abelian. Among the subgroups of $GL(n, R)$ are the **real special linear group**

$$SL(n, R) = \{A \in GL(n, R) : \det A = 1\}$$

and the **orthogonal group**

$$O(n) = \{A \in GL(n, R) : AA^t = E\}$$

where $A^t = (A_{ji})$ is the **transpose** of the $n \times n$ matrix $A = (A_{ij})$.

Similarly the **complex general linear** group

$$GL(n, \mathcal{C}) = \{A = (A_{ij}), 1 \leq i, j \leq n : A_{ij} \in \mathcal{C} \text{ and } \det A \neq 0\},$$

where $\mathcal{C}$ is the field of complex numbers, is a group under matrix multiplication. Among its subgroups are the **complex special linear group**

$$SL(n, \mathcal{C}) = \{A \in GL(n, \mathcal{C}) : \det A = 1\},$$

and the **unitary group**

$$U(n) = \{A \in GL(n, \mathcal{C}) : A\bar{A}^t = E\}$$

where $\bar{A} = \{\bar{A}_{ij}\}$ is the **complex** conjugate of $A = \{A_{ij}\}$. For $n = 1$, $U(1) = \{z : |z| = 1\}$ is the **circle group**.

It is not difficult to show that every finite group is isomorphic to a subgroup of S_n. Furthermore, every finite group is isomorphic to a (finite) subgroup of $GL(n, R)$.

(6) The group Z_n. This is the finite abelian group whose elements are the integers $0, 1, 2, \cdots, n-1$ for fixed $n \geq 1$, and group multiplication is addition mod n. Thus 0 is the identity element and $n - k$ is the inverse of $k = 1, \ldots, n-1$. This group is isomorphic to the multiplicative group of 1×1 matrices

$$C_n = \{\exp(2\pi i k/n) : k = 0, 1, \cdots, n-1\},$$

where

$$k \longleftrightarrow \exp(2\pi i k/n)$$

is the isomorphism.

(7) The affine or $ax+b$ group. The affine group G_A is the subgroup of $GL(n, R)$ consisting of matrices

$$(a, b) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \quad a, b \in R, \ a > 0.$$

Clearly the group product is

$$(a_1, b_1) \cdot (a_2, b_2) = \begin{pmatrix} a_1 a_2 & a_1 b_2 + b_1 \\ 0 & 1 \end{pmatrix}$$
$$= (a_1 a_2, a_1 b_2 + b_1),$$

the identity element is $(1, 0)$, and the inverse of (a, b) is $(1/a, -b/a)$.

(8) The (three-dimensional real) **Heisenberg group**

$$H_R = \left\{ (x_1, x_2, x_3) = \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} : x_i \in R \right\}.$$

This is a subgroup of $GL(3, R)$ with group product

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_2).$$

The identity element is the identity matrix $(0, 0, 0)$ and $(x_1, x_2, x_3)^{-1} = (-x_1, -x_2, x_1 x_2 - x_3)$. Note that H_R is non abelian. The **center** of H_R, i.e., the subgroup C_R of all elements which commute with every element of H_R,

$$C_R = \{h \in H_R : hg = gh, \forall g \in H_R\}$$

consists of the elements $(0, 0, x_3), x_3 \in R$. A finite analog of H_R is

$$H_n = \left\{ \begin{pmatrix} 1 & a_1 & a_3 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix} : a_i \in Z_n \right\},$$

a finite group under matrix multiplication where addition and multiplication of the number a_i is carried out mod n.

(9) Let V be a finite dimensional vector space, real or complex, and denote by $GL(V)$ the set of all invertible linear transformations of $\mathbf{T}$ of V onto V. Then $GL(V)$ is a group with the product of $\mathbf{T}_1, \mathbf{T}_2 \in GL(V)$ given by $\mathbf{T}_1\mathbf{T}_2$ where $(\mathbf{T}_1\mathbf{T}_2)\mathbf{v} = \mathbf{T}_1(\mathbf{T}_2\mathbf{v})$ for each $\mathbf{v} \in V$, i.e. , $\mathbf{T}_1\mathbf{T}_2$ is the usual product of linear transformations. The identity element is the operator $\mathbf{E}$ such that $\mathbf{E}\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in G$. The inverse of $\mathbf{T} \in GL(V)$ is the inverse linear operator $\mathbf{T}^{-1}$ where $\mathbf{T}\mathbf{v} = \mathbf{w}$ for $\mathbf{v}, \mathbf{w} \in V$ if and only if $\mathbf{T}^{-1}\mathbf{w} = \mathbf{v}$.

Suppose $\dim V = n$ and let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be a basis for V. The matrix of the operator $\mathbf{T}$ with respect to the basis is $T = (T_{ij})$ where $\mathbf{T}\mathbf{v}_j = \sum_{i=1}^n T_{ij}\mathbf{v}_i, 1 \le j \le n$. It is easy to see that this correspondence defines an isomorphism between the group $GL(V)$ and $GL(n)$.

Many of the most important applications of groups to the sciences are expressed in terms of the group representation, to which we now turn.

3.2 Group representations.

Definition. A **representation (rep)** of a group G with **representation space** V is a homomorphism $\mathbf{T} : g \to \mathbf{T}(g)$ of G into $GL(V)$. The **dimension** of the representation is the dimension of V.

It follows from this definition that

$$\mathbf{T}(g_1)\mathbf{T}(g_2) = \mathbf{T}(g_1 g_2), \quad \mathbf{T}(g)^{-1} = \mathbf{T}(g^{-1}), \qquad (3.2)$$
$$\mathbf{T}(e) = \mathbf{E}, \quad g_1, g_2, g \in G,$$

(Initially we shall consider only finite-dimensional reps of groups on complex rep spaces. Later we will lift these finiteness restrictions.)

Definition. An **n-dimensional matrix rep** of G is a homomorphism $T : g \to T(g)$ of G into $GL(n, \mathcal{C})$.

The $n \times n$ matrices $T(g), g \in G$, satisfy multiplication properties analogous to (3.2). Any group rep $\mathbf{T}$ of G with rep space V defines many matrix reps, since if $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is a basis of V, the matrices $T(g) = (T(g)_{kj})$ defined by

$$\mathbf{T}(g)\mathbf{v}_k = \sum_{j=1}^{n} T(g)_{jk}\mathbf{v}_j, \quad 1 \le k \le n \qquad (3.3)$$

form an n-dimensional matrix rep of G. Every choice of basis for V yields a new matrix rep of G defined by $\mathbf{T}$. However, any two such matrix reps T, T' are equivalent in the sense that there exists a matrix $S \in GL(n, \mathcal{C})$ such that

$$T'(g) = ST(g)S^{-1} \qquad (3.4)$$

for all $g \in G$. Indeed, if T, T' correspond to the bases $\{\mathbf{v}_i\}, \{\mathbf{v}'_i\}$ respectively, then for S we can take the matrix (S_{ji}) defined by

$$\mathbf{v}_i = \sum_{j=1}^{n} S_{ji}\mathbf{v}'_j, \quad i = 1, \cdots, n. \qquad (3.5)$$

Definition. Two complex n-dimensional matrix reps T and T' are **equivalent** ($T \approx T'$) if there exists an $S \in GL(n, \mathcal{C})$ such that (3.4) holds.

Thus equivalent matrix reps can be viewed as arising from the same operator rep. Conversely, given an n-dimensional matrix rep $T(g)$ we can define many n-dimensional operator reps of G. If V is an n-dimensional vector space with basis $\{\mathbf{v}_i\}$ we can define the group rep $\mathbf{T}$ by (3.3), i.e., we define the operator $\mathbf{T}(g)$ by the right-hand side of (3.3). Every choice of a vector space V and a basis $\{\mathbf{v}_i\}$ for V yields a new operator rep defined by T. However, if V, V' are two such n-dimensional vector spaces with bases $\{\mathbf{v}_i\}, \{\mathbf{v}'_i\}$ respectively, then the reps $\mathbf{T}$ and $\mathbf{T}'$ are related by

$$\mathbf{T}'(g) = \mathbf{S}\mathbf{T}(g)\mathbf{S}^{-1}, \qquad (3.6)$$

where $\mathbf{S}$ is an invertible operator from V onto V' defined by

$$\mathbf{S}\mathbf{v}_i = \mathbf{v}'_i \quad 1 \le i \le n.$$

Definition. Two n-dimensional group reps $\mathbf{T}, \mathbf{T}'$ of G on the spaces V, V' are **equivalent** ($\mathbf{T} \cong \mathbf{T}'$) if there exists an invertible linear transformation $\mathbf{S}$ of V onto V' such that (3.6) holds.

Clearly, there is a one-to-one correspondence between classes of equivalent operator reps and classes of equivalent matrix reps. In order to determine all possible reps of a group G it is enough to find one rep $\mathbf{T}$ in each equivalence class. It is a matter of choice whether we study operator reps or matrix reps.

The following are examples of group reps:

(1) The matrix groups $GL(n, \mathbb{C}), SL(n, \mathbb{C})$ are n-dimensional matrix reps of themselves.

(2) Let G be a finite group of order n. We formally define an n-dimensional vector space R_G consisting of all elements of the form

$$\sum_{g \in G} x(g) \cdot g, \quad x(g) \in \mathbb{C}.$$

Two vectors $\sum x(g) \cdot g$ and $\sum y(g) \cdot g$ are equal if and only if $x(g) = y(g)$ for all $g \in G$. The sum of two vectors and the scalar multiple of a vector are defined by

$$\begin{aligned} \sum x(g) \cdot g + \sum y(g) \cdot g &= \sum [x(g) + y(g)] \cdot g, \\ \alpha \sum x(g) \cdot g &= \sum \alpha x(g) \cdot g, \quad \alpha \in \mathbb{C}. \end{aligned} \tag{3.7}$$

The zero vector of R_G is $\theta = \sum 0 \cdot g$. The vectors $1 \cdot g_0$, $g_0 \in G$, form a natural basis for R_G. (From now on, we make the identification $1 \cdot g = g \in R_G$.) We define the **product** of two elements $x = \sum x(g) \cdot g, \quad y = \sum y(h) \cdot h$ in a natural manner:

$$\begin{aligned} xy = (\sum x(g) \cdot g)(\sum y(h) \cdot h) &= \sum_{g,h \in G} x(g)y(h) \cdot gh \\ &= \sum_{k \in G} xy(k) \cdot k, \end{aligned} \tag{3.8}$$

where

$$xy(k) = \sum_{h \in G} x(h)y(h^{-1}k). \tag{3.9}$$

(Here $xy(g)$ is called the **convolution** of the functions $x(g), y(g)$.) It is easy to verify the following relations:

$$\begin{aligned} &(x + y)z = xz + yz, \quad x(y + z) = xy + xz, \quad x, y, z \in R_G, \\ &(xy)z = x(yz), \quad \alpha(xy) = (\alpha x)y = x(\alpha y), \\ &ex = xe = x, \quad \alpha \in \mathbb{C}, \end{aligned} \tag{3.10}$$

where e is the identity element of G. Thus R_G is an algebra, called the **group ring** of G.

The mapping $\mathbf{L}$ of G into $GL(R_G)$ given by

$$\mathbf{L}(g)x = gx, \quad x \in R_G \tag{3.11}$$

defines an n-dimensional rep of G, the **(left) regular** rep. Indeed,

$$\begin{aligned}\mathbf{L}(g_1g_2)x &= g_1g_2x = \mathbf{L}(g_1)g_2x = \mathbf{L}(g_1)\mathbf{L}(g_2)x\\ \mathbf{L}(e)x &= ex = x\end{aligned}$$

and the $\mathbf{L}(g)$ are linear operators. Similarly, the **(right) regular** rep of G is defined by

$$\mathbf{R}(g)x = xg^{-1}, \quad x \in R_G,\ g \in G. \tag{3.12}$$

Let $\mathbf{T}$ be a rep of the finite group G on a finite-dimensional inner product space V. The rep $\mathbf{T}$ is **unitary** if for all $g \in G$

$$\langle \mathbf{T}(g)\mathbf{v}, \mathbf{T}(g)\mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle, \quad \mathbf{v}, \mathbf{w} \in V, \tag{3.13}$$

i.e., if the operators $\mathbf{T}(g)$ are unitary. Recall that an **orthonormal (ON)** basis for the n-dimensional space V is a basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ such that $\langle \mathbf{v}_i, \mathbf{v}_j\rangle = \delta_{ij}$, where $\langle \cdot, \cdot\rangle$ is the inner product on V. The matrices $T(g)$ of the operators $\mathbf{T}(g)$ with respect to an ON basis $\{\mathbf{v}_i\}$ are unitary matrices

$$\overline{T(g)}_{ji} = T(g^{-1})_{ij} = [T(g)^{-1}]_{ij}.$$

Hence, they form a unitary matrix rep of G. The following theorem shows that for finite groups at least, we can always restrict ourselves to unitary reps.

Theorem 3.1. *Let $\mathbf{T}$ be a rep of G on the inner product space V. Then $\mathbf{T}$ is equivalent to a unitary rep on V.*

Proof. First we define a new inner product $(\cdot, \cdot)$ on V with respect to which $\mathbf{T}$ is unitary. For $\mathbf{u}, \mathbf{v} \in V$ let

$$(\mathbf{u}, \mathbf{v}) = \frac{1}{n(G)} \sum_{g\in G} \langle \mathbf{T}(g)\mathbf{u}, \mathbf{T}(g)\mathbf{v}\rangle. \tag{3.14}$$

(Thus $(\mathbf{u}, \mathbf{v})$ is an average of the numbers $\langle \mathbf{T}(g)\mathbf{u}, \mathbf{T}(g)\mathbf{v}\rangle$ taken over the group.) It is easy to check that $(\cdot, \cdot)$ is an inner product on V. Furthermore,

$$\begin{aligned}(\mathbf{T}(h)\mathbf{u}, \mathbf{T}(h)\mathbf{v}) &= \frac{1}{n(G)} \sum_{g\in G} \langle \mathbf{T}(gh)\mathbf{u}, \mathbf{T}(gh)\mathbf{v}\rangle\\ &= \frac{1}{n(G)} \sum_{g'\in G} \langle \mathbf{T}(g')\mathbf{u}, \mathbf{T}(g')\mathbf{v}\rangle\\ &= (\mathbf{u}, \mathbf{v}),\end{aligned}$$

where the next to last equality follows from the fact that if g runs through the elements of G exactly once, then so does gh. Clearly, $\mathbf{T}$ is unitary with respect to the inner product $(\cdot,\cdot)$. Now let $\{\mathbf{u}_i\}$ be an ON basis of V with respect to $(\cdot,\cdot)$ and let $\{\mathbf{v}_i\}$ be an ON basis with respect to $\langle\cdot,\cdot\rangle$. Define the nonsingular linear operator $\mathbf{S}: V \to V$ by $\mathbf{S}\mathbf{u}_i = \mathbf{v}_i, 1 \le i \le n$. Then for $\mathbf{w} = \sum \alpha_i \mathbf{u}_i$ and $\mathbf{x} = \sum \beta_j \mathbf{u}_j$ we find

$$\begin{aligned}\langle \mathbf{Sw}, \mathbf{Sx}\rangle &= \sum_{i,j} \alpha_i \overline{\beta_j} \langle \mathbf{Su}_i, \mathbf{Su}_j\rangle \\ &= \sum_i \alpha_i \overline{\beta_i} = (\mathbf{w}, \mathbf{x})\end{aligned}$$

so $\langle \mathbf{w}, \mathbf{x}\rangle = (\mathbf{S}^{-1}\mathbf{w}, \mathbf{S}^{-1}\mathbf{x})$ and

$$\begin{aligned}\langle \mathbf{ST}(g)\mathbf{S}^{-1}\mathbf{w}, \mathbf{ST}(g)\mathbf{S}^{-1}\mathbf{x}\rangle &= (\mathbf{T}(g)\mathbf{S}^{-1}\mathbf{w}, \mathbf{T}(g)\mathbf{S}^{-1}\mathbf{x}) \\ &= (\mathbf{S}^{-1}\mathbf{w}, \mathbf{S}^{-1}\mathbf{x}) = \langle \mathbf{w}, \mathbf{x}\rangle.\end{aligned}$$

Thus, the rep $\mathbf{T}'(g) = \mathbf{ST}(g)\mathbf{S}^{-1}$ is unitary on V. □

It follows that we can always assume that a rep $\mathbf{T}$ on V is unitary.

Now we study the decomposition of a finite-dimensional rep of a finite group G into irreducible components.

Definition. A subspace W of V is **invariant** under $\mathbf{T}$ if $\mathbf{T}(g)\mathbf{w} \in W$ for every $g \in G$, $\mathbf{w} \in W$.

If W is invariant we can define a rep $\mathbf{T}' = \mathbf{T}|W$ of G on W by

$$\mathbf{T}'(g)\mathbf{w} = \mathbf{T}(g)\mathbf{w}, \quad \mathbf{w} \in W.$$

This rep is called the **restriction** of $\mathbf{T}$ to W. If $\mathbf{T}$ is unitary so is $\mathbf{T}'$.

Definition. The rep $\mathbf{T}$ is **reducible** if there is a proper subspace W of V which is invariant under $\mathbf{T}$. Otherwise, $\mathbf{T}$ is **irreducible (irred)**.

A rep is irred if the only invariant subspaces of V are $\{\boldsymbol{\theta}\}$, the zero vector, and V itself.

Every reducible rep $\mathbf{T}$ can be decomposed into irred reps in an almost unique manner. In proving this, we can assume that $\mathbf{T}$ is unitary.

If W is a proper subspace of the inner product space V and

$$W^{\perp} = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{w}\rangle = 0, \quad \text{all } \mathbf{w} \in W\} \tag{3.15}$$

is the subspace of all vectors perpendicular to W, it is easy to show that $V = W \oplus W^{\perp}$, (V is the **direct sum** of W and $W^{\perp}$). That is, every $\mathbf{v} \in V$ can be written uniquely in the form

$$\mathbf{v} = \mathbf{w} + \mathbf{w}', \quad \mathbf{w} \in W, \quad \mathbf{w}' \in W^{\perp}.$$

Theorem 3.2. *If $\mathbf{T}$ is a reducible unitary rep of G on V and W is a proper invariant subspace of W, then $W^\perp$ is also a proper invariant subspace of V. In this case we write $\mathbf{T} = \mathbf{T}' \oplus \mathbf{T}''$ and say that $\mathbf{T}$ is the* **direct sum** *of $\mathbf{T}'$ and $\mathbf{T}''$, where $\mathbf{T}', \mathbf{T}''$ are the (unitary) restrictions of $\mathbf{T}$ to $W, W^\perp$, respectively.*

Proof. We must show $\mathbf{T}(g)\mathbf{u} \in W^\perp$ for every $g \in G, \mathbf{u} \in W^\perp$. Now for every $\mathbf{w} \in W$,

$$\langle \mathbf{T}(g)\mathbf{u}, \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{T}(g^{-1})\mathbf{w}\rangle = 0$$

since $\mathbf{T}(g^{-1})\mathbf{w} \in W$. The first equality follows from (3.13) and unitarity. Thus $\mathbf{T}(g)\mathbf{u} \in W^\perp$. □

Suppose $\mathbf{T}$ is reducible and V_1 is a proper invariant subspace of V of smallest dimension. Then, necessarily, the restriction $\mathbf{T}_1$ of $\mathbf{T}$ to V_1 is irred and we have the direct sum decomposition $V = V_1 \oplus V_1^\perp$, where $V_1^\perp$ is invariant under $\mathbf{T}$. If $V_1^\perp$ is not irred we can find a proper irred subspace V_2 of smallest dimension such that $V_1^\perp = V_2 \oplus V_2^\perp$, by repeating the above argument. We continue in this fashion until eventually we obtain the direct sum decomposition

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_\ell \quad \text{or} \quad \mathbf{T} = \mathbf{T}_1 \oplus \mathbf{T}_2 \oplus \cdots \oplus \mathbf{T}_\ell$$

where the V_i are mutually orthogonal proper invariant subspaces of V which transform irreducibly under the restrictions $\mathbf{T}_i$ of $\mathbf{T}$ to V_i. (The decomposition process comes to an end after a finite number of steps because V is finite-dimensional.) Some of the $\mathbf{T}_i$ may be equivalent. If a_1 of the reps $\mathbf{T}_i$ are equivalent to $\mathbf{T}_1, a_2$ to $\mathbf{T}_2, \cdots, a_k$ to $\mathbf{T}_k$ and $\mathbf{T}_1, \cdots, \mathbf{T}_k$ are pairwise nonequivalent, we write

$$\mathbf{T} = \sum_{j=1}^{k} \oplus a_j \mathbf{T}_j. \tag{3.16}$$

Theorem 3.3. *Every finite-dimensional unitary rep of a finite group can be decomposed into a direct sum of irred unitary reps*

The above decomposition is not unique since the irred subspaces $V_1, \cdots, V_\ell$ are not uniquely determined. However, it can be shown that the integers a_j in (3.16) are uniquely determined, [B7], [G1], [M5].

3.3 Shur's lemmas. The following two theorems (Shur's lemmas) are crucial for the analysis of irred reps.

Theorem 3.4. *Let $\mathbf{T}, \mathbf{T}'$ be irred reps of the group G on the finite-dimensional vector spaces V, V' respectively and let $\mathbf{A}$ be a nonzero linear transformation mapping V into V', such that*

$$\mathbf{T}'(g)\mathbf{A} = \mathbf{A}\mathbf{T}(g) \tag{3.17}$$

for all $g \in G$. Then $\mathbf{A}$ is a nonsingular linear transformation of V onto V', so $\mathbf{T}$ and $\mathbf{T}'$ are equivalent.

Proof. Let $N_{\mathbf{A}}$ be the **null space** and $R_{\mathbf{A}}$ the **range** of $\mathbf{A}$:

$$N_{\mathbf{A}} = \{\mathbf{v} \in V : \mathbf{A}\mathbf{v} = \boldsymbol{\theta}\}, \quad R_A = \{\mathbf{v}' \in V' : \mathbf{v}' = \mathbf{A}\mathbf{v} \text{ for some } \mathbf{v} \in V\}.$$

The subspace N_A of V is invariant under $\mathbf{T}$ since $\mathbf{AT}(g)\mathbf{v} = \mathbf{T}'(g)\mathbf{Av} = \boldsymbol{\theta}$ for all $g \in G, \mathbf{v} \in N_A$. Since $\mathbf{T}$ is irred, $N_\mathbf{A}$ is either V or $\{\boldsymbol{\theta}\}$. The first possibility implies $\mathbf{A}$ is the zero operator, which contradicts the hypothesis. Therefore, $N_\mathbf{A} = \{\boldsymbol{\theta}\}$. The subspace $R_\mathbf{A}$ of V' is invariant under $\mathbf{T}'$ because $\mathbf{T}'(g)\mathbf{Av} = \mathbf{AT}(g)\mathbf{v} \in R_\mathbf{A}$ for all $\mathbf{v} \in V$. But $\mathbf{T}'$ is irred so $R_\mathbf{A}$ is either V' or $\{\boldsymbol{\theta}\}$. If $R_\mathbf{A} = \{\boldsymbol{\theta}\}$ then $\mathbf{A}$ is the zero operator, which is impossible. Therefore $R_\mathbf{A} = V'$ which implies that $\mathbf{T}$ and $\mathbf{T}'$ are equivalent. □

Corollary 3.1. *Let $\mathbf{T}, \mathbf{T}'$ be nonequivalent finite-dimensional irred reps of G. If $\mathbf{A}$ is a linear transformation from V to V' which satisfies (3.17) for all $g \in G$, then $\mathbf{A}$ is the zero operator.*

While Theorems 3.1 – 3.4 and Corollary 3.1 apply also for real vector spaces, the following results are true only for complex reps.

Theorem 3.5. *Let $\mathbf{T}$ be a rep of the group G on the finite-dimensional complex vector space V, $(\dim V \geq 1)$. Then $\mathbf{T}$ is irred if and only if the only transformations $\mathbf{A} : V \to V$ such that*

$$\mathbf{T}(g)\mathbf{A} = \mathbf{AT}(g) \tag{3.18}$$

for all $g \in G$ are $\mathbf{A} = \lambda\mathbf{E}$ where $\lambda \in \mathbb{C}$ and $\mathbf{E}$ is the identity operator on V.

Proof. It is well known that a linear operator on a finite-dimensional **complex** vector space always has at least one eigenvalue. Let λ be an eigenvalue of an operator $\mathbf{A}$ which satisfies (3.18) and define the eigenspace C_λ by

$$C_\lambda = \{\mathbf{v} \in V : \mathbf{Av} = \lambda\mathbf{v}\}.$$

Clearly C_λ is a subspace of V and $\dim C_\lambda > 0$. Furthermore, C_λ is invariant under $\mathbf{T}$ because

$$\mathbf{AT}(g)\mathbf{v} = \mathbf{T}(g)\mathbf{Av} = \lambda\mathbf{T}(g)\mathbf{v}$$

for $\mathbf{v} \in C_\lambda$, $g \in G$, so $\mathbf{T}(g)\mathbf{v} \in C_\lambda$. If $\mathbf{T}$ is irred then $C_\lambda = V$ and $\mathbf{Av} = \lambda\mathbf{v}$ for all $\mathbf{v} \in V$.

Conversely, suppose $\mathbf{T}$ is reducible. Then there exists a proper invariant subspace V_1 of V and by Theorem 3.2, a proper invariant subspace V_2 such that $V = V_1 \oplus V_2$. Any $\mathbf{v} \in V$ can be written uniquely as $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ with $\mathbf{v}_j \in V_j$. We define the projection operator $\mathbf{P}$ on V by $\mathbf{Pv} = \mathbf{v}_1 \in V_1$. Then $\mathbf{PT}(g)\mathbf{v} = \mathbf{T}(g)\mathbf{Pv} = \mathbf{T}(g)\mathbf{v}_1$ (verify this), and $\mathbf{P}$ is clearly not a multiple of $\mathbf{E}$. □

Choosing a basis for V and a basis for V' we can immediately translate Shur's lemmas into statements about irred matrix reps

Corollary 3.2. *Let T and T' be $n \times n$ and $m \times m$ complex irred matrix reps of the group G, and let A be an $m \times m$ matrix such that*

$$T'(g)A = AT(g) \tag{3.19}$$

for all $g \in G$. If T and T' are nonequivalent then A equals the zero matrix. (In particular, this is true for $n \neq m$.) If $T \equiv T'$ then $A \equiv \lambda E_n$ where $\lambda \in \mathbb{C}$ and E_n is the $n \times n$ identity matrix.

Note that the proofs of Shur's lemmas use only the concept of irreducibility and the fact that the rep spaces are finite-dimensional. The homomorphism property of reps and the fact that G is finite are not needed.

3.4 Orthogonality relations for finite group representations. Now let G be a finite group again and select one irred rep $\mathbf{T}^{(\mu)}$ of G in each equivalence class of irred reps Then every irred rep is equivalent to some $\mathbf{T}^{(\mu)}$ and the reps $\mathbf{T}^{(\mu_1)}, \mathbf{T}^{(\mu_2)}$ are nonequivalent if $\mu_1 \neq \mu_2$. The parameter μ indexes the equivalence classes of irred reps. (We will soon show that there are only a finite number of these classes.) Introduction of a basis in each rep space $V^{(\mu)}$ leads to a matrix rep $T^{(\mu)}$. The $T^{(\mu)}$ form a complete set of irred $n_\mu \times n_\mu$ matrix reps of G, one from each equivalence class. Here $n_\mu = \dim V^{(\mu)}$. If we wish, we can choose the $T^{(\mu)}$ to be unitary.

The following procedure leads to an extremely useful set of relations in rep theory, the **orthogonality relations**. Given two irred matrix reps $T^{(\mu)}, T^{(\nu)}$ of G choose an arbitrary $n_\mu \times n_\nu$ matrix B and form the $n_\mu \times n_\nu$ matrix

$$A = N^{-1} \sum_{g \in G} T^{(\mu)}(g) B T^{(\nu)}(g^{-1}) \tag{3.20}$$

where $N = n(G)$. (Here, A is just the average of the matrices $T^{(\mu)}(g)BT^{(\nu)}(g^{-1})$ over the group G.) We will show that A satisfies

$$T^{(\mu)}(h)A = AT^{(\nu)}(h) \tag{3.21}$$

for all $h \in G$. Indeed,

$$\begin{aligned} T^{(\mu)}(h)A &= N^{-1} \sum_{g \in G} T^{(\mu)}(h) T^{(\mu)}(g) B T^{(\nu)}(g^{-1}) \\ &= N^{-1} \sum_{g \in G} T^{(\mu)}(hg) B T^{(\nu)}((hg)^{-1}) T^{(\nu)}(h) \\ &= AT^{(\nu)}(h). \end{aligned}$$

We have used the fact that as g runs over each of the elements of G exactly once, so does $g' = hg$. This result and Corollary 3.1 imply that if $\mu \neq \nu$ then A is the zero matrix, whereas if $\mu = \nu$ then $A = \lambda E_{n_\mu}$ for some $\lambda \in \mathcal{C}$. Hence, $A = \lambda(\mu, B)\delta_{\mu\nu} E_{n_\mu}$ where $\delta_{\mu\nu}$ is the Kronecker delta, and the coefficient λ depends on μ and B.

To derive all possible consequences of this identity it is enough to let B run through the $n_\mu \times n_\nu$ matrices $B^{(\ell,m)} = (B^{(\ell,m)}_{jk})$, where

$$B^{(\ell,m)}_{jk} = \delta_{j\ell}\delta_{km}, \quad 1 \leq j, \ell \leq n_\mu, \quad 1 \leq k, m \leq n_\nu.$$

Making these substitutions, we obtain

$$\sum_{g \in G} T^{(\mu)}_{i\ell}(g) T^{(\nu)}_{ms}(g^{-1}) = N\lambda \delta_{\mu\nu}\delta_{is}, \quad 1 \leq i, \ell \leq n_\mu, \quad 1 \leq m, s \leq n_\nu. \tag{3.22}$$

Here, λ may depend on μ, ℓ and m, but not on i or s. To evaluate λ, set $\nu = \mu$, $s = i$, and sum on i to obtain

$$\begin{aligned} n_\nu N\lambda = \sum_{g \in G} \sum_{i=1}^{n_\mu} T^{(\mu)}_{mi}(g^{-1}) T^{(\mu)}_{i\ell}(g) &= \sum_{g \in G} T^{(\mu)}_{mi}(e) \\ &= N\delta_{mi} \end{aligned}$$

since $N = n(G)$ and $T^{(\mu)}_{mi}(e) = \delta_{mi}$. Therefore, $\lambda = \delta_{mi}/n_\mu$. We can simplify (3.22) slightly if we assume (as we can) that all of the matrix reps $T^{(\nu)}(g)$ are unitary. Then

$$T^{(\nu)}_{ms}(g^{-1}) = \overline{T}^{(\nu)}_{sm}(g)$$

and (3.22) reduces to

$$\sum_{g\in G} T^{(\mu)}_{i\ell}(g)\overline{T}^{(\nu)}_{sm}(g) = \frac{N}{n_\mu}\delta_{is}\delta_{\ell m}\delta_{\mu\nu}. \tag{3.23}$$

Expressions (3.22), (3.23) are the **orthogonality relations** for matrix elements of irred reps of G. We can write these relations in a basis-free manner. Suppose the irred unitary reps $\mathbf{T}^{(\mu)}, \mathbf{T}^{\nu}$ act on the rep spaces $V^{(\mu)}, V^{(\nu)}$ with ON bases $\{\mathbf{v}^{(\mu)}_i\}, \{\mathbf{v}^{(\nu)}_s\}$, respectively. Then we have $T^{(\mu)}_{i\ell}(g) = \langle \mathbf{T}^{(\mu)}(g)\mathbf{v}^{(\mu)}_\ell, \mathbf{v}^{(\mu)}_i\rangle$ with a similar expression for $\mathbf{T}^{(\nu)}$. Expanding arbitrary vectors $\mathbf{f}, \mathbf{h} \in V^{(\mu)}$ in the basis $\{\mathbf{v}^{(\mu)}_\ell\}$ and $\mathbf{f}', \mathbf{h}' \in V^{(\nu)}$ in the basis $\{\mathbf{v}^{(\nu)}_s\}$ and using (3.23) we find

$$\sum_{g\in G}\langle \mathbf{T}^{(\mu)}(g)\mathbf{f}, \mathbf{h}\rangle\langle \mathbf{f}', \mathbf{T}^{(\nu)}(g)\mathbf{h}'\rangle = \langle \mathbf{f}', \mathbf{h}\rangle\langle \mathbf{f}, \mathbf{h}'\rangle\delta_{\mu\nu}\frac{N}{n_\mu}. \tag{3.24}$$

That is, the left hand side of (3.24) vanishes unless $\mu = \nu$, in which case the inner products on the right hand side correspond to the space $V^{(\mu)} = V^{(\nu)}$.

To better understand the orthogonality relations it is convenient to consider the elements x of the group ring R_G as complex-valued functions $x(g)$ on the group G. The relation between this approach and the definition of R_G as given in example is provided by the correspondence

$$x = \sum_{g\in G} x(g)\cdot g \longleftrightarrow x(g). \tag{3.25}$$

The elements of the N-tuple $(x(g_1), \cdots, x(g_N))$, where g_i ranges over G, can be regarded as the components of $x \in R_G$ in the natural basis provided by the elements of G. Furthermore the one-to-one mapping (3.25) leads to the relations

$$\begin{aligned} x + y &\leftrightarrow x(g) + y(g), \quad \alpha x \leftrightarrow \alpha x(g) \\ xy &\leftrightarrow xy(g) = \sum_{g'\in G} x(g')y(g'^{-1}g) \end{aligned} \tag{3.26}$$

where the expression defining $xy(g)$ is called the **convolution product** of $x(g)$ and $y(g)$. The ring of functions just constructed is algebraically isomorphic to R_G with the isomorphism given by (3.26). Under this isomorphism the element $h = 1\cdot h \in R_G$ is mapped into the function

$$h(g) = \begin{cases} 1 \text{ if } g = h \\ 0 \text{ otherwise .} \end{cases}$$

Now consider the right regular rep on R_G. Writing

$$\mathbf{R}(h)x = \sum_{g\in G}[\mathbf{R}(h)x](g)\cdot g = xh^{-1} = \sum_g x(gh)\cdot g,$$

we obtain

$$[\mathbf{R}(h)x](g) = x(gh), \quad h \in G, \tag{3.27}$$

as the action of $\mathbf{R}(h)$ on our new model of R_G. From Theorem 3.1, there is an inner product on the N-dimensional vector space R_G with respect to which the right regular rep $\mathbf{R}$ is unitary. Indeed, the following inner product works:

$$(x,y) = N^{-1}\sum_{g\in G} x(g)\overline{y(g)}, \quad x, y \in R_G. \tag{3.28}$$

Now note that for fixed μ, i, j with $1 \le i, j \le n_\mu$ the matrix element $T^{(\mu)}_{ij}(g)$ defines a function on G, hence an element of R_G. Furthermore, comparing (3.28) with (3.23), we see that the functions

$$\varphi^{(\mu)}_{ij}(g) = n^{1/2}_\mu T^{(\mu)}_{ij}(g), \quad 1 \le i, j \le n_\mu, \tag{3.29}$$

where μ ranges over all equivalence classes of irred reps of G, form an ON set in R_G. Since R_G is N-dimensional the ON set can contain at most N elements. Thus there are only a finite number, say ξ, of nonequivalent irred reps of G. Each irred matrix rep μ yields n^2_μ vectors of the form (3.29). The full ON set $\{\varphi^{(\mu)}_{ij}\}$ spans a subspace of R_G of dimension

$$n_1^2 + n_2^2 + \cdots + n_\xi^2 \le N. \tag{3.30}$$

The inequality (3.30) is a strong restriction on the possible number and dimensions of irred reps of G. This result can be further strengthened by showing that the ON set $\{\varphi^{(\mu)}_{ij}\}$ is actually a **basis** for R_G. Since the dimension N of R_G is equal to the number of basis vectors, we obtain the equality

$$n_1^2 + n_2^2 + \cdots + n_\xi^2 = N. \tag{3.31}$$

To prove this result, let V be the subspace of R_G spanned by the ON set $\{\varphi^{(\mu)}_{ij}\}$. From (3.29) and the homomorphism property of the matrices $T^{(\mu)}(g)$ there follows

$$[\mathbf{R}(h)\varphi^{(\mu)}_{ij}](g) = \varphi^{(\mu)}_{ij}(gh) = \sum_{k=1}^{n_\mu} T^{(\mu)}_{kj}(h)\varphi^{(\mu)}_{ik}(g) \in V. \tag{3.32}$$

Thus V is invariant under $\mathbf{R}$. According to Theorem 3.2, $V^\perp$ is also invariant under R and $R_G = V \oplus V^\perp$. (Here $V^\perp$ is defined with respect to the inner product (3.28).) If $V^\perp \ne \{\boldsymbol{\theta}\}$ then it contains a subspace W transforming under some irred rep $\mathbf{T}^{(\nu)}$ of G. Thus, there exists an ON basis $x_1, \cdots, x_{n_\nu}$ for W such that

$$[\mathbf{R}(g)x_i](h) = x_i(hg) = \sum_{j=1}^{n_\mu} T^{(\nu)}_{ji}(g)x_j(h), \quad 1 \le i \le n_\mu. \tag{3.33}$$

Setting $h = e$ in (3.33) we find

$$x_i(g) = \sum_j x_j(e)T^{(\nu)}_{ji}(g) = \sum_j x_j(e)\varphi^{(\nu)}_{ji}(g)/n^{1/2}_\nu,$$

so $x_i \in V$. Thus $W \subseteq V \cap V^\perp$. This is possible only if $W = \{\boldsymbol{\theta}\}$. Therefore, $V^\perp = \{\boldsymbol{\theta}\}$ and $V = R_G$.

Theorem 3.6. *The functions*

$$\{\varphi_{ij}^{(\mu)}(g)\}, \quad \mu = 1, \cdots, \xi, \quad 1 \le i, j \le n_\mu,$$

form an ON basis for R_G. Every function $x \in R_G$ can be written uniquely in the form

$$x(g) = \sum_{i,j,\mu} a_{ij}^{\mu} \varphi_{ij}^{(\mu)}(g), \quad a_{ij}^{\mu} = (x, \varphi_{ij}^{(\mu)}). \tag{3.34}$$

The series (3.34) is the "generalized Fourier expansion" for the functions $x \in R_G$ and the a_{ij}^{μ} are the "generalized Fourier coefficients". Furthermore we have the "generalized Plancherel formula"

$$(x, y) = \sum_{i,j,\mu} (x, \varphi_{ij}^{(\mu)})(\varphi_{ij}^{(\mu)}, y). \tag{3.35}$$

We can also write expansion (3.34) in a basis-free form through the use of (3.29) and the homomorphism property of the matrices $T^{(\mu)}(g)$:

$$\begin{aligned}
x(g) &= \sum_{i,j,\mu} n_\mu (x, T_{ij}^{(\mu)}) T_{ij}^{(\mu)}(g) \\
&= \frac{1}{N} \sum_{i,j,\mu} n_\mu \sum_{h \in G} x(h) \overline{T}_{ij}^{(\mu)}(h) T_{ij}^{(\mu)}(g) \\
&= \frac{1}{N} \sum_{i,j,\mu} n_\mu \sum_{h \in G} x(h) T_{ji}^{(\mu)}(h^{-1}) T_{ij}^{(\mu)}(g) \\
&= \frac{1}{N} \sum_{j,\mu} n_\mu \sum_{h \in G} x(h) T_{jj}^{(\mu)}(h^{-1} g) \\
&= \sum_{j,\mu} n_\mu (x, T_{jj}^{(\mu)}(g^{-1} \cdot)) \\
&= \sum_{\mu} n_\mu (x, \text{tr } T^{(\mu)}(g^{-1} \cdot))
\end{aligned}$$

Here tr $T^{(\mu)}(h)$ is the **trace** *of the matrix $T^{(\mu)}(h)$.*

Relation (3.35) can be written in the basis-free form

$$(x, y) = \sum_{\mu} n_\mu (s_x^{(\mu)}, y)$$

where $s_x^{(\mu)}(h) = (x, \text{tr } T^{(\mu)}(h^{-1} \cdot))$.

3.5 Representations of Abelian groups. The representation theory of Abelian (not necessarily finite) groups is especially simple.

Lemma 3.1. *Let G be an Abelian group and let $\mathbf{T}$ be a finite-dimensional irred rep of G on a complex vector space V. Then $\mathbf{T}$, hence V, is one-dimensional.*

Proof. Suppose $\mathbf{T}$ is irred on V and $\dim V > 1$. There must exist a $g \in G$ such that $\mathbf{T}(g)$ is not a multiple of the identity operator, for otherwise V would be reducible. Let λ be an eigenvalue of $\mathbf{T}(g)$ and let C_λ be the eigenspace

$$C_\lambda = \{\mathbf{v} \in V : \mathbf{T}(g)\mathbf{v} = \lambda \mathbf{v}\}.$$

Clearly, C_λ is a proper subspace of V. If $h \in G$ and $\mathbf{w} \in C_\lambda$ then

$$\mathbf{T}(g)(\mathbf{T}(h)\mathbf{w}) = \mathbf{T}(h)(\mathbf{T}(g)\mathbf{w}) = \lambda(\mathbf{T}(h)\mathbf{w}),$$

since G is Abelian, so C_λ is invariant under the operator $\mathbf{T}(h)$. Therefore, $\mathbf{T}$ is reducible. Contradiction! □

Example. $Z_N, N > 1$

This is the Abelian group of order N, example (6) in §3.1, with addition of integers mod N as group multiplication. Since $n_j = 1$ for each irred rep of Z_N, it follows from (3.31) that Z_N has exactly $\xi = N$ distinct irred reps. Since the N elements of Z_N can be represented as $g_0^k, k = 0, 1, 2, \cdots, N-1$ where $g_0^N = e$, we have $[\mathbf{T}^{(\mu)}(g_0)]^N = \mathbf{T}^{(\mu)}(g_0^N) = \mathbf{T}^{(\mu)}(e) = 1$ for each irred rep $T^{(\mu)}$. Thus $T^{(\mu)}(g_0)$ is an Nth root of unity and it uniquely determines $\mathbf{T}^{(\mu)}(g)$ for all $g \in Z_N$. Since there are exactly N such roots, which we label as

$$\mathbf{T}^{(\mu)}(g_0) = \exp(2\pi i \mu/N), \quad \mu = 0, 1, \cdots, N-1,$$

the possible irred reps are

$$\mathbf{T}^{(\mu)}(g_0^k) = \exp(2\pi i k\mu/N) \quad k, \mu = 0, 1, \cdots, N-1. \tag{3.36}$$

It follows from Theorem 3.6 that every function $x \in R_{Z_N}$ has the finite Fourier expansion

$$x(k) = \sum_{\mu=0}^{N-1} \hat{x}(\mu) \exp(2\pi i k\mu/N) \tag{3.37}$$

where

$$\hat{x}(\mu) = (x, T^{(\mu)}) = \frac{1}{N} \sum_{\ell=0}^{N-1} x(\ell) \exp(-2\pi i \ell\mu/N).$$

Furthermore, the orthogonality relations are

$$(T^{(\nu)}, T^{(\mu)}) = \delta_{\nu\mu} = \frac{1}{N} \sum_{\ell=0}^{N-1} \exp[2\pi i \ell(\nu - \mu)/N)] \tag{3.38}$$

and the Plancherel formula reads

$$\frac{1}{N} \sum_{k=0}^{N-1} x(k)\overline{y}(k) = \sum_{\mu=0}^{N-1} \hat{x}(\mu)\overline{\hat{y}}(\mu), \quad x, y \in R_{Z_N}.$$

Since finite non-Abelian groups do not occur in the sequel, we refer the reader to standard textbooks for examples of unitary irred reps of these groups, [B7], [M5].

3.6 Exercises.

3.1 The **center** of a group G is the subgroup

$$C = \{h \in G : hg = gh, \ \forall g \in G\}.$$

Compute the center of the Heisenberg group H_R and the center of the affine group G_A.

3.2 Let V be an n-dimensional complex vector space with basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$. The **matrix** of the operator $\mathbf{T} \in GL(V)$ with respect to this basis is $T = (T_{ij})$ where $\mathbf{T}\mathbf{v}_j = \sum_{i=1}^{n} T_{ij}\mathbf{v}_i$, $1 \le j \le n$. Verify that the correspondence $\mathbf{T} \leftrightarrow T$ is an isomorphism of the groups $GL(V)$ and $GL(n)$.

3.3 Show that the map $g \to \mathbf{R}(g)$ is a rep of the group G on the group ring R_G where $\mathbf{R}(g)x = xg^{-1}$ for $g \in G$, $x \in R_G$.

3.4 Verify explicitly that the Hermitian form $(\mathbf{u}, \mathbf{v})$, (3.14), defines an inner product on the vector space V. (Among other things, you must show that $(\mathbf{u}, \mathbf{u}) = 0$ if and only if $\mathbf{u} = \boldsymbol{\theta}$.)

3.5 Let $T_1(g)$ and $T_2(g)$ be $n \times n$ matrix reps of the group G with **real** matrix elements. These reps are **real equivalent** if there is a real nonsingular matrix S such that $T_1(g)S = ST_2(g)$ for all $g \in G$. Show that T_1 and T_2 are complex equivalent if and only if they are real equivalent. (Hint: Write $S = A + iB$, where A and B are real, and show that $A + tB$ is invertible for some real number t.)

3.6 Show that the matrix elements of two real irred reps of a group G which are not real equivalent satisfy an orthogonality relation. Show that every real irred rep is real equivalent to a rep by real orthogonal matrices.

§4. REPRESENTATION THEORY FOR INFINITE GROUPS

4.1 Linear Lie groups. We will now indicate how some of the basic results in the rep theory of finite groups can be extended to infinite groups. A fundamental tool in the rep theory of finite groups is the averaging of a function or operator over the group by taking a sum. We will now introduce a new class of groups, the linear Lie groups, in which one can (explicitly) integrate over the group manifold and, at least for compact linear Lie groups, can prove close analogs of Theorems 3.1–3.5.

Let W be an open connected set containing $\mathbf{e} = (0, \cdots, 0)$ in the space F_n of all (real or complex) n-tuples $\mathbf{g} = (g_1, \cdots, g_n)$. (The reader can assume W is an open sphere with center $\mathbf{e}$.)

Definition. An n-dimensional **local linear Lie group** G is a set of $m \times m$ non-singular matrices $A(\mathbf{g}) = A(g_1, \cdots, g_n)$, defined for each $\mathbf{g} \in W$, such that

(1) $A(\mathbf{e}) = E_m$ (the identity matrix)
(2) The matrix elements of $A(\mathbf{g})$ are analytic functions of the parameters $g_1, \cdots, g_n$ and the map $\mathbf{g} \to A(\mathbf{g})$ is one-to-one.
(3) The n matrices $\frac{\partial A(\mathbf{g})}{\partial g_j}$, $j = 1, \cdots, n$, are linearly independent for each $\mathbf{g} \in W$. That is, these matrices span an n-dimensional subspace of the m^2-dimensional space of all $m \times m$ matrices.
(4) There exists a neighborhood W' of $\mathbf{e}$ in F_n, $W' \subseteq W$, with the property that for every pair of n-tuples $\mathbf{g}, \mathbf{h}$ in W' there is an n-tuple $\mathbf{k}$ in W satisfying

$$A(\mathbf{g})A(\mathbf{h}) = A(\mathbf{k}) \tag{4.1}$$

where the operation on the left is matrix multiplication.

From the implicit function theorem one can show that (4) implies that there exists a nonzero neighborhood V of $\mathbf{e}$ such that $\mathbf{k} = \boldsymbol{\varphi}(\mathbf{g}, \mathbf{h})$ for all $\mathbf{g}, \mathbf{h} \in V$ where $\boldsymbol{\varphi}$ is an analytic vector-valued function of its an arguments and $\mathbf{g}, \mathbf{h}, \mathbf{k}$ are related by (4.1), [HS], [R4].

Let G be a local linear group of $m \times m$ matrices. We will now construct a (connected, global) **linear Lie group** $\tilde{G}$ containing G. Algebraically, $\tilde{G}$ is the abstract subgroup of $GL(m, \mathcal{C})$ generated by the matrices of G. That is, $\tilde{G}$ consists of all possible products of finite sequences of elements in G. In addition, the elements of $\tilde{G}$ can be parametrized analytically. If $B \in \tilde{G}$ we can introduce coordinates in a neighborhood of B by means of the map $\mathbf{g} \to BA(\mathbf{g})$ where $\mathbf{g}$ ranges over a suitably small neighborhood Z of $\mathbf{e}$ in F_n. In particular, the coordinates of B will be $\mathbf{e} = (0, \cdots, 0)$. Proceeding in this way for each $B \in \tilde{G}$ we can cover $\tilde{G}$ with local coordinate systems as "coordinate patches". The same group element C will have many different sets of coordinates, depending on which coordinate patch containing C we happen to consider. Suppose C lies in the intersection of coordinate patches around B_1 and B_2, respectively. Then C will have coordinates $\mathbf{g}_1, \mathbf{g}_2$ respectively, where $C = B_1A(\mathbf{g}_1) = B_2A(\mathbf{g}_2)$. Since

$$A(\mathbf{g}_1) = B_1^{-1}B_2A(\mathbf{g}_2), \quad A(\mathbf{g}_2) = B_2^{-1}B_1A(\mathbf{g}_1)$$

it follows that in a suitably small neighborhood of $\mathbf{e}$: (a) the coordinates $\mathbf{g}_2$ are analytic functions $\mathbf{g}_2 = \boldsymbol{\rho}(\mathbf{g}_1)$ of the coordinates $\mathbf{g}_1$, (b) $\boldsymbol{\rho}$ is one-to-one, and (c) the

Jacobian of the coordinate transformation is nonzero. This makes $\tilde{G}$ into an analytic manifold. (In addition to the coordinate neighborhoods described above, we can always add more coordinate neighborhoods to $\tilde{G}$ provided they satisfy conditions (a) – (c) on the overlap with any of the original coordinate systems.) We leave it to the reader to show that $\tilde{G}$ is **connected**. That is, any two elements A, B in $\tilde{G}$ can be connected by an analytic curve $C(t)$ lying entirely in $\tilde{G}$.

In general an n-dimensional (global) **linear Lie group** K is an abstract matrix group which is also an n-dimensional local linear group G. Clearly, $K \sqsupseteq \tilde{G}$. Indeed, $\tilde{G}$ is the connected component of K containing the identity matrix. The group K need not be connected.

Examples (3), (5) ($GL(n, R)$, $SL(n, R)$, $O(n)$, $GL(n, \mathcal{C})$, $SL(n, \mathcal{C})$, $U(n)$), (7), (8) (H_R) from §3.1 are all linear Lie groups. To illustrate, we can write $A \in GL(n, R)$ for A "close" to the identity matrix as $A_{jk} = \delta_{jk} + g_{jk}, 1 \leq j, k \leq n$, and verify that conditions (1) – (4) of the definition of a local linear Lie group are satisfied for the coordinates $\mathbf{g} = \{g_{jk}\} \in R_{n^2}$. It follows that $GL(n, R)$ is a real global n^2-dimensional linear Lie group.

4.2 Invariant measures on Lie groups. Let G be a real n-dimensional global Lie group of $m \times m$ matrices. A function $f(B)$ on G is **continuous** at $B \in G$ if it is a continuous function of the parameters $(g_1, \cdots, g_n)$ in a local coordinate system for G at B. (Clearly, if f is continuous with respect to one local coordinate system at B it is continuous with respect to all coordinate systems.) If f is continuous at every $B \in G$ then it is a **continuous function** on G. We shall show how to define an infinitesimal volume element dA in G with respect to which the associated integral over the group is left-invariant, i.e.,

$$\int_G f(BA)dA = \int_G f(A)dA, \quad B \in G, \tag{4.2}$$

where f is a continuous function on G such that either of the integrals converges. In terms of local coordinates $\mathbf{g} = (g_1, \cdots, g_n)$ at A,

$$dA = w(\mathbf{g})dg_1 \cdots dg_n = w(\mathbf{g})d\mathbf{g}, \tag{4.3}$$

where the positive continuous function w is called a **weight function**. If $\mathbf{k} = (\mathbf{k}_1, \cdots, \mathbf{k}_n)$ is another set of local coordinates at A then,

$$dA = \tilde{w}(\mathbf{k})dk_1 \cdots dk_n, \quad \tilde{w}(k) = w(\mathbf{g}(\mathbf{k}))|\det(\partial g_i/\partial k_j)|,$$

where the determinant is the Jacobian of the coordinate transformation. (For a precise definition of integrals on manifolds see [S4].)

Two examples of left-invariant integrals are well known. The group R' (example (3) in §3.1) with elements

$$A(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad x \in R \tag{4.4}$$

is isomorphic to the real line. The continuous functions on R' are just the continuous functions $f(x)$ on the real line. Here, dx is a left-invariant measure. Indeed by a simple change of variable we have

$$\int_{-\infty}^{\infty} f(y+x)dx = \int_{-\infty}^{\infty} f(x)dx, \quad y \in R$$

where f is any continuous function on R such that the integral converges. Since R is Abelian, dx is also right-invariant.

A second example is the circle group $U(1) = \{e^{i\phi}\}$, the Abelian group of 1×1 unitary matrices. The continuous functions on $U(1)$ can be written $f(\phi)$, where f is continuous for $0 \leq \phi \leq 2\pi$ and periodic with period 2π. The measure $d\phi$ is left-invariant (right invariant) since

$$\int_0^{2\pi} f(\alpha + \phi) d\phi = \int_0^{2\pi} f(\phi) d\phi.$$

We now show how to construct a left-invariant measure for the n-dimensional real linear Lie group G. Let $A(\mathbf{g})$ be a parametrization of G in a neighborhood of the identity element and chosen such that $A(\theta) = E_m$. Set $C_j = \partial_{g_j} A(\mathbf{g})|_{\mathbf{g}=\theta}$, $1 \leq j \leq n$. Then by property (3) of a local linear Lie group, the $m \times m$ matrices $\{C_j\}$ are linearly independent. Given any other parametrization $A'(\mathbf{h})$ near the identity and such that $A'(\mathbf{h}^0) = E_m$ we have $A'(\mathbf{h}) = A(\mathbf{g}(\mathbf{h}))$ for some analytic function $\mathbf{g}(\mathbf{h})$ and $C'_j = \partial_{h_j} A'(\mathbf{h})|_{\mathbf{h}=\mathbf{h}^0} = \sum_{\ell=1}^n \partial_{g_\ell} A(\mathbf{g})|_{\mathbf{g}=\theta} \left(\frac{\partial g_\ell}{\partial h_j}(\mathbf{h}^0)\right) = \sum_{\ell=1}^n \alpha_\ell^{(j)} C_\ell$. Here the $\{C'_j\}$ must be linearly independent so $\det(\alpha_\ell^{(j)}) \neq 0$. In other words, the **tangent space** at the identity element of G is n-dimensional and once we chose a fixed basis $\{C_j\}$ for the tangent space, any coordinate system in a neighborhood of E_m will determine n linearly independent n-tuples $\alpha^{(1)}, \cdots, \alpha^{(n)}$ which generate a parallelepiped with volume

$$V = |\det(\alpha_k^{(i)})| > 0$$

in the tangent space. Now suppose $A'(\mathbf{h})$ is a parametrization of G in a neighborhood of the group element A_0 and such that $A'(\mathbf{h}^0) = A_0$. Then the matrices $C'_j = \partial_{h_j} A'(\mathbf{h})|_{\mathbf{h}=\mathbf{h}^0}$, $j = 1, \cdots, n$, are linearly independent. Furthermore, the matrices in a neighborhood of $E_m \in G$ can be represented as $A_0^{-1} A'(\mathbf{h}) = A(\mathbf{g}(\mathbf{h}))$. Differentiating this expression with respect to h_j and setting $\mathbf{h} = \mathbf{h}^0$ we find

$$A_0^{-1} C'_j = \sum_{\ell=1}^n \alpha_\ell^{(j)} C_\ell$$

for n linearly independent n-tuples $\alpha^{(1)}, \cdots \alpha^{(n)}$. This defines a parallelepiped in the tangent space to G at A_0 as the volume of its image in the tangent space at E_m:

$$V_{A_0}^{(\mathbf{h})} = |\det(\alpha_\ell^{(j)})| > 0.$$

By construction, our volume element is left-invariant. Indeed if $B \in G$ then

$$(BA_0)^{-1} \frac{\partial}{\partial h_j} [BA_0(\mathbf{h})]|_{\mathbf{h}=\boldsymbol{\theta}} = A_0^{-1} B^{-1} B C'_j = A_0^{-1} C'_j = \sum_{\ell=1}^n \alpha_\ell^{(j)} C_\ell,$$

so $V_{BA_0} = V_{A_0}$. We define the measure $d_\ell A$ on G by

$$d_\ell A = V_A(g) dg_1 \cdots dg_n. \tag{4.5}$$

Expression (4.5) actually makes sense independent of local coordinates. If $\mathbf{k} = (k_1, \cdots, k_n)$ is another local coordinate system at A then

$$A^{-1}\frac{\partial A}{\partial k_\ell} = \sum_j A^{-1}C'_j \frac{\partial g_j}{\partial k_\ell} = \sum_{j,s} \frac{\partial g_j}{\partial k_\ell}\alpha_s^{(j)} C_s$$

so

$$V_a(k) = |\det\left(\sum_j \frac{\partial g_j}{\partial k_\ell}\alpha_s^{(j)}\right) = |\det\left(\frac{\partial g_j}{\partial k_\ell}\right)| \cdot |\det(\alpha_s^{(j)})|.$$

Thus

$$\begin{aligned} V_A(\mathbf{k})dk_1\cdots dk_n &= V_A(\mathbf{g})|\det(\partial g_j/\partial k_\ell)|dk_1\cdots dk_n \\ &= V_A(\mathbf{g})dg_1\cdots dg_n. \end{aligned} \tag{4.6}$$

This shows that the integral

$$\int_G f(A)d_\ell A = \int_G f(g_1,\cdots,g_n)V_A(\mathbf{g})dg_1\cdots dg_n$$

is well-defined, provided it converges. Furthermore,

$$\begin{aligned} \int_G f(BA)d_\ell A &= \int_G f(BA(\mathbf{g}))V_A(\mathbf{g})d\mathbf{g} = \int_G f(BA(\mathbf{g}))V_{BA}(\mathbf{g})d\mathbf{g} \\ &= \int_G f(A)V_A(\mathbf{g})dg = \int_G f(A)d_\ell A, \end{aligned} \tag{4.7}$$

where the third equality follows from the fact that BA runs over G if A does.

By analogous procedures one can also define a right-invariant measure $d_r A$ in G. Writing

$$(\partial A/\partial g_j)A^{-1} = B_j = \sum_k \beta_k^{(j)} C_k,$$

we define

$$W_A(\mathbf{g}) = |\det(\beta_k^{(j)})|, \quad d_r A = W_A(\mathbf{g})dg_1\cdots dg_n. \tag{4.8}$$

The reader can verify that $d_r A$ is right-invariant on G.

Since $A(A^{-1}\partial A/\partial g_j)A^{-1} = (\partial A/\partial g_j)A^{-1}$, we have

$$W_A(\mathbf{g}) = |\det \tilde{A}| \cdot V_A(\mathbf{g}), \tag{4.9}$$

where $\tilde{A}$ is the automorphism $C \to ACA^{-1}$ of the tangent space at the identity. (That is, we consider $\tilde{A}$ as an $n\times n$ matrix acting on the n-dimensional tangent space.) Thus, if $\det\tilde{A} = 1$ then $d_\ell A = d_r A$ and there exists a two-sided invariant measure on G.

It can be shown that a much larger class of groups (the locally compact topological (groups) possesses left-invariant (right-invariant) measures. Furthermore, the left-invariant (right-invariant) measure of a group is unique up to a constant factor. That is, if δA and $\delta' A$ are left-invariant measures on G then there exists a constant $c > 0$ such that $\delta A = c\delta' A$, [G1], [N2].

To illustrate our construction consider the Heisenberg matrix group H_R, example (8) in §3.1. Here H_R is a 3-dimensional Lie group which can be covered by a single coordinate patch $\mathbf{x} = (x_1, x_2, x_3)$.

$$A(\mathbf{x}) = \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}$$

Clearly, the matrices

$$C_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, C_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, C_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

form a basis for the tangent space at the identity element of H_R. Now

$$\begin{aligned} A^{-1}dA &= \begin{pmatrix} 0 & dx_1 & dx_3 - x_1 dx_2 \\ 0 & 0 & dx_2 \\ 0 & 0 & 0 \end{pmatrix} \\ &= C_1 dx_1 + (C_2 - x_1 C_3)dx_2 + C_3 dx_3 \end{aligned}$$

where dA is the differential of $A(\mathbf{x})$. Thus,

$$V_A(\mathbf{x}) = |\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -x_1 \\ 0 & 0 & 1 \end{pmatrix} | = 1$$

and $d_\ell A = dx_1 dx_2 dx_3$. Similarly,

$$\begin{aligned} (dA)A^{-1} &= \begin{pmatrix} 0 & dx_1 & -x_2 dx_1 + dx_3 \\ 0 & 0 & dx_2 \\ 0 & 0 & 0 \end{pmatrix} \\ &= (C_1 - x_2 C_3)dx_1 + C_2 dx_2 + C_3 dx_3, \end{aligned}$$

so

$$W_A(\mathbf{x}) = |\det \begin{pmatrix} 1 & 0 & -x_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} | = 1$$

and $d_r A = dx_1 dx_2 dx_3$.

The affine group G_A, example (7) of §3.1 is 2-dimensional and can again be covered by a single coordinate patch:

$$A(a, b) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \quad a > 0.$$

A basis for the tangent space at the identity is

$$C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and

$$A^{-1}dA = \begin{pmatrix} da/a & db/a \\ 0 & 0 \end{pmatrix} = \frac{1}{a}(C_1 da + C_2 db).$$

Therefore

$$V_A(a,b) = |\det \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{pmatrix} | = \frac{1}{a^2}$$

and $d_\ell A = dadb/a^2$. On the other hand,

$$\begin{aligned} (dA)A^{-1} &= \begin{pmatrix} da/a & -bda/a & +db \\ 0 & 0 \end{pmatrix} \\ &= \frac{1}{a}(C_1 - bC_2)da + C_2 db. \end{aligned}$$

Therefore

$$W_A(a,b) = |\det \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ 0 & 1 \end{pmatrix} | = \frac{1}{a}$$

and $d_r A = dadb/a$. In this case the left- and right-invariant measures are distinct.

4.3 Orthogonality relations for compact Lie groups. Now we are ready to extend the orthogonality relations to representations of a larger class of groups, the compact linear Lie groups. We say that a sequence of $m \times m$ complex matrices $\{A^{(j)}\}$ is a **Cauchy sequence** if each of the sequences of matrix elements $\{A^{(j)}_{ik}\}, 1 \le i,k \le m$ is Cauchy. Clearly, every Cauchy sequence of matrices converges to a unique matrix $A = (A_{ik}), A_{ik} = \lim_{j\to\infty} A^{(j)}_{ik}$. A set U of $m \times m$ matrices is **bounded** if there exists a constant $K > 0$ such that $|A_{ik}| \le K$ for $1 \le i,k \le m$ and all $A \in U$. The set U is **closed** provided every Cauchy sequence in U converges to a matrix in U.

Definition. A (global) group of $m \times m$ matrices is **compact** if it is a bounded, closed subset of the set of all $m \times m$ matrices.

As an example we show that the orthogonal group $O(3,R)$ is compact. If $A \in O(3,R)$ then $A^t A = E_3$, i.e.,

$$\sum_{j=1}^{3} A_{j\ell} A_{jk} = \delta_{\ell k}.$$

Setting $\ell = k$ we obtain $\sum_i (A_{ik})^2 = 1$, so $|A_{ik}| \le 1$ for all i,k. Thus the matrix elements of A are bounded. Let $\{A^{(j)}\}$ be a Cauchy sequence in $O(3,R)$ with limit A. Then $E_3 = \lim_{j\to\infty} (A^{(j)})^t A^{(j)} = A^t A$ so $A \in O(3,R)$ and $O(3,R)$ is compact.

Now suppose G is a real, compact, linear Lie group of dimension n. It follows from the Heine-Borel Theorem [R4] that the group manifold of G can be covered by a finite number of bounded coordinate patches. Thus, for any continuous function $f(A)$ on G, the integral

$$\int_G f(A)d_\ell A = \int_G f(A)V_A(\mathbf{g})d\mathbf{g}$$

will converge (since the domain of integration is bounded.) In particular the integral

$$V_G = \int_G 1 d_\ell A, \tag{4.10}$$

called the **volume** of G, converges. The preceding remarks also hold for the right-invariant measure $d_r A$. Moreover, we can show $d_\ell A = d_r A$ for compact groups.

Theorem 4.1. *If G is a compact linear Lie group then $d_\ell A = d_r A$.*

Proof. By (4.9), $d_r A = |\det \tilde{A}| d_\ell A$, where $\tilde{A}$ is the mapping $C \to ACA^{-1}$ of the tangent space at the identity of G. (We can think of $\tilde{A}$ as an $m^2 \times m^2$ matrix rep of G.) Since G is compact the matrices $A, A^{-1} \in G$ are uniformly bounded. Thus the matrices $\tilde{A}$ are bounded and there exists a constant $M > 0$ such that $|\det \tilde{A}| \leq M$ for all $A \in G$. Now fix A and suppose $|\det \tilde{A}| = s \neq 1$. Then

$$|\det \tilde{A}^j| = |\det \tilde{A}|^j = s^j, \quad j = 0 \pm 1, \pm 2, \cdots.$$

Choosing j appropriately we get $s^j > M$, which is impossible. Therefore $s = 1$ for all $A \in G$ and $d_\ell A = d_r A$. □

For G compact we write $dA = d_\ell A = d_r A$ where the measure dA is both left- and right-invariant.

Using the invariant measure for compact groups we can mimic the proofs of most of the results for finite groups obtained in Chapter 3. In particular we can show that any finite-dimensional rep of a compact group can be decomposed into a direct sum of irred reps and can obtain orthogonality relations for the matrix elements.

For finite groups K these results were proved using the average of a function over K. If f is a function on K then the **average** of f over K is

$$AV(f(k)) = \frac{1}{n(K)} \sum_{k \in K} f(k).$$

If $h \in K$ then

$$AV(f(hk)) = AV(f(kh)) = AV(f(k)). \tag{4.11}$$

Furthermore

$$AV(a_1 f_1(k) + a_2 f_2(k)) = a_1 AV(f_1(k)) + a_2 AV(f_2(k)), \quad AV(1) = 1. \tag{4.12}$$

Properties (4.11) and (4.12) are sufficient to prove most of the fundamental results on the reps of finite groups. Now let G be a compact linear Lie group and let f be a continuous function on G. We define

$$AV(f(A)) = \frac{1}{V_G} \int_G f(A)dA = \int_G f(A)\delta A \tag{4.13}$$

where dA is the invariant measure on G, $V_G = \int_G 1dA$ is the volume of G, and $\delta A = V_G^{-1}dA$ is the **normalized** invariant measure. Then

$$AV(f(BA)) = \int_G f(BA)\delta A = \int_G f(A)\delta A = AV(f(A)),$$
$$AV(f(AB)) = AV(f(A)), \quad AV(1) = \int_G \delta A = 1, \quad B \in G. \tag{4.14}$$

since δA is both left- and right-invariant. Thus, $AV(f(A))$ also satisfies properties (4.11), (4.12).

In order to mimic the finite group constructions we need to limit ourselves to **continuous** reps of G, i.e., reps $\mathbf{T}$ such that the operators $\mathbf{T}(A)$ are continuous functions of the group parameters of $A \in G$.

Theorem 4.2. *Let* $\mathbf{T}$ *be a continuous rep of the compact linear Lie group* G *on the finite-dimensional inner product space* V. *Then* $\mathbf{T}$ *is equivalent to a unitary rep on* V.

Proof. Let $\langle \cdot, \cdot \rangle$ be the inner product on V. We define another inner product $(\cdot, \cdot)$ on V, with respect to which $\mathbf{T}$ is unitary. For $\mathbf{u}, \mathbf{v} \in V$ define

$$(\mathbf{u}, \mathbf{v}) = \int_G \langle \mathbf{T}(A)\mathbf{u}, \mathbf{T}(A)\mathbf{v} \rangle \delta A = AV[\langle \mathbf{T}(A)\mathbf{u}, \mathbf{T}(A)\mathbf{v} \rangle]. \tag{4.15}$$

(The integral converges since the integrand is continuous and the domain of integration is finite.) It is straightforward to check that $(\cdot, \cdot)$ is an inner product. In particular, the positive definite property follows from the fact that the weight function is strictly positive (except possibly on a set of Lebesgue measure 0.) Now

$$(\mathbf{T}(B)\mathbf{u}, \mathbf{T}(B)\mathbf{v}) = AV[\langle \mathbf{T}(AB)\mathbf{u}, \mathbf{T}(AB)\mathbf{v} \rangle] = AV[\langle \mathbf{T}(A)\mathbf{u}, \mathbf{T}(A)\mathbf{v} \rangle] = (\mathbf{u}, \mathbf{v}),$$

so $\mathbf{T}$ is unitary with respect to $(\cdot, \cdot)$. The remainder of the proof is identical with that of Theorem 3.1. □

Thus with no loss of generality, we can restrict ourselves to the study of unitary reps.

Theorem 4.3. *If* $\mathbf{T}$ *is a unitary rep of* G *on* V *and* W *is an invariant subspace of* V *then* $W^\perp$ *is also an invariant subspace under* $\mathbf{T}$.

Theorem 4.4. *Every finite-dimensional, continuous, unitary rep of a compact linear Lie group can be decomposed into a direct sum of irred unitary reps.*

The proofs of these theorems are identical with corresponding proofs for finite groups.

Let $\{\mathbf{T}^{(\mu)}\}$ be a complete set of nonequivalent finite-dimensional unitary irred reps of G, labeled by the parameter μ. (Here we consider only reps of G on **complex** vector spaces.) Initially we have no way of telling how many distinct values μ can take. (It will turn out that μ takes on a countably infinite number of values, so that we can choose $\mu = 1, 2, \cdots$.) We introduce an ON basis in each rep space $V^{(\mu)}$ to obtain a unitary $n_\mu \times n_\mu$ matrix rep $T^{(\mu)}$ of G.

Now we mimic the construction of the orthogonality relations for finite groups. Given the matrix reps $T^{(\mu)}, T^{(\nu)}$, choose an arbitrary $n_\mu \times n_\nu$ matrix C and form the $n_\mu \times n_\nu$ matrix

$$D = AV[T^{(\mu)}(A)CT^{(\nu)}(A^{-1})] = \int_G T^{(\mu)}(A)CT^{(\nu)}(A^{-1})\delta A.$$

Just as in the corresponding construction for finite groups, one can easily verify that

$$T^{(\mu)}(B)D = DT^{(\nu)}(B)$$

for all $B \in G$. Recall that the Shur lemmas are valid for finite-dimensional reps of all groups, not just finite groups. Thus if $\mu \neq \nu$, i.e., $T^{(\mu)}$ not equivalent to $T^{(\nu)}$, then D is the zero matrix. If $\mu = \nu$ then $D = \lambda E_{n_\mu}$ for some $\lambda \in \mathbb{C}$:

$$D(C, \mu, \nu) = \lambda(\mu, C)\delta_{\mu\nu}E_{n_\mu}.$$

Letting C run over all $n_\mu \times n_\nu$ matrices, we obtain the independent identities

$$\int_G T^{(\mu)}_{i\ell}(A)T^{(\nu)}_{ks}(A^{-1})\delta A = \lambda(\mu, \ell, k)\delta_{\mu\nu}\delta_{is}, \tag{4.16}$$

for the matrix elements $T^{(\mu)}_{i\ell}(A)$. To evaluate λ we set $\nu = \mu$, $s = i$ and sum on i:

$$\sum_{i=1}^{n_\mu} \lambda = n_\mu \lambda = \int_G \sum_{i=1}^{n_\mu} T^{(\mu)}_{ki}(A^{-1})T^{(\mu)}_{i\ell}(A)\delta A = \delta_{k\ell}.$$

Therefore $\lambda = \delta_{k\ell}/n_\mu$. Since the matrices $T^{(\mu)}(A)$ are unitary, (4.16) becomes

$$\int_G T^{(\mu)}_{i\ell}(A)\overline{T^{(\nu)}_{sk}}(A)\delta A = (\delta_{is}/n_\mu)\delta_{\ell k}\delta_{\mu\nu}, \ 1 \leq i, \ell \leq n_\mu, \ 1 \leq s, k \leq n_\nu.$$

These are the **orthogonality relations** for matrix elements of irred reps of G.

4.4 The Peter-Weyl theorem. In the case of finite groups K we were able to relate the orthogonality relations to an inner product on the group ring R_K. We can consider R_K as the space of all functions $f(k)$ on K. Then

$$\langle f_1, f_2 \rangle = \frac{1}{n(K)} \sum_{k \in K} f_1(k)\overline{f_2}(k)$$

defines an inner product on R_K with respect to which the functions $\{n_\mu^{1/2} T^{(\mu)}_{i\ell}(g)\}$ form an ON basis. We extend this idea to a compact linear Lie group G as follows: Let $L_2(G)$ be the space of all functions on G which are (Lebesgue) square-integrable:

$$L_2(G) = \{f(A) : \int_G |f(A)|^2 \delta A < \infty\}. \tag{4.18}$$

With respect to the inner product

$$\langle f_1, f_2\rangle = \int_G f_1(A)\overline{f_2}(A)\delta A, \tag{4.19}$$

$L_2(G)$ is a Hilbert space, [G1], [N2]. Note that every continuous function on G belongs to $L_2(G)$. Let

$$\varphi^{(\mu)}_{ij}(A) = n^{1/2}_\mu T^{(\mu)}_{ij}(A). \tag{4.20}$$

It follows from (4.17) and (4.19) that $\{\varphi^{(\mu)}_{ij}\}$, where $1 \le i, j \le n_\mu$ and μ ranges over all equivalence classes of irred reps, forms an ON set in $L_2(G)$.

For finite groups we know that the set $\{\varphi^{(\mu)}_{ij}\}$ is an ON **basis** for the group ring, and every function f on the group can be written as a unique linear combination of these basis functions. Similarly one can show that for G compact the set $\{\varphi^{(\mu)}_{ij}\}$ is an ON basis for $L_2(G)$. Thus, every $f \in L_2(G)$ can be expanded uniquely in the (generalized) **Fourier series**

$$f(A) \sim \sum_\mu \sum_{i,k=1}^{n\mu} c^\mu_{ik}\varphi^{(\mu)}_{ik}(A) \tag{4.21}$$

where

$$c^\mu_{ik} = \langle f, \varphi^{(\mu)}_{ik}\rangle. \tag{4.22}$$

Furthermore we have the **Plancherel equality**

$$\langle f_1, f_2\rangle = \sum_\mu \sum_{i,k=1}^{n\mu} c^{\mu(1)}_{ik}\overline{c}^{\mu(2)}_{ik}.$$

(For $f_1 \equiv f_2$ this is called **Parseval's equality**.) Convergence of the right-hand side of (4.21) to the left-hand side is meant in the sense of the Hilbert space norm. We will not here take up the question of pointwise convergence. See, however, [DS2], [DM2], [K2], [R1].

Theorem 4.5. *(Peter-Weyl). If G is a compact linear Lie group, the set $\{\varphi^{(\mu)}_{ij}\}$ is an ON basis for $L_2(G)$.*

The proof of this theorem depends heavily on facts about symmetric completely continuous operators in Hilbert space and will not be given here. For the details see [G1] or [N2].

Corollary 4.1. *A compact linear Lie group G has a countably infinite (not finite) number of equivalence classes of irred reps $\{\mathbf{T}^{(\mu)}\}$. Thus, we can label the reps so that $\mu = 1, 2, \cdots$.*

Proof. The functions $\{\varphi^{(\mu)}_{jk}\}$ form an ON basis for $L_2(G)$. Since $L_2(G)$ is a separable, infinite-dimensional Hilbert space there are a countably infinite number of basis vectors [G1], [N2]. □

We illustrate the **Peter-Weyl theorem** for an important example, the circle group $U(1)$, example 5 in §3.1. Since $U(1) = \{e^{2\pi i\phi}\}$, $(i = \sqrt{-1})$, is compact and Abelian its irred matrix reps are continuous functions $x(\phi)$ such that

$$x(\phi_1 + \phi_2) = x(\phi_1)x(\phi_2), \quad \phi_1, \phi_2 \in R, \tag{4.23}$$

and $x(\phi + 1) = x(\phi)$. The functional equation (4.23) has only the solutions $x(\phi) = e^{a\phi}$ and the periodicity of x implies $a = 2\pi i m$, where m is an integer. Therefore, there are an infinite number of irreducible unitary representations of $U(1)$:

$$x_m(\phi) = e^{2\pi i m\phi}, \quad m = 0, \pm 1, \pm 2, \cdots .$$

The normalized invariant measure on $U(1)$ is $d\phi$. The space $L_2(U(1))$ is just the space $L_2([0,1])$ consisting of all measurable functions $f(\phi)$ with period 1 such that $\int_0^1 |f(\phi)|^2 d\phi < \infty$. By the Peter-Weyl theorem the functions $\{e^{2\pi i m\phi}\}$ form an ON basis for $L_2([0,1])$. Every $f \in L_2([0,1])$ can be expressed uniquely in the form

$$f(\phi) \sim \sum_{m=-\infty}^{\infty} c_m e^{2\pi i m\phi}, \quad c_m = \int_0^1 f(\phi) e^{-2\pi i m\phi} d\phi. \tag{4.24}$$

Furthermore,

$$\int_0^1 |f(\phi)|^2 d\phi = \sum_{m=-\infty}^{\infty} |c_m|^2. \tag{4.25}$$

Here (4.24) is the well-known Fourier series expansion of a periodic function and (4.25) is Parseval's equality.

4.5 The rotation group and spherical harmonics. A second very important example of the orthogonality relations for compact linear Lie groups and the Peter-Weyl theorem in the rotation group $SO(3) = SO(3, R)$. Here we will give some basic facts about the irreducible representations of $SO(3)$ and refer to the literature for most of the proofs, [B7], [GMS], [M5], [V].

Recall that $SO(3)$ has a convenient realization as the group of all 3×3 real matrices A such that $A^t A = E_3$ and $\det A = 1$, example 5, §3.1. This is the natural realization of $SO(3)$ as the group of all rotations in R_3 which leave the origin fixed. One convenient parametrization of $SO(3)$ is in terms of the **Euler angles**. Recall that a rotation through angle φ about the z axis is given by

$$R_z(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3)$$

and rotations through angle φ about the x and y axis are given by

$$R_x(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} \in SO(3),$$

$$R_y(\varphi) = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix} \in SO(3),$$

respectively. Differentiating each of these curves in $SO(3)$ with respect to φ and setting $\varphi = 0$ we find the following linearly independent matrices in the tangent space at the identity:

$$L_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{4.26}$$

One can check from the definition $A^t A = E_3$ that the tangent space at the identity is at most three-dimensional, so the matrices (4.26) form a basis for this space.

The Euler angles φ, θ, ψ for $A \in SO(3)$ are given by

$$\begin{aligned} A(\varphi, \theta, \psi) &= R_z(\varphi) R_x(\theta) R_z(\psi) \\ &= \begin{pmatrix} \cos\varphi\cos\psi - \sin\varphi\sin\psi\cos\theta, & -\cos\varphi\sin\psi - \sin\varphi\cos\psi\cos\theta, & \sin\varphi\sin\theta \\ \sin\varphi\cos\psi + \cos\varphi\sin\psi\cos\theta, & -\sin\varphi\sin\psi + \cos\varphi\cos\psi\cos\theta, & -\cos\varphi\sin\theta \\ \sin\psi\sin\theta, & \cos\psi\sin\theta, & \cos\theta \end{pmatrix}. \end{aligned} \tag{4.27}$$

It can be shown that every $A \in SO(3)$ can be represented in the form (4.27) where the Euler angles run over the domain

$$0 \le \varphi < 2\pi, \quad 0 \le \theta \le \pi, \quad 0 \le \psi < 2\pi. \tag{4.28}$$

The representation of A by Euler angles is unique except for the cases $\theta = 0, \pi$ where only the sum $\varphi + \psi$ is determined by A, but this exceptional set is only one-dimensional and doesn't contribute to an integral over the group manifold. The invariant measure on $SO(3)$ can be computed directly from the formulas of §4.2. Let $A(\varphi, \theta, \psi) \in SO(3)$. Then

$$\begin{aligned} A^{-1}\frac{\partial A}{\partial \varphi} &= \sin\psi\sin\theta L_x + \cos\psi\sin\theta L_y + \cos\theta L_z, \\ A^{-1}\frac{\partial A}{\partial \theta} &= \cos\psi L_x - \sin\psi L_y, \\ A^{-1}\frac{\partial A}{\partial \psi} &= L_z. \end{aligned}$$

Thus,

$$V_A(\varphi, \theta, \psi) = |\det \begin{pmatrix} \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta \\ \cos\psi & -\sin\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} | = \sin\theta$$

and

$$dA = \sin\theta d\varphi d\theta d\psi. \tag{4.29}$$

Since $SO(3)$ is compact, dA is both left- and right-invariant. The volume of $SO(3)$ is

$$V_{SO(3)} = \int_{SO(3)} dA = \int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta = 8\pi^2. \tag{4.30}$$

The irred unitary reps of $SO(3)$ are denoted $\mathbf{T}^{(\ell)}, \ell = 0, 1, 2, \cdots$, where $\dim \mathbf{T}^{(\ell)} = 2\ell + 1$. (In particular $T^{(0)}(A) = 1$ and $T^{(1)}(A) = A$.) Expressed in terms of an ON basis for the rep space $V^{(\ell)}$ consisting of simultaneous eigenfunctions for the operators $\mathbf{T}^{(\ell)}(R_z(\varphi))$, the matrix elements are

$$\begin{aligned} T^{\ell}_{km}(\varphi, \theta, \psi) &= i^{k-m}\left[\frac{(\ell+m)!(\ell-k)!}{(\ell+k)!(\ell-m)!}\right]^{1/2} \times \\ & e^{i(k\varphi+m\psi)}\frac{[\sin\theta]^{m-k}(1+\cos\theta)^{\ell+k-m}}{2^{\ell}\Gamma(m-k+1)}{}_2F_1\left(\begin{matrix}-\ell-k, m-1; \frac{\cos\theta-1}{\cos\theta+1}\\ m-k+1\end{matrix}\right) \\ &= i^{k-m}\left[\frac{(\ell+m)!(\ell-k)!}{(\ell+k)!(\ell-m)!}\right]^{1/2} e^{i(k\varphi+m\psi)}P_{\ell}^{-k,m}(\cos\theta), \\ & -\ell \le k, m \le \ell. \end{aligned} \tag{4.31}$$

Here ${}_2F_1\binom{a,b}{c,x}$ is the Gaussian hypergeometric function and $\Gamma(z)$ is the gamma function [EMOT1], [V], [WW]. A generating function for the matrix elements is

$$g(A, z) = \frac{(\beta z + \bar{\alpha})^{\ell-m}(\alpha z - \bar{\beta})^{\ell+m}}{[(\ell-m)!(\ell+m)!]^{1/2}} = \sum_{k=-\ell}^{\ell} T^{\ell}_{km}(A)\frac{(-1)^{k-m}z^{\ell+k}}{[(\ell-k)!(\ell+k)!]^{1/2}}$$

where

$$\alpha = e^{i(\varphi+\psi)/2}\cos\frac{\theta}{2}, \quad \beta = ie^{i(\phi-\psi)/2}\sin\frac{\theta}{2}.$$

The group property

$$T^{\ell}_{km}(A_1A_2) = \sum_{j=-\ell}^{\ell} T^{\ell}_{kj}(A_1)T^{\ell}_{jm}(A_2)$$

defines an **addition theorem** obeyed by the matrix elements. The unitary property of the operator $\mathbf{T}^{(\ell)}(A)$ implies

$$T^{\ell}_{km}(A^{-1}) = \overline{T^{\ell}_{mk}}(A),$$

or in Euler angles,

$$(-1)^{m-k}P_{\ell}^{-k,m}(\cos\theta) = \frac{(\ell+k)!(\ell-m)!}{(\ell-k)!(\ell+m)!}P_{\ell}^{-m,k}(\cos\theta).$$

Also, $|T^{\ell}_{km}(A)| \le 1$ or

$$|P_{\ell}^{-k,m}(\cos\theta)| \le \left[\frac{(\ell+k)!(\ell-m)!}{(\ell+m)!(\ell-k)!}\right]^{1/2}, 0 \le \theta \le \pi.$$

The matrix elements $T^\ell_{om}(\varphi,\theta,\psi)$, are proportional to the spherical harmonics $Y^m_\ell(\theta,\psi)$. Indeed

$$T^\ell_{om}(\varphi,\theta,\psi) = i^m \left(\frac{4\pi}{2\ell+1}\right)^{1/2} Y^m_\ell(\theta,\psi) = i^m \left[\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P^m_\ell(\cos\theta)e^{im\psi}$$

where the $P^m_\ell(\cos\theta)$ are the associated Legendre functions [EMOT1], [GMS], [M4], [M5]. Moreover,

$$T^\ell_{oo}(\varphi,\theta,\psi) = P_\ell(\cos\theta)$$

where $P_\ell(\cos\theta)$ is the Legendre polynomial.

According to the general theory of §4.3, the matrix elements $T^\ell_{km}(A)$ satisfy the orthogonality relations

$$\int_{SO(3)} T^\ell_{km}(A)\overline{T^{\ell'}_{k'm'}}(A)dA = \frac{8\pi^2}{2\ell+1}\delta_{kk'}\delta_{mm'}\delta_{\ell\ell'}. \tag{4.32}$$

Thus

$$\int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\ T^\ell_{km}(\varphi,\theta,\psi)\overline{T^{\ell'}_{k'm'}}(\varphi,\theta,\psi)\sin\theta = \frac{8\pi^2}{2\ell+1}\delta_{kk'}\delta_{mm'}\delta_{\ell\ell'}.$$

The ψ and φ integrations are trivial, while the θ integration gives

$$\int_0^{\pi} P^{k,m}_\ell(\cos\theta)P^{k,m}_{\ell'}(\cos\theta)\sin\theta d\theta = \frac{2}{2\ell+1}\frac{(\ell-k)!(\ell-m)!}{(\ell+k)!(\ell+m)!}\delta_{\ell\ell'}.$$

For $k=m=0$ these are the orthogonality relations for the Legendre polynomials. (Note: By definition, $P^{0,-m}_\ell(\cos\theta) = P^m_\ell(\cos\theta), P^{0,0}_\ell(\cos\theta) = P_\ell(\cos\theta)$, where P^m_ℓ, P_ℓ are Legendre functions.)

By the Peter-Weyl theorem, the functions

$$\varphi^\ell_{km}(\varphi,\theta,\psi) = (2\ell+1)^{1/2}T^\ell_{km}(\varphi,\theta,\psi),$$
$$-\ell \le k,m \le \ell, \quad \ell = 0,1,2,\cdots$$

constitute an ON basis for $L_2(SO(3))$. If $f \in L_2(SO(3))$ then

$$f(\varphi,\theta,\psi) = \sum_{\ell=0}^{\infty}\sum_{k,m=-\ell}^{\ell} a^\ell_{km}\varphi^\ell_{km}(\varphi,\theta,\psi) \tag{4.33}$$

where

$$a^\ell_{km} = (f,\varphi^\ell_{km}) = \frac{1}{8\pi^2}\int_0^{2\pi} d\psi \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \times \times f(\varphi,\theta,\psi)\overline{\varphi^\ell_{km}}(\varphi,\theta,\psi)\sin\theta. \tag{4.34}$$

Some particular cases of (4.33) are of special interest. Suppose $f(\theta,\psi) \in L_2(SO(3))$ is independent of the variable φ. If we think of (θ,ψ) as latitude and longitude, we can consider f as a function on the unit sphere $S_2 \cong SO(3)/U(1)$, square-integrable with respect to the area measure on S_2. Since the φ-dependence of $\varphi^\ell_{km}(\varphi,\theta,\psi)$ is $e^{ik\varphi}$, it follows from (4.34) that $a^\ell_{km} = 0$ unless $k = 0$; the only possible nonzero coefficients are a^ℓ_{om}. Now

$$\varphi^\ell_{om}(\varphi,\theta,\psi) = (4\pi)^{1/2} Y^m_\ell(\theta,\psi)$$

where Y^m_ℓ is a spherical harmonic. Thus,

$$f(\theta,\psi) = \sum_{\ell=0}^{\infty} \sum_{m=-\infty}^{\infty} c^\ell_m Y^m_\ell(\theta,\psi) \tag{4.35}$$

where

$$c^\ell_m = \int_0^{2\pi} d\psi \int_0^{\pi} d\theta f(\theta,\psi)\overline{Y^m_\ell}(\theta,\psi)\sin\theta, \quad (Y^m_\ell, Y^{m'}_{\ell'}) = \delta_{\ell\ell'}\delta_{mm'}.$$

This is the expansion of a function on the sphere as a linear combination of spherical harmonics. Again, (4.35) converges in the norm of $L_2(SO(3))$, not necessarily pointwise.

If $f(\theta) \in L_2(SO(3))$ is a function of θ alone then the coefficients a^ℓ_{km} are zero unless $k = m = 0$. Here,

$$\varphi^\ell_{oo}(\varphi,\theta,\psi) = (2\ell+1)^{1/2} P_\ell(\cos\theta), \quad \ell = 0,1,2,\cdots$$

where

$$P_\ell(x) = {}_2F_1\left(\begin{matrix}\ell+1, & -\ell, \\ 1 & \end{matrix} \frac{1-x}{2}\right) = \left(\frac{1+x}{2}\right)^\ell {}_2F_1\left(\begin{matrix}-\ell, & -\ell, \\ 1 & \end{matrix} \frac{1-x}{x+1}\right)$$

is a Legendre polynomial of order ℓ. The coefficient of x^ℓ in the expansion of $P_\ell(x)$ is nonzero and $P_\ell(1) = 1$. The expansion of $f(\theta)$ becomes

$$\begin{aligned} f(\theta) &= \sum_{\ell=\theta}^{\infty} c_\ell P_\ell(\cos\theta), \\ c_\ell &= \frac{1}{2}(2\ell+1)\int_0^{\pi} f(\theta)P_\ell(\cos\theta)\sin\theta d\theta, \\ \int_0^{\pi} P_\ell(\cos\theta)&P_k(\cos\theta)\sin\theta d\theta = \frac{2\delta_{k\ell}}{2\ell+1}. \end{aligned} \tag{4.37}$$

4.6 Fourier transforms and their relation to Fourier series. Abelian groups G, (not necessarily compact) are another class of groups concerning which one can make general statements about the decomposition of $L_2(G)$ in terms of unitary representations [B7], [DM2], [G1], [K2], [N2], [R1], [V]. We will not go into this theory but consider only a single, very important, example where the results are familiar to everyone: The group R of real numbers t with addition of numbers as group multiplication. Here R is isomorphic to the matrix group R', examples (1) and (3), Section §3.1, and dt is the invariant measure. The unitary irred reps of R are one-dimensional, hence continuous functions $\chi(t)$ such that

$$\chi(t_1+t_2)=\chi(t_1)\chi(t_2), \quad t_1,t_2\in R. \tag{4.38}$$

This functional equation has only the solutions $\chi(t)=e^{at}$ and the unitarity requirement implies $a=2\pi i\omega$ where ω is real. Given a function $s\in L_2(R)$ we have

$$s(t)=\int_{-\infty}^{\infty} S(\omega)e^{2\pi i\omega t}d\omega$$

where the **Fourier coefficients** $S(\omega)$ are defined by

$$\begin{aligned} S(\omega) &= \int_{-\infty}^{\infty} s(t)e^{-2\pi i\omega t}dt \\ &= (s,\chi_\omega), \end{aligned}$$

and $\chi_\omega(t)=e^{2\pi i\omega t}$ is the irred rep. Parseval's equality is

$$\int_{-\infty}^{\infty}|s(t)|^2dt=\int_{-\infty}^{\infty}|S(\omega)|^2d\omega.$$

Formally, the orthogonality relations are

$$(\chi_\omega,\chi_{\omega'})=\int_{-\infty}^{\infty}\exp[2\pi it(\omega-\omega')]dt=\delta(\omega-\omega')$$

where $\delta(\omega)$ is the Dirac delta function. Note that the sum over irred reps, familiar for compact groups, is here replaced by an integral over irred reps.

It is illuminating to compare the Fourier transform on R with the corresponding results for $U(1)$ and Z_n. Assume that $f\in L_2(R)$ belongs to the Schwartz class, i.e., f is in $C^\infty(R)$ and there exist constants $C_{n,q}$ (depending on f) such that $|t^n\frac{d^q}{dt^q}f|\le C_{n,q}$ on R for each $n,q=0,1,2,\cdots$. Then the projection operator P maps f to a continuous function in $L_2([0,1])$ with period one:

$$P[f](x)=\sum_{m=-\infty}^{\infty} f(x+m) \tag{4.39}$$

Expanding $P[f](x)$ into a Fourier series we find

$$P[f](x)=\sum_{m=-\infty}^{\infty} c_n e^{2\pi inx}$$

where

$$c_n=\int_0^1 P[f](x)e^{-2\pi inx}dx=\int_{-\infty}^{\infty} f(x)e^{-2\pi inx}dx=\hat f(n)$$

and $\hat{f}(\omega)$ is the Fourier transform of $f(x)$. Thus,

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx}, \tag{4.40}$$

and we see that $P[f](x)$ tells us the value of $\hat{f}$ at the integer points $\omega = n$, but not in general at the non-integer points. (For $x = 0$, equation (4.40) is known as the Poisson summation formula, [DM2]. If we think of f as a signal, we see that **periodization** (4.39) of f results in a loss of information. However, if f vanishes outside of $[0, 1)$ then $P[f](x) \equiv f(x)$ for $0 \le x < 1$ and

$$f(x) = \sum_n \hat{f}(n)e^{2\pi inx}, \quad 0 \le x < 1 \tag{4.41}$$

without error. Now suppose (4.41) holds, so that there is no periodization error. For an integer $N > 1$ we sample the signal at the points $a/N, a = 0, 1, \cdots, N-1$:

$$f\left(\frac{a}{N}\right) = \sum_n \hat{f}(n)e^{2\pi ina/N}, \quad 0 \le a < N. \tag{4.42}$$

From the Euclidean algorithm we have $n = b + cN$ where $0 \le b < N$ and b, c are integers. Thus

$$f\left(\frac{a}{N}\right) = \sum_{b=0}^{N-1} \left[\sum_c \hat{f}(b + cN)\right] e^{2\pi iab/N}, \quad 0 \le a < N. \tag{4.43}$$

Note that the quantity in brackets is the projection of $\hat{f}$ at integer points to a periodic function of period N. Furthermore, the expansion (4.43) is essentially the finite Fourier expansion (3.37). However, simply sampling the signal at the points a/N tells us only $\sum_c \hat{f}(b + cN)$ not (in general) $\hat{f}(b)$. This is known as **aliasing error**.

Although we will not work out the details in these notes, a similar approach to the foregoing (with periodizing and aliasing error) is appropriate and useful for Fourier analysis on the Heisenberg and affine groups.

4.7 Exercises.

4.1 Construct a real irred two-dimensional rep of the circle group $U(1)$.

4.2 Show how to decompose any real finite-dimensional rep of $U(1)$ as a direct sum of real irred reps.

4.3 Verify that the measure $d_r A$, (4.8), is right-invariant on the group G.

4.4 Prove: If G is a compact linear Lie group then $d(A^{-1}) = dA$, i.e., $\int_G f(A^{-1})dA = \int_G f(B)dA$. Hint: Show that $V_{A^{-1}}(\mathbf{g}) = |\det(-\tilde{A})|V_A(\mathbf{g})$ where $\tilde{A}$ is the automorphism $A \to AAA^{-1}$ of $n \times n$ matrices A.

4.5 Assuming that $\chi(t)$ is a continuously differentiable function of t, use a differential equations argument to show that the only nonzero solutions of the functional equation

$$\chi(t_1 + t_2) = \chi(t_1)\chi(t_2), \quad t_1, t_2 \in R$$

are $\chi(t) = e^{at}$, where a is a constant.

4.6 Assuming only that that $\chi(t)$ is a real continuous function of t, show that the only nonzero solutions of the functional equation

$$\chi(t_1 + t_2) = \chi(t_1)\chi(t_2), \quad t_1, t_2 \in R$$

are $\chi(t) = e^{at}$, where a is a constant. Hint: Set $\chi(t) = e^{\phi(t)}$ so that $\phi(t_1 + t_2) = \phi(t_1) + \phi(t_2)$. Then determine $\phi(t)$ for t rational from $\phi(1)$.

4.7 Use the Poisson summation formula for the Gauss kernel $f(x) = (2\pi t)^{-\frac{1}{2}} e^{-\frac{x^2}{2t}}$ to derive the identity

$$\frac{1}{\sqrt{2\pi t}} \sum_{n=-\infty}^{\infty} e^{-\frac{n^2}{2t}} = \sum_{n=-\infty}^{\infty} e^{-2\pi^2 n^2 t}.$$

Hint: $\hat{f}(\omega) = e^{-2\pi^2\omega^2 t}$.

§5. REPRESENTATIONS OF THE HEISENBERG GROUP

5.1 Induced representations of H_R. Recall that the Heisenberg group H_R can be realized as the linear Lie group of matrices

$$A(\mathbf{x}) = \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.1}$$

with group product

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_2),$$

and invariant measure

$$d_\ell A = d_r A = dx_1 dx_2 dx_3.$$

To motivate the construction of the unitary irred reps of H_R we review the essentials of the Frobenius construction of induced representations. Let G be a linear Lie group and H a linear Lie subgroup of G. If $\mathbf{T}$ is a rep of G on the vector space W we can obtain a rep $\mathbf{T}_H$ of H by restricting $\mathbf{T}$ to H,

$$\mathbf{T}_H(B) = \mathbf{T}(B), \quad B \in H.$$

On the other hand the method of Frobenius allows one to construct a rep of G from a rep of H. Let $\mathbf{T}$ be a finite-dimensional unitary rep of H on the inner product space V. Denote by U^G the vector space of all functions $\mathbf{f}(A)$ with domain G and range contained in V where addition and scalar multiplication of functions are the vector operations. Here, for a fixed $A \in G$, $\mathbf{f}(A)$ is a vector in V. Let V^G be the subspace of U^G defined by

$$V^G = \{\mathbf{f} \in U^G : \mathbf{f}(BA) = \mathbf{T}(B)\mathbf{f}(A) \text{ for all } B \in H, A \in G\}. \tag{5.2}$$

We define a rep $\mathbf{T}^G$ of G on V^G by

$$[\mathbf{T}^G(A)]\mathbf{f}(A') = \mathbf{f}(A'A), \quad A, A' \in G, \quad \mathbf{f} \in V^G. \tag{5.3}$$

It is clear that V^G is invariant under G and the operators $\mathbf{T}^G(A)$ satisfy the homomorphism property. Here, $\mathbf{T}^G$ is called an **induced representation**. If $(\cdot, \cdot)$ is the inner product on V and H is compact we can initially define the inner product $\langle \cdot, \cdot \rangle$ on V^G by

$$\langle \mathbf{f}_1, \mathbf{f}_2 \rangle = \int_G (\mathbf{f}_1(A), \mathbf{f}_2(A)) d_r A$$

where $d_r A$ is the right-invariant measure on G. Then we restrict the operators $\mathbf{T}^G$ to the subspace $V'^G \subseteq V^G$ of functions $\mathbf{f}$ such that $\langle \mathbf{f}, \mathbf{f} \rangle < \infty$. If $A' \in G$ and $\mathbf{f} \in V'^G$ we have

$$\langle \mathbf{T}^G(A')\mathbf{f}_1, \mathbf{T}^G(A')\mathbf{f}_2 \rangle = \int_G (\mathbf{f}_1(AA'), \mathbf{f}_2(AA')) d_r A$$
$$= \int_G (\mathbf{f}_1(A), \mathbf{f}_2(A)) d_r A = \langle \mathbf{f}_1, \mathbf{f}_2 \rangle$$

so $\mathbf{T}^G$ is unitary. However, note from (5.2) that $(\mathbf{f}_1(BA), \mathbf{f}_2(BA)) = (\mathbf{f}_1(A), \mathbf{f}_2(A))$ for all $B \in H$, i.e., the inner product $(\cdot, \cdot)$ is constant on the right cosets HA. Thus if H is noncompact the above integral will be undefined (or V'^G will contain only the zero function). If the coset space $X = H\backslash G$ admits a G-invariant measure, i.e., a measure $d\mu(x)$ such that $d\mu(xA) = d\mu(x)$ for all $x \in X, A \in G$, then we can define the inner product on V'^G as

$$\langle \mathbf{f}_1, \mathbf{f}_2 \rangle = \int_X (\mathbf{f}_1(x), \mathbf{f}_2(x)) d\mu(x) \tag{5.4}$$

and the operators $\mathbf{T}^G : \mathbf{f}(x) \to \mathbf{T}(B)\mathbf{f}(xA)$ will still be unitary. Here, we choose an $A'_x \in G$ in each right coset $HA' \leftrightarrow x$, so that $A'_x A = BA'_{xA}$ and $\mathbf{f}(x) \equiv \mathbf{f}(A'_x)$. (Note: It is always possible to find local coordinates $\mathbf{f} = (g_1, \cdots, g_n)$ on G such that $\mathbf{h} = (g_1, \cdots, g_m)$, $m \le n$, are local coordinates on H and $\mathbf{x} = (g_{m+1} \cdots, g_n)$ are local coordinates on X.) In general, no such invariant measure $d\mu$ exists, but it does exist in many cases. In particular, if both G and H are **unimodular**, i.e., if the left-invariant and right-invariant measures for each of these groups are the same, and H is a closed subgroup of G then it can be shown that an (unique up to a constant multiplier) invariant measure $d\mu$ exists and that in local coordinates

$$dA = V_G(\mathbf{g}) dg_1 \cdots dg_n, \quad A \in G$$
$$dB = V_H(\mathbf{h}) dg_1 \cdots dg_m, \quad B \in H$$

and

$$d\mu(\mathbf{x}) = V_x(\mathbf{x}) dg_{m+1} \cdots dg_n$$

where $dA = d\mu dB$. Thus $V_x(\mathbf{x}) = V(\mathbf{g})/V_h(\mathbf{h})$.

5.2 The Schrödinger representation. Let us consider the case where $G = H_R$ and H is the subgroup of matrices $\{A(0, x_2, x_3)\}$. Since $A(0, x_2, x_3)A(0, x'_2, x'_3) = A(0, x_2 + x'_2, x_3 + x'_3)$ it is clear that H is Abelian and that the operators $\mathbf{T}_\lambda[A(0, x_2, x_3)] = e^{2\pi i \lambda x_3}$ define a one-dimensional unitary irred rep of H. Furthermore the left-invariant and right-invariant measure on H is $dx_2 dx_3$. Since

$$A(x_1, x_2, x_3) = A(0, x_2, x_3)A(x_1, 0, 0)$$

it is clear that the coset space $H\backslash G$ can be parametrized by the coordinate $x_1 = t$, and the (scalar) functions in V'^G can be taken as $f(t)$. If $A(x_1, x_2, x_3)$ acts on this function, it transforms to

$$f(t + x_1, x_2, x_3 + tx_2) \equiv \mathbf{T}_\lambda(B) f(t + x_1)$$

where $B = A(0, x_2, x_3 + tx_2)$, (5.2). Thus the action of H_R restricted to the coset space is

$$\mathbf{T}^\lambda(A(\mathbf{x}))f(t) \equiv \mathbf{T}^\lambda[x_1, x_2, x_3]f(t) = e^{2\pi i \lambda (x_3 + tx_2)} f(t + x_1). \tag{5.5}$$

One can verify directly that $\mathbf{T}^\lambda$ is a rep, i.e.,

$$\mathbf{T}^\lambda(A_1)\mathbf{T}^\lambda(A_2) = \mathbf{T}^\lambda(A_1 A_2)$$

and that it is unitary with respect to the inner product

$$\langle f_1, f_2 \rangle = \int_{-\infty}^{\infty} f_1(t)\overline{f_2}(t)dt. \tag{5.6}$$

For $\lambda = 0$ this rep is reducible but (as we will show later) for $\lambda \neq 0$ it is irred in the sense that there is no nontrivial closed subspace of $L_2(R)$ which is invariant under the operators $\mathbf{T}^\lambda(A)$, $A \in H_R$. This rep is called the **Schrödinger representation** of the Heisenberg group.

Note that the mapping

$$A(x_1, x_2, x_3) \xrightarrow{\rho} (x_1, x_2)$$

is a homomorphism of H_R onto the Abelian group $R_2 : (x_1, x_2) \cdot (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$. The unitary irred reps of R_2 are clearly of the form $\chi_{\alpha_1,\alpha_2}(x_1, x_2) = e^{2\pi i(\alpha_1 x_1 + \alpha_2 x_2)}$ for real constants α_1, α_2. It follows that the matrices $T^{\alpha_1,\alpha_2}(A(\mathbf{x})) = e^{2\pi i(\alpha_1 x_1 + \alpha_2 x_2)}$ define one-dimensional unitary irred reps of H_R. It can be shown that $\mathbf{T}^\lambda$ and T^{α_1,α_2} are the only irred unitary reps of H_R, [S2].

5.3 Square integrable representations. Assuming for the time being that $\mathbf{T}^\lambda$ is irred for real $\lambda \neq 0$ let us see if there is an analog for H_R of the orthogonality and completeness relations for matrix elements which hold for linear compact Lie groups, and for compact topological groups in general. Noncompact groups G can have infinite dimensional irred unitary reps. That is, the rep space is a separable infinite dimensional Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. The rep operators $\mathbf{T}(g), g \in G$ are **unitary** i.e., each $\mathbf{T}(g)$ is a linear mapping of H onto itself which preserves inner product: $\langle \mathbf{T}(g)f_1, \mathbf{T}(g)f_2 \rangle = \langle f_1, f_2 \rangle$, for all $f_1, f_2 \in H$. The rep is **irreducible** if there is no proper closed subspace of H which is invariant under the operator $\mathbf{T}(g), g \in G$. The reps $\mathbf{T}, \mathbf{T}'$ of G on the Hilbert spaces H, H', respectively, are **equivalent** if there is a bounded invertible linear operator $\mathbf{S} : H \to H'$ such that $\mathbf{S}\mathbf{T}(g) = \mathbf{T}'(g)\mathbf{S}$ for all $g \in G$, i.e., $\mathbf{S}\mathbf{T}(g)\mathbf{S}^{-1} = \mathbf{T}'(g)$.

With these definitions the analog of Theorems 3.2, 3.4 are true and can be proven with ease. Moreover, the following analog of Theorem 3.5 holds:

Theorem 5.1. *Let $\mathbf{T}$ be a unitary rep of the group G on the separable Hilbert space H. Then $\mathbf{T}$ is irred if and only if the only bounded operator $\mathbf{S}$ on H satisfying*

$$\mathbf{T}(g)\mathbf{S} = \mathbf{S}\mathbf{T}(g) \tag{5.7}$$

is $\mathbf{S} = \lambda\mathbf{E}$ where $\mathbf{E}$ is the identity operator on H.

Sketch of the proof. Part of this result is easy to prove. Suppose $\mathbf{S} = \lambda\mathbf{E}$ is the only solution of (5.7) and let M be a closed subspace of H which is invariant under $\mathbf{T} : \mathbf{T}(g)\mathbf{f} \in M$ for all $g \in G, \mathbf{f} \in M$. Then $M^\perp$ is also invariant under $\mathbf{T}$ and $H = M \oplus M^\perp$. Thus for every $\mathbf{f} \in H$ we have the unique decomposition $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$, $\mathbf{f}_1 \in M$, $\mathbf{f}_2 \in M^\perp$ where $\mathbf{T}(g)\mathbf{f}_1 \in M$ and $\mathbf{T}(g)\mathbf{f}_2 \in M^\perp$. Now define the **projection operator** $\mathbf{P}$ by $\mathbf{P}\mathbf{f} = \mathbf{f}_1$. Clearly $\mathbf{P}$ is a bounded operator on H and $\mathbf{T}(g)\mathbf{P} = \mathbf{P}\mathbf{T}(g)$ for all $g \in G$. By hypothesis, $\mathbf{P} = \lambda\mathbf{E}$. If $\lambda = 0$ then M is the zero subspace; if $\lambda \neq 0$ then $\lambda = 1$ and $M = H$. Thus M is not a proper invariant subspace and $\mathbf{T}$ is irred.

The converse is somewhat more difficult. Suppose $\mathbf{T}$ is irred and $\mathbf{S}$ is a bounded linear operator satisfying (5.7). Then the adjoint $\mathbf{S}^*$ of $\mathbf{S}$ also satisfies (5.7) so, without loss of generality, one can assume that $\mathbf{S}$ is self-adjoint. Then, by the spectral theorem for self-adjoint operators [AG1], [AG2], [G1], [N2], [RN], the projection operators $\mathbf{P}_\lambda$ in the spectral family associated with $\mathbf{S}$ must all commute with each $\mathbf{T}(g)$. By hypothesis then, each $\mathbf{P}_\lambda$ is either the zero operator or the identity operator. Hence $\mathbf{S} = \lambda\mathbf{E}$ for some real number λ. □

Note: At this point it is appropriate to mention that if the unitary irred reps $\mathbf{T}, \mathbf{T}'$ of a group G are equivalent, then they are **unitary equivalent**, i.e., there is a unitary operator $\mathbf{U}$ such that $\mathbf{U}\mathbf{T}(g) = \mathbf{T}'(g)\mathbf{U}$ for all $g \in G$. (Here $\mathbf{U}$ maps the Hilbert space H onto the Hilbert space H' and preserves inner product: $\langle \mathbf{U}\mathbf{f}_1, \mathbf{U}\mathbf{f}_2\rangle' = \langle \mathbf{f}_1, \mathbf{f}_2\rangle$ for all $\mathbf{f}_1, \mathbf{f}_2 \in H$. In particular, this means that $\mathbf{U}^* = \mathbf{U}^{-1}$ where the **adjoint** $\mathbf{S}^* : H' \to H$ of a bounded operator $\mathbf{S} : H \to H'$ is defined by $\langle \mathbf{S}\mathbf{f}, \mathbf{f}'\rangle' = \langle \mathbf{f}, \mathbf{S}^*\mathbf{f}'\rangle$ for all $f \in H, f' \in H'$.) Indeed, if $\mathbf{T} \sim \mathbf{T}'$ then there is a nonzero bounded invertible operator $\mathbf{S}: \ H \to H'$ such that $\mathbf{S}\mathbf{T}(g) = \mathbf{T}(g)\mathbf{S}$. Taking the adjoint of both sides of this equation and using the fact that $\mathbf{T}^*(g) = \mathbf{T}^{-1}(g), \mathbf{T}^{'*}(g) = \mathbf{T}^{'-1}(g)$ we have $\mathbf{S}^*\mathbf{T}^{'-1}(g) = \mathbf{T}^{-1}(g)S^*$. Eliminating $\mathbf{T}'(g)$ from the two equations we find $\mathbf{S}\mathbf{T}(g)\mathbf{S}^{-1} = \mathbf{S}^{*-1}\mathbf{T}(g)\mathbf{S}^*$ or $(\mathbf{S}^*\mathbf{S})\mathbf{T}(g) = \mathbf{T}(g)(\mathbf{S}^*\mathbf{S})$. Since $\mathbf{T}$ is irred it follows from Theorem 5.1 that $\mathbf{S}^*\mathbf{S} = \alpha\mathbf{E}$ where $\mathbf{E}$ is the identity operator and α is a constant. Since $\alpha||\mathbf{f}||^2 = \langle \mathbf{S}^*\mathbf{S}\mathbf{f}, \mathbf{f}\rangle = \langle \mathbf{S}\mathbf{f}, \mathbf{S}\mathbf{f}\rangle = ||\mathbf{S}\mathbf{f}||^2 > 0$ for $\mathbf{f} \neq \boldsymbol{\theta}$, then $\alpha = \beta^2$ is a positive constant. Setting $\mathbf{U} = \beta^{-1}\mathbf{S}$ we have $\mathbf{U}^*\mathbf{U} = \alpha^{-1}\mathbf{S}^*\mathbf{S} = \mathbf{E}$, so $\mathbf{U}$ is unitary and $\mathbf{U}\mathbf{T}(g) = \mathbf{T}'(g)\mathbf{U}$.

Based on these results we can mimic the proof of the orthogonality relations (4.17) for any linear Lie group which is unimodular, i.e., such that $d_r A = d_\ell A$. There are two differences, however: (1) In order that the integrals (4.17) exist we must limit ourselves to those irred reps $\mathbf{T}^{(\mu)}$ of G whose matrix elements $T^{(\mu)}_{jk}(A)$ are square integrable. (2) We cannot normalize the measure in general, since the volume of G may be infinite. Thus we obtain the result

$$\int_G T^{(\mu)}_{i\ell}(A)\overline{T}^{(\nu)}_{sk}(A)dA = \frac{\delta_{is}}{d(\mu)}\delta_{\ell k}\delta_{\mu\nu} \tag{5.8}$$

where $T^{(\mu)}_{i\ell}(A)$ are the matrix elements of $\mathbf{T}^{(\mu)}$ with respect to some ON basis and $\mathbf{T}^{(\mu)}, \mathbf{T}^{(\nu)}$ are unitary irred reps of G whose matrix elements are square integrable. The constant $d(\mu) > 0$ is called the **degree** of $\mathbf{T}^{(\mu)}$. (It can be shown that if one matrix element $T^{(\mu)}_{ij}(A)$ of $\mathbf{T}^{(\mu)}$ is square integrable, then all possible matrix elements $\langle \mathbf{T}^{(\mu)}(A)\mathbf{f}, \mathbf{g}\rangle$ are square integrable.) Furthermore, (5.8) can be written in a basis-free form analogous to (3.24).

Theorem 5.2. *For $\lambda \neq 0$ the representation $\mathbf{T}^\lambda$ of H_R on $L_2(R)$ is irreducible.*

Proof. We will present the basic ideas of the proof, omitting some of the technical details. Our aim will be to show that if $\mathbf{L}$ is a bounded operator on $L_2(R)$ which commutes with the operators $\mathbf{T}^\lambda(A)$ for some $\lambda \neq 0$ and all $A \in H_R$ then $\mathbf{L} = \kappa\mathbf{E}$ for some constant κ, where $\mathbf{E}$ is the identity operator. (It follows from this that $\mathbf{R}_\lambda$

is irred, for if M were a proper closed subspace of $L_2(R)$, invariant under $\mathbf{T}^\lambda$, then the self-adjoint projection operator $\mathbf{P}$ on M would commute with the operators $\mathbf{T}^\lambda(A)$. This is impossible since $\mathbf{P}$ could not be a scalar multiple of $\mathbf{E}$.)

Suppose the bounded operator $\mathbf{L}$ satisfies

$$\mathbf{L}\mathbf{T}^\lambda[x_1, x_2, x_3] = \mathbf{T}^\lambda[x_1, x_2, x_3]\mathbf{L}$$

for all real x_i. First consider the case $x_1 = x_3 = 0$: $\mathbf{L}$ commutes with the operation of multiplication by functions of the form $e^{i\lambda bt}$ for real b. Clearly $\mathbf{L}$ must also commute with multiplication by finite sums of the form $\sum_{b_j} c_j e^{2\pi i\lambda b_j t}$ and, by using the well-known fact that trigonometric polynomials are dense in the space of measurable functions, $\mathbf{L}$ must commute with multiplication by any bounded function $f(t)$ on $(-\infty, \infty)$. Now let Q be a bounded closed internal in $(-\infty, \infty)$ and let $\chi_Q \in L_2(R)$ be the **characteristic function** of Q:

$$\chi_Q(t) = \begin{cases} 1 & \text{if } t \in Q \\ 0 & \text{if } t \notin Q \end{cases}.$$

Let $f_Q \in L_2(R)$ be the function $f_Q = \mathbf{L}\chi_Q$. Since $\chi_Q^2 = \chi_Q$ we have $f_Q(t) = \mathbf{L}\chi_Q(t) = \mathbf{L}\chi_Q^2(t) = \chi_Q(t)L\chi_Q(t) = \chi_Q(t)f_Q(t)$ so f_Q is nonzero only for $t \in Q$. Furthermore, if Q' is a closed interval with $Q' \subseteq Q$ and $f_{Q'} = \mathbf{L}\chi_{Q'}$ then $f_{Q'}(t) = \mathbf{L}\chi_{Q'}\chi_Q(t) = \chi_{Q'}(t)\mathbf{L}\chi_Q(t) = \chi_{Q'}(t)f_Q(t)$ so $f_{Q'}(t) = f_Q(t)$ for $t \in Q'$ and $f_{Q'}(t) = 0$ for $t \notin Q'$. It follows that there is a unique function $f(t)$ such that $\chi_{\tilde{Q}} f \in L_2(R)$ and $\chi_{\tilde{Q}}(t)f(t) = \mathbf{L}\chi_{\tilde{Q}}(t)$ for any closed bounded interval $\tilde{Q}$ in $(-\infty, \infty)$. Now let $\boldsymbol{\varphi}$ be a C^∞ function which is zero in the exterior of $\tilde{Q}$. Then $\mathbf{L}\boldsymbol{\varphi}(t) = \mathbf{L}(\boldsymbol{\varphi}\chi_{\tilde{Q}}(t)) = \boldsymbol{\varphi}(t)\mathbf{L}\chi_{\tilde{Q}}(t) = \boldsymbol{\varphi}(t)f(t)\chi_{\tilde{Q}}(t) = f(t)\boldsymbol{\varphi}(t)$, so $\mathbf{L}$ acts on $\boldsymbol{\varphi}$ by multiplication by the function $f(t)$. Since as $\tilde{Q}$ runs over all finite subintervals of $(-\infty, \infty)$ the functions $\boldsymbol{\varphi}$ are dense in $L_2(R)$, it follows that $\mathbf{L} = f(t)\mathbf{E}$.

Now we use the hypothesis that $\mathbf{L}\mathbf{T}^\lambda[x_1, 0, 0]\boldsymbol{\varphi}(t) = \mathbf{T}^\lambda[x_1, 0, 0]\mathbf{L}\boldsymbol{\varphi}(t)$ for all x_1 and $\boldsymbol{\varphi} \in L_2(R) : f(t)\varphi(t + x_1) = f(t + x_1)\varphi(t + x_1)$. Thus $f(t) = f(t + x_1)$ almost everywhere, which implies that $f(t)$ is a constant. $\square$

5.4 Orthogonality of radar cross-ambiguity functions.

The results of §5.3 do not directly apply to the unitary rep $\mathbf{T}^\lambda$ of H_R because this rep fails to be square integrable. Indeed it is evident from (5.5) that the x_3-dependence of the matrix element $\langle \mathbf{T}^\lambda(\mathbf{x})f_1, f_2\rangle$ is of the form $e^{2\pi i\lambda x_3}$, so the integral

$$\iiint \langle \mathbf{T}^\lambda(\mathbf{x})f_1, f_2\rangle \langle \mathbf{T}^\lambda(\mathbf{x})f_3, f_4\rangle dx_1 dx_2 dx_3$$

will diverge. All is not lost because we can factor out the **center** of H_R and consider only the factor space. The center C consists of all elements of H_R which commute with every element of H_R. Clearly

$$C = \{A(0, 0, x_3) = A(x_3), \quad x_3 \in R\}.$$

Now $\mathbf{T}^\lambda$ is irred and $\mathbf{T}^\lambda(x_3)\mathbf{T}^\lambda(\mathbf{x}) = \mathbf{T}^\lambda(\mathbf{x})\mathbf{T}^\lambda(x_3)$ for all $A(\mathbf{x})$. From Theorem 5.1, $\mathbf{T}^\lambda(x_3)$ must be a multiple of the identity operator on $L_2(R)$. Indeed we know that $\mathbf{T}^\lambda(x_3) = e^{2\pi i\lambda x_3}\mathbf{E}$. Now

$$A(\mathbf{x}) = A(x_1, x_2, 0)A(x_3),$$

and since $\mathbf{T}^\lambda$ is irred, the unitary operators $\mathbf{T}^\lambda(x_1, x_2, 0)$, $x_1, x_2 \in R$, must act irreducibly on $L_2(R)$. Furthermore the measure $dx_1 dx_2$ is (two-sided) invariant under the action of H_R. Thus we can repeat the arguments leading to (5.8) for reps $\mathbf{T}^{(\mu)} = \mathbf{T}^{(\nu)} \equiv \mathbf{T}^\lambda$ and measure $dx_1 dx_2$ to obtain (with the assumption, correct as we shall see, that the matrix elements $T^\lambda_{jk}(A)$ are square integrable):

$$\iint_{-\infty}^{\infty} T^\lambda_{j\ell}(x_1, x_2, 0)\overline{T^\lambda_{sk}}(x_1, x_2, 0)dx_1 dx_2 = \frac{\delta_{js}\delta_{\ell k}}{d(\lambda)}. \tag{5.9}$$

In fact, the structure of H_R is so simple that one can use ordinary Fourier analysis to evaluate the left-hand side of (5.9) for arbitrary matrix elements. The result is

$$\iint_{-\infty}^{\infty} dx_1 dx_2 \langle \mathbf{T}^\lambda(x_1, x_2)f_1, f_2\rangle\langle f_4, \mathbf{T}^\lambda(x_1, x_2)f_3\rangle = \langle f_1, f_3\rangle\langle f_4, f_2\rangle \tag{5.10}$$

where

$$\langle \mathbf{T}^\lambda(x_1, x_2)f_1, f_2\rangle = \int_{-\infty}^{\infty} e^{2\pi i\lambda t x_2} f_1(t + x_1)\overline{f_2}(t)dt.$$

(This shows explicitly that the matrix elements $\langle \mathbf{T}^\lambda(x_1, x_2)f_1, f_2\rangle$ are square integrable.

At this point we recall that the narrow-band cross-ambiguity function can be written in the form

$$\psi_{nm}\,(-x_1, x_2/2) = e^{\pi i x_1 x_2} \int_{-\infty}^{\infty} f_n(t + x_1)\overline{f}_m(t)e^{2\pi i t x_2}dt \tag{5.11}$$

so that the cross-ambiguity function differs from the matrix element $\langle \mathbf{T}^1(x_1, x_2, 0)f_n, f_m\rangle$ of H_R by the simple multiplicative factor $e^{\pi i x_1 x_2}$. Thus the results of group representation theory can be brought to bear on the radar ambiguity function. (**For the purposes of computation of the ambiguity function, the phase factor $e^{\pi i x_1 x_2}$ is of no concern and we henceforth will identify the matrix element itself with the ambiguity function.**)

Note the special case of (5.10) where $f_j \equiv f$ and $||f|| = 1$:

$$\iint_{-\infty}^{\infty} dx_1 dx_2 |\langle \mathbf{T}^\lambda(x_1, x_2)f, f\rangle|^2 = 1. \tag{5.10$'$}$$

For $\lambda = 1$ this is the **radar uncertainty relation**. (The maximum of $| < \mathbf{T}^\lambda(x_1, x_2)f, f > |$ is 1 and occurs for $x_1 = x_2 = 0$. However, in view of (5.10′) the graph of this function cannot be too "peaked" around the maximum.)

From (5.10) we see that if $\{f_n,\ n = 0, 1, 2\cdots\}$ is an ON basis for $L_2(R)$ then the matrix elements $\{\langle \mathbf{T}^\lambda(x_1, x_2)f_n, f_m\rangle\}$ form an ON set in $L_2(R^2)$. This set is actually an ON **basis** for $L_2(R^2)$, see Exercise 5.2. **This allows us to expand a moving target distribution function in the narrow-band case as a series in this ON basis, [W5].**

5.5 The Heisenberg commutation relations. To motivate our next example we make a brief digression to study the **Heisenberg commutation relations** of quantum mechanics. Recall that the matrices

$$C_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

form a basis for the tangent space at the identity element of H_R. Defining the **commutator** $[\cdot,\cdot]$ of two 3×3 matrices A and B by $[A, B] = AB - BA$, we see that

$$[C_1, C_2] = C_3, \quad [C_1, C_3] = \Theta, \quad [C_2, C_3] = \Theta \tag{5.12}$$

where Θ is the zero matrix. (Indeed, one can show that the tangent space at the identity for any local linear Lie group G is closed under the commutator operation: if A and B belong to the tangent space, then so does $[A, B]$. The tangent space equipped with the commutator operation is called the **Lie algebra** of G. The Lie algebra contains essential information about G. Indeed one can reconstruct the connected component of the identity element in G just from a knowledge of the Lie algebra. The lack of commutivity in the group operations corresponds to the nonvanishing of the Lie algebra commutators. For more details on the relationship between Lie groups and Lie algebras see [G1], [HS], [M4], [M5].)

Recall that $C_j = \partial_{x_j} A(\mathbf{x})|_{\mathbf{x}=\theta}$. Corresponding to the representation $\mathbf{T}^\lambda$ of H_R we can define the analogous operators $\mathbf{C}_j$ where

$$\mathbf{C}_j f(t) = \partial_{x_j} \mathbf{T}^\lambda(\mathbf{x}) f(t)|_{\mathbf{x}=\theta}, \; j = 1, 2, 3$$

and $f \in L_2(R)$ is a C^∞ function with compact support. From (5.5) we see that

$$\mathbf{C}_1 = \frac{d}{dt}, \quad \mathbf{C}_2 = 2\pi i \lambda t, \quad \mathbf{C}_3 = 2\pi i \lambda. \tag{5.13}$$

Defining the **commutator** of operators $\mathbf{A}, \mathbf{B}$ by $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$ we verify that

$$\begin{aligned} [\mathbf{C}_1, \mathbf{C}_2] &= \left[\frac{d}{dt}, 2\pi i \lambda t\right] = 2\pi i \lambda = \mathbf{C}_3, \\ [\mathbf{C}_1, \mathbf{C}_3] &= \left[\frac{d}{dt}, 2\pi i \lambda\right] = \mathbf{O}, \\ [\mathbf{C}_2, \mathbf{C}_3] &= [2\pi i \lambda t, 2\pi i \lambda] = \mathbf{O}, \end{aligned} \tag{5.14}$$

where $\mathbf{O}$ is the zero operator, in analogy with (5.12). Applying these operators on the domain D of C^∞ functions with compact support, a dense subdomain of $L_2(R)$, we see that they are skew-adjoint; i.e., $\mathbf{C}_j^* = -\mathbf{C}_j$, $j = 1, 2, 3$, where the adjoint $\mathbf{C}^*$ of $\mathbf{C}$ is defined by

$$\langle \mathbf{C} f_1, f_2 \rangle = \int_{-\infty}^{\infty} (\mathbf{C} f_1(t)) \overline{f_2}(t) dt = \langle f_1, \mathbf{C}^* f_2 \rangle \tag{5.15}$$

for all $f_1, f_2 \in D$. (In the case of $\mathbf{C}_1$ we have to integrate by parts.)

The skew-adjoint operators

$$\mathbf{C}_1 = \frac{d}{dt}, \quad \mathbf{C}_2 = 2\pi i\lambda t, \quad \mathbf{C}_3 = 2\pi i\lambda \tag{5.16}$$

satisfy the **Heisenberg commutation relations** (5.14) and are reminiscent of the **annihilation and creation operators for bosons**, familiar from quantum theory. In this theory there is a separable Hilbert space H, an **annihilation operator** $\mathbf{a}$ and its adjoint the **creation operator** $\mathbf{a}^*$ such that

$$[\mathbf{a}^*, \mathbf{a}] = -\mathbf{E} \tag{5.17}$$

where $\mathbf{E}$ is the identity operator on H. Here $\mathbf{a}$ and $\mathbf{a}^*$, and the relation (5.17) are well-defined on some dense subspace D of H and map D into itself. It is further assumed that the equation $\mathbf{a}\boldsymbol{\psi} = \boldsymbol{\theta}$, $\boldsymbol{\psi} \in H$, has a unique solution in D, up to a multiplicative factor. The normalized solution $\boldsymbol{\psi}_0$, $||\boldsymbol{\psi}_0|| = 1$ is called the **vacuum state**. Finally it is assumed that the closure of the subspace generated by applying $\mathbf{a}$ and $\mathbf{a}^*$ to $\boldsymbol{\psi}_0$ recursively, is H itself. With these assumptions one can construct explicitly an ON basis for H, a basis of eigenvectors of the **number of particles operator** $\mathbf{N} = \mathbf{a}^*\mathbf{a}$.

To make the commutation relations of the $\mathbf{C}$-operators agree with (5.17) and to assure that $\mathbf{a}^*$ is the adjoint of $\mathbf{a}$ we have, in essence, only one choice:

$$\begin{aligned} \mathbf{a}^* &= \frac{1}{\sqrt{2}}\left(\frac{d}{dt} - t\right) = \frac{1}{\sqrt{2}}\left(\mathbf{C}_1 + \frac{i}{2\pi\lambda}\mathbf{C}_2\right) \\ \mathbf{a} &= \frac{1}{\sqrt{2}}\left(-\frac{d}{dt} - t\right) = \frac{1}{\sqrt{2}}\left(-\mathbf{C}_1 + \frac{i}{2\pi\lambda}\mathbf{C}_2\right) \\ \mathbf{E} &= -\frac{i}{2\pi\lambda}\mathbf{C}_3. \end{aligned} \tag{5.18}$$

To find the vacuum state $\boldsymbol{\psi}_0(t)$ we solve the equation $\mathbf{a}\boldsymbol{\psi}_0 = \boldsymbol{\theta}$. The solution of this first order differential equation is easily seen to be

$$\boldsymbol{\psi}_0(t) = \pi^{-1/4}e^{-t^2/2}$$

where the constant factor is chosen so $||\boldsymbol{\psi}_0|| = 1$. Since

$$\mathbf{N} = \mathbf{a}^*\mathbf{a} = -\frac{1}{2}\frac{d^2}{dt^2} + \frac{t^2}{2} - \frac{1}{2} \tag{5.19}$$

we have $\mathbf{N}\boldsymbol{\psi}_0 = \boldsymbol{\theta}$. (We can take D to be the space of all functions $p(t)e^{-t^2/2}$ where p is a polynomial.) Now let $\boldsymbol{\psi}$ be a normalized eigenvector of $\mathbf{N}$ with eigenvalue μ. The commutation relations (5.17) imply

$$\begin{aligned} \mathbf{N}(\mathbf{a}^*\boldsymbol{\psi}) &= (\mu + 1)\mathbf{a}^*\boldsymbol{\psi}, \\ \mathbf{N}(\mathbf{a}\boldsymbol{\psi}) &= (\mu - 1)\mathbf{a}\boldsymbol{\psi}. \end{aligned} \tag{5.20}$$

Thus $\mathbf{a}^*$ and $\mathbf{a}$ are raising and lowering operators: Given an eigenvector with eigenvalue μ we can obtain a ladder of eigenvectors with eigenvalues $\mu+n$, n an integer. Now

$$\langle \mathbf{a}^*\boldsymbol{\psi}, \mathbf{a}^*\boldsymbol{\psi}\rangle = \langle \mathbf{a}\mathbf{a}^*\boldsymbol{\psi}, \boldsymbol{\psi}\rangle = \langle (\mathbf{a}^*\mathbf{a}+\mathbf{E})\boldsymbol{\psi}, \boldsymbol{\psi}\rangle = \mu + 1. \tag{5.21}$$

Thus for $\mu \geq 0$, $||\mathbf{a}^*\boldsymbol{\psi}|| = \sqrt{\mu+1} > 0$, so the process of constructing eigenvectors of $\mathbf{N}$ by applying recursively the creation operator to the vacuum state can be continued indefinitely. Indeed from (5.20) and (5.21) we can define normalized eigenvectors $\boldsymbol{\psi}_n$ with eigenvalues n recursively by

$$\mathbf{a}^*\boldsymbol{\psi}_n = (n+1)^{1/2}\boldsymbol{\psi}_{n+1},\ n = 0, 1, 2, \cdots . \tag{5.22}$$

The commutation relations imply the formulas

$$\mathbf{N}\boldsymbol{\psi}_n = n\boldsymbol{\psi}_n, \quad \mathbf{a}\boldsymbol{\psi}_n = n^{1/2}\boldsymbol{\psi}_{n-1}. \tag{5.23}$$

Substituting expressions (5.18) into (5.22) and (5.23), we obtain a second-order differential equation and two recurrence formulas for the special functions $\boldsymbol{\psi}_n(t)$. We can obtain a generating function for the $\boldsymbol{\psi}_n$ from the first-order operator $\mathbf{a}^* = \frac{1}{\sqrt{2}}\left(\frac{d}{dt} - t\right)$. Note that $\mathbf{a}^* = \frac{1}{\sqrt{2}} e^{t^2/2}\left(\frac{d}{dt}\right)e^{-t^2/2}$. Hence by Taylor's theorem,

$$\begin{aligned} e^{\alpha\mathbf{a}^*}\boldsymbol{\psi}(t) &= \sum_{k=0}^{\infty} \frac{\alpha^k}{k!}(\mathbf{a}^*)^k = e^{t^2/2}\sum_{k=0}^{\infty}\frac{\left(\frac{\alpha}{\sqrt{2}}\right)^k}{k!}\left(\frac{d}{dt}\right)^k [e^{-t^2/2}\boldsymbol{\psi}(t)] \\ &= e^{t^2/2}e^{-(t+\frac{\alpha}{\sqrt{2}})^2/2}\boldsymbol{\psi}(t+2^{-1/2}\alpha) \\ &= \exp(-\frac{\alpha^2}{4} - 2^{-1/2}\alpha t)\boldsymbol{\psi}(t+2^{-1/2}\alpha) \end{aligned}$$

for any analytic function $\boldsymbol{\psi}$. On the other hand, from (5.22) we have

$$e^{\alpha\mathbf{a}^*}\boldsymbol{\psi}_n(t) = \sum_{k=0}^{\infty}\left[\frac{(n+k)!}{n!}\right]^{1/2}\frac{\alpha^k}{k!}\boldsymbol{\psi}_{n+k}(t).$$

Comparing these equations, we find the identity

$$\exp\left(-\frac{\beta^2}{2} - \beta t\right)\boldsymbol{\psi}_n(t+\beta) = \sum_{k=0}^{\infty}\left[\frac{2^k(n+k)!}{n!}\right]^{1/2}\frac{\beta^k}{k!}\boldsymbol{\psi}_{n+k}(t). \tag{5.24}$$

In the special case $n = 0$, the generating function yields

$$\pi^{-1/4}\exp\left(-\beta^2 - 2\beta t - \frac{1}{2}t^2\right) = \sum_{k=0}^{\infty}\frac{2^{k/2}\beta^k}{(k!)^{1/2}}\boldsymbol{\psi}_k(t). \tag{5.25}$$

Comparing this with the well-known generating function

$$\exp(-\beta^2 + 2\beta t) = \sum_{k=0}^{\infty}\frac{\beta^k}{k}H_k(t)$$

for the Hermite polynomials $H_k(t)$, [EMOT1], [M4], [M5], [M6], [V], [WW], we obtain

$$\psi_k(t) = \pi^{-1/4}(k!)^{-1/2}(-1)^k 2^{-k/2} e^{-t^2/2} H_k(t). \tag{5.26}$$

The above series converge for all t and β. Since the $\{\psi_n(t)\}$ form an ON set in $L_2(R)$ we easily obtain the formula

$$\int_{-\infty}^{\infty} H_n(t)H_k(t)e^{-t^2} dt = \pi^{1/2} 2^n n! \delta_{nk}.$$

We sketch a proof of the fact that the $\{\psi_n(t)\}$ form an ON basis for $L_2(R)$. It is enough to show that this set is *dense* in $L_2(R)$, i.e., if

$$< g, \psi_n >= 0, \quad \text{for } g \in L_2(R), \quad n = 0, 1, 2, \cdots , \tag{5.27}$$

then $g(t) = 0$ almost everywhere. Since $H_n(t)$ is a polynomial of order n in t with the coefficient of t^n nonzero, conditions (5.27) are equivalent to

$$\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} t^n g(t)\ dt = 0, \quad n = 0, 1, 2, \cdots .$$

Now consider the function

$$G(z) = \int_{-\infty}^{\infty} e^{izt} e^{-\frac{t^2}{2}} g(t)\ dt.$$

Since $g \in L_2(R)$, $G(z)$ is an (entire) analytic function of z and its derivatives can be obtained by differentiating under the integral sign:

$$\frac{d^n G(z)}{dz^n} = G^{(n)}(z) = i^n \int_{-\infty}^{\infty} e^{izt} t^n e^{-\frac{t^2}{2}} g(t)\ dt.$$

Now

$$G(z) = \sum_{n=0}^{\infty} \frac{G^{(n)}(0)}{n!} z^n \equiv 0.$$

Since $G(z)$ is the Fourier transform of $e^{-\frac{t^2}{2}} g(t)$, it follows that $g(t) = 0$ almost everywhere.

5.6 The Bargmann-Segal Hilbert space. Our next task is to compute the matrix elements $T^\lambda_{jk}(A) = (\mathbf{T}^\lambda(A)\psi_k, \psi_j)$ with respect to this basis. However, a computation of these matrix elements using the direct evaluation of the integral is not very enlightening. A better approach is to use a simpler model of the representation $\mathbf{T} \equiv \mathbf{T}^\lambda$, motivated by another solution of the Heisenberg commutation relations $[\mathbf{a}^*, \mathbf{a}] = -\mathbf{E}$. Rather than the solution (5.18) we can try

$$\mathbf{a}^* = z, \quad \mathbf{a} = \frac{d}{dz}. \tag{5.28}$$

This will work provided we can define a Hilbert space F on which $\mathbf{a}$ and $\mathbf{a}^*$ act and such that $\mathbf{a}^*$ is the adjoint of $\mathbf{a}$. (We will construct an ON basis $\{\mathbf{j}_n\}$ for F corresponding to the basis $\{\boldsymbol{\psi}_n\}$.) Since $\mathbf{a}\mathbf{j}_0(z) = 0$ implies that $\mathbf{j}_0(z)$ is constant, in order to mimic successfully the construction (5.22), (5.23) we see that the $\mathbf{j}_n$ must be proportional to z^n, so the elements of F must be functions $\mathbf{j}(z)$. If z were a real variable it would not be possible to find an inner product $(\mathbf{f}_1, \mathbf{f}_2) = \int \mathbf{f}_1(z)\overline{\mathbf{f}_2}(z)\rho(z)dz$ with respect to which $\mathbf{a}^*$ is the adjoint of $\mathbf{a}$. However, if we take z to be a complex variable and search for an inner product of the form $(\mathbf{f}_1, \mathbf{f}_2) = \iint_{-\infty}^{\infty} \mathbf{f}_1(z)\overline{\mathbf{f}_2}(z)\rho(z, \bar{z})dxdy$ we will be successful. (Here $z = x + iy$ and the region of integration is the plane R_2. We assume that the weight function ρ is nonnegative.) Integrating by parts in the formula $(\mathbf{f}_1, \mathbf{a}\mathbf{f}_2) = (\mathbf{a}^*\mathbf{f}_1, \mathbf{f}_2)$, assuming that the boundary terms vanish and that the formula holds identically in $\mathbf{f}_1, \mathbf{f}_2$ we obtain the condition $-\partial_{\bar{z}}\rho(z, \bar{z}) = z\rho(z, \bar{z})$. Thus $\rho(z, \bar{z}) = \pi^{-1}e^{-z\bar{z}}$ where we have chosen the constant π^{-1} so that $(1, 1) = 1$. It follows from the analogs of (5.22), (5.23) that

$$\begin{aligned} &\mathbf{a}\mathbf{j}_n = \sqrt{n}\mathbf{j}_{n-1}, \quad \mathbf{a}^*\mathbf{j}_n = \sqrt{n+1}\mathbf{j}_{n+1}, \\ &\mathbf{N}\mathbf{j}_n = n\mathbf{j}_n, \quad (\mathbf{j}_n, \mathbf{j}_m) = \delta_{nm} \\ &n, m = 0, 1, 2, \cdots \end{aligned} \tag{5.29}$$

where $\mathbf{j}_{-1} \equiv \boldsymbol{\theta}$ and

$$\mathbf{j}_n(z) = \frac{z^n}{\sqrt{n!}}, \quad n = 0, 1, 2, \cdots . \tag{5.30}$$

For any two functions $\mathbf{f}(z) = \sum_{k=0}^{\infty} a_k z^k$ and $\mathbf{h}(z) = \sum_{k=0}^{\infty} b_k z^k$ in F we find

$$(\mathbf{f}, \mathbf{h}) = \sum_{k=0}^{\infty} k! a_k \bar{b}_k \tag{5.31}$$

and

$$||\mathbf{f}||^2 = (\mathbf{f}, \mathbf{f}) = \sum_{k=0}^{\infty} k! |a_k|^2. \tag{5.32}$$

From (5.32), $\mathbf{f}$ belongs to the Hilbert space F if and only if $\sum_{k=0}^{\infty} k!|a_k|^2 < \infty$. Clearly if $\mathbf{f} \in F$ then there is a constant $C \geq 0$ such that $|a_k| \leq C/\sqrt{k!}$ for all k. By the ratio test, the series $\sum_k z^k/\sqrt{k!}$ converges for all complex z, so $\mathbf{f}$ is an entire function, i.e., the power series expansion for $\mathbf{f}$ has an infinite radius of convergence.

The space F was introduced by Segal and studied in detail by Bargmann [B2]. We mention here some of the special properties of F.

Define the function $\mathbf{e}_b \in F$ for some complex constant b by $\mathbf{e}_b(z) = \exp(\bar{b}z) = \sum_{k=0}^{\infty} (\bar{b}z)^k/k!$. It follows from (5.31) that for any $\mathbf{f} \in F$,

$$\begin{aligned} (\mathbf{f}, \mathbf{e}_b) &= \sum_{k=0}^{\infty} a_k b^k = \mathbf{f}(b), \\ (\mathbf{e}_b, \mathbf{e}_b) &= \mathbf{e}_b(b) = e^{\bar{b}b}. \end{aligned}$$

Thus $\mathbf{e}_b \in F$ acts like a delta function!

From the Schwarz inequality, $|\mathbf{f}(b)| = |(\mathbf{f}, \mathbf{e}_b)| \leq ||\mathbf{e}_b|| \cdot ||\mathbf{f}|| = e^{\bar{b}b/2}||\mathbf{f}||$. Thus, if $\mathbf{f}, \mathbf{h} \in F$, then $|\mathbf{f}(b) - \mathbf{h}(b)| \leq e^{\bar{b}b/2}||f - h||$ which shows that convergence in the norm of F implies pointwise convergence, uniform on any compact set in $\mathcal{C}$.

Now we construct a representation of H_R on F, using the annihilation and creation operators (5.28). Comparing with (5.18) we see that the standard basis for the Lie algebra of H_R is

$$\begin{aligned} \mathbf{C}_1 &= \frac{1}{\sqrt{2}}(\mathbf{a}^* - \mathbf{a}) = \frac{1}{\sqrt{2}}\left(z - \frac{d}{dz}\right) \\ \mathbf{C}_2 &= -\frac{2\pi\lambda i}{\sqrt{2}}(\mathbf{a}^* + \mathbf{a}) = -\frac{2\pi\lambda i}{\sqrt{2}}\left(z + \frac{d}{dz}\right) \\ \mathbf{C}_3 &= 2\pi\lambda i \mathbf{E} \end{aligned} \tag{5.33}$$

Mimicking the derivation of (5.24) by exponentiating these operators, we obtain the following candidates for operators defining a unitary rep of H_R on F:

$$\begin{aligned} \mathbf{T}'(x_1, 0, 0)\mathbf{f}(z) &= \exp\left(-\frac{x_1^2}{4} + 2^{-1/2}x_1 z\right)\mathbf{f}(z - 2^{-1/2}x_1), \\ \mathbf{T}'(0, x_2, 0)\mathbf{f}(z) &= \exp\left(-\pi^2\lambda^2 x_2^2 - 2^{1/2}i\pi\lambda x_2 z\right)\mathbf{f}(z - 2^{1/2}i\pi\lambda x_2), \\ \mathbf{T}'(0, 0, x_3)\mathbf{f}(z) &= e^{2\pi\lambda i x_3}\mathbf{f}(z). \end{aligned} \tag{5.34}$$

Since $A(\mathbf{x}) = A(0, x_2, 0)A(x_1, 0, 0)A(0, 0, x_3)$ we construct the operators $\mathbf{T}'(\mathbf{x}) \equiv \mathbf{T}'(A(\mathbf{x}))$ by

(5.35)

$$\begin{aligned} \mathbf{T}'(\mathbf{x})\mathbf{f}(z) &= [\mathbf{T}'(0, x_2, 0)\mathbf{T}'(x_1, 0, 0)\mathbf{T}'(0, 0, x_3)\mathbf{f}](z) \\ &= \exp\left[-\frac{(x_1^2 + 4\pi^2\lambda^2 x_2^2)}{4} + \frac{(x_1 - 2i\pi\lambda x_2)}{\sqrt{2}}z - i\pi\lambda x_1 x_2 + 2i\pi\lambda x_3\right] \times \\ &\qquad \mathbf{f}\left(z - \frac{[x_1 + 2i\pi\lambda x_2]}{\sqrt{2}}\right). \end{aligned}$$

It is straightforward to check that the operators $\mathbf{T}'(\mathbf{x})$ define a unitary rep of H_R on F. As one would expect, this rep is equivalent to the rep $\mathbf{T} \equiv \mathbf{T}^\lambda$ of H_R on $L_2(R)$. To establish the equivalence we construct the unitary operator $\mathbf{A} : L_2(R) \to F$ which maps the ON basis vector $\boldsymbol{\psi}_k(t)$ of $L_2(R)$, (5.26), to the ON basis vector $\mathbf{j}_k(z) = z^k/\sqrt{k!}$ of F:

$$\begin{aligned} [\mathbf{A}\boldsymbol{\psi}](z) &= \int_{-\infty}^{\infty} A(z, t)\boldsymbol{\psi}(t)dt, \quad \boldsymbol{\psi} \in L_2(R), \\ A(z, t) &= \sum_{k=0}^{\infty} \mathbf{j}_k(z)\boldsymbol{\psi}_k(t) = \pi^{-1/4}\exp[-(z^2 + t^2)/2 - \sqrt{2}zt]. \end{aligned} \tag{5.36}$$

The last identity follows from (5.25) with $z = \sqrt{2}\beta$. Since $A(z, \cdot) \in L_2(R)$ for each $z \in \mathcal{C}$ the integral in (5.36) is always defined. Now if $\boldsymbol{\psi} \in L_2(R)$ then it can be expanded uniquely in the form $\boldsymbol{\psi} = \sum_{n=0}^{\infty} c_n \boldsymbol{\psi}_n$. From (5.36) and the orthogonality of the basis $\{\boldsymbol{\psi}_k\}$ for $L_2(R)$ we have

$$\mathbf{f} = \mathbf{A}\boldsymbol{\psi} = \sum_{n=0}^{\infty} c_n \mathbf{j}_n \in F$$

where the $\{\mathbf{j}_k\}$ form an ON basis for F. Clearly the operator $\mathbf{A}$ is unitary. A correct expression for $\mathbf{A}^{-1} : F \to L_2(R)$ is a bit more involved:

$$\mathbf{A}^{-1}\mathbf{f}(t) = \lim_{\substack{\mu \to 1 \\ \mu < 1}} \int A\overline{(\mu z, t)} \mathbf{f}(z) \rho(z, \bar{z}) dx dy,$$

see [B2], [M4]. (It is necessary to insert the parameter $\mu < 1$ because $A(z, t)$ for fixed t does not belong to F.)

Now we can verify explicitly the relations

$$\begin{aligned} \mathbf{T}'(x_1, 0, 0)\mathbf{A} &= \mathbf{A}\mathbf{T}(x_1, 0, 0), \\ \mathbf{T}'(0, x_2, 0)\mathbf{A} &= \mathbf{A}\mathbf{T}(0, x_2, 0), \\ \mathbf{T}'(0, 0, x_3)\mathbf{A} &= \mathbf{A}\mathbf{T}(0, 0, x_3), \end{aligned}$$

where $\mathbf{T} \equiv \mathbf{T}^\lambda$ is given by (5.5). Since $\mathbf{A}$ is invertible, it follows that $\mathbf{T}'(\mathbf{x}) = \mathbf{A}\mathbf{T}(\mathbf{x})\mathbf{A}^{-1}$, so $\mathbf{T}'$ is equivalent to $\mathbf{T}$.

Since our chosen ON basis for F consists simply of powers of z, it is relatively easy to compute the matrix elements $T_{k\ell}(\mathbf{x}) = (\mathbf{T}'(\mathbf{x})\mathbf{j}_\ell, \mathbf{j}_k)$. (It is immediate that $(\mathbf{T}'(\mathbf{x})\mathbf{j}_\ell, \mathbf{j}_k) = \langle \mathbf{T}(x)\boldsymbol{\psi}_\ell, \boldsymbol{\psi}_k \rangle$ so these matrix elements are exactly the same as those which could be computed using the ON basis $\{\boldsymbol{\psi}_k\}$ for $L_2(R)$.) We define the generating function

$$G(\mathbf{x}; u, v) = (\mathbf{T}'(\mathbf{x})\mathbf{e}_{\bar{u}}, \mathbf{e}_v) = \sum_{n,m=0}^{\infty} (\mathbf{T}'(\mathbf{x})\mathbf{j}_m, \mathbf{j}_n) \frac{u^m v^n}{\sqrt{m! n!}}. \tag{5.37}$$

Due to the delta function property of $\mathbf{e}_{\bar{v}}$ we obtain

(5.38)

$$\begin{aligned} (\mathbf{T}'(\mathbf{x})\mathbf{e}_{\bar{u}}, \mathbf{e}_v) &= [\mathbf{T}(\mathbf{x})\mathbf{e}_{\bar{u}}](v) \\ &= \exp\left[-\frac{(x_1^2 + 4\pi^2\lambda^2 x_2^2)}{4} + \frac{(x_1 - 2\pi\lambda i x_2)}{\sqrt{2}} v - \pi\lambda i x_1 x_2 + 2\pi\lambda i x_3\right] \\ &\quad \times \exp\left[u\left(v - \frac{[x_1 + 2\pi\lambda i x_2]}{\sqrt{2}}\right)\right]. \end{aligned}$$

Introducing polar coordinates

$$x_1 = r\cos\theta, \quad 2\pi\lambda x_2 = r\sin\theta$$

and equating coefficients of $u^m v^n$ in (5.37) and (5.38) we obtain the explicit expression

$$\begin{aligned} T_{k\ell}(\mathbf{x}) = \exp\left[2\pi\lambda i x_3 + i(k-\ell)\theta - \pi\lambda i x_1 x_2\right] e^{-r^2/4} \left(\frac{k!}{\ell!}\right)^{1/2} \times \\ \left(\frac{r}{2^{1/2}}\right)^{\ell-k} L_k^{(\ell-k)}(r^2/2) \end{aligned} \tag{5.39}$$

where $L_k^{(\alpha)}(x)$ is the associated Laguerre polynomial [EMOT1], [M3], [M4]. An alternate expression is

$$T_{k\ell}(\mathbf{x}) = \exp\left[2\pi\lambda i x_3 + i(k-\ell)\theta - \pi\lambda i x_1 x_2\right] e^{-r^2/4}\left(\frac{\ell!}{k!}\right)^{1/2} \times \left(-\frac{r}{2^{1/2}}\right)^{k-\ell} L_\ell^{(k-\ell)}(r^2/2). \tag{5.40}$$

Note that the skew adjoint operator $-i\mathbf{N} = -iz\frac{d}{dz}$ is well defined on F and can be exponentiated to yield the unitary operator $\mathbf{U}'(\alpha)$:

$$\mathbf{U}'(\alpha)\mathbf{f}(z) = \exp\left(-i\alpha z\frac{d}{dz}\right)\mathbf{f}(z) = \mathbf{f}(e^{-i\alpha}z), \quad \mathbf{f} \in F. \tag{5.41}$$

Here the eigenvectors of $\mathbf{U}$ acting on F are just the ON basis vectors $\mathbf{j}_n$:

$$\mathbf{U}'(\alpha)\mathbf{j}_n = e^{-in\alpha}\mathbf{j}_n. \tag{5.42}$$

(One can extend the rep $\mathbf{T}'$ of the three parameter group H_R to the four-parameter **oscillator group** generated by $\mathbf{T}'(\mathbf{x})$ and $\mathbf{U}(\alpha)$. See [M3], [M4] for the details.) Using the unitary transformation $\mathbf{A}$ one can transform the $\mathbf{U}'(\alpha)$ to unitary operators on $L_2(R)$:

$$\mathbf{U}(\alpha)\boldsymbol{\psi}(t) = \lim_{n\to\infty}\int_{-n}^{n} \frac{e^{i\epsilon(\pi/4-\beta/2)}}{(2\pi|\sin\alpha|)^{1/2}} \times \exp\left[i\cot\alpha(t^2+\tau^2)/2 - it\tau/\sin\alpha\right]\boldsymbol{\psi}(\tau)d\tau. \tag{5.43}$$

Here, $\mathbf{U}(\alpha) = \mathbf{A}^{-1}\mathbf{U}'(\alpha)\mathbf{A}$ and $\alpha = 2k\pi + \epsilon\beta$, k an integer, $\epsilon = \pm 1$, $0 < \beta < \pi$. (See [B2] for details.) Note that $\mathbf{U}(\pi/2)$ is just the (ordinary) Fourier transform on $L_2(R)$. Thus, the Fourier transform is embedded in a one parameter group of transformations. the infinitesimal generator of this one-parameter group is the second-order differential operator

$$-i\mathbf{N} = i\left(\frac{1}{2}\frac{d^2}{dt^2} - \frac{t^2}{2} + \frac{1}{2}\right).$$

Furthermore, $\mathbf{U}(\alpha)\boldsymbol{\psi}_n = e^{-in\alpha}\boldsymbol{\psi}_n$.

It follows from the orthogonality relations (5.10) that the matrix elements (5.39) form an ON set in $L_2(R^2)$. (See [M3] or [M4] for a direct proof of this fact.) These orthogonality relations reduce to the following orthogonality relations for associated Laguerre polynomials:

$$\int_0^\infty L_m^{(k)}(r^2)L_n^{(k)}(r^2)e^{-r^2}r^{2k+1}dr = \frac{(n+k)!}{2n!}\delta_{mn}, \tag{5.44}$$

valid for all integers $m, n \geq 0$ and all integers k such that $n+k \geq 0, m+k \geq 0$. (By switching between (5.39) and (5.40) depending on whether $k-\ell \leq 0$ or $k-\ell > 0$ we can always take $k \geq 0$ in (5.44).) Since for each $k \geq 0$ the associated Laguerre polynomials are known to be complete in $L_2(0,\infty)$ with weight function $e^{-r^2}r^{2k+1}$, it follows that the matrix elements $\{T_{mn}\}$ form an ON basis for $L_2(R^2)$, with the usual Lebesgue weight function 1. (Indeed, this follows from Exercise 5.2, without any knowledge of the completeness properties of the Laguerre polynomials.) Thus, the matrix elements of $\mathbf{T}$ with respect to any ON basis for $L_2(R)$ will form an ON basis for $L_2(R^2)$. **(In other words, the cross-ambiguity functions with respect to an ON basis of signals $\{s_k\}$ form an ON basis for $L_2(R^2)$.)**

5.7 The lattice representation of H_R. There is another realization of the irred unitary rep $\mathbf{T}^\lambda$ that we shall find useful: an induced rep of H_R from the subgroup H' where

$$H' = \left\{ A(a_1, a_2, y_3) = \begin{pmatrix} 1 & a_1 & y_3 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

a_1, a_2 are integers and $y_3 \in R$. Note that the operators $\mathbf{T}_0(a_1, a_2, y_3) = e^{2\pi i y_3}$ define a one-dimensional unitary rep of H'. (Here, the fact that a_1, a_2 are integers is crucial in verifying that $\mathbf{T}_0$ is a rep of H'.) We will study the rep $\tilde{\mathbf{T}}$ of H_R induced from the rep $\mathbf{T}_0$ of H', (5.2 - 5.4). Here $\tilde{\mathbf{T}}$ is defined on the space V of functions $\mathbf{f}$ on H_R such that $\mathbf{f}(BA) = \mathbf{T}_0(B)\mathbf{f}(A)$ for all $B \in H', A \in H_R$, i.e.,

$$\mathbf{f}(a_1 + x_1,\ a_2 + x_2,\ y_3 + x_3 + a_1 x_2) = e^{2\pi i y_3}\mathbf{f}(x_1, x_2, x_3). \tag{5.45}$$

The operators $\tilde{\mathbf{T}}(A), A \in H_R$ act on V according to

$$[\tilde{\mathbf{T}}(A)\mathbf{f}](A') = \mathbf{f}(A'A). \tag{5.46}$$

We see from (5.45) that for any $A(x_1, x_2, x_3)$ we can always choose $B(a_1, a_2, y_3)$ such that $BA = A'(x_1', x_2', 0)$ where $0 \le x_1' < 1, 0 \le x_2' < 1$. Thus $\mathbf{f}$ can be restricted to $X = H' \backslash H_R$ with coordinates $(x_1', x_2', 0)$. Moreover, setting $x_3 = 0, y_3 = -a_1 x_2$ in (5.45) we have the periodicity condition

$$\boldsymbol{\varphi}(a_1 + x_1, a_2 + x_2) = e^{-2\pi i a_1 x_2}\boldsymbol{\varphi}(x_1, x_2) \tag{5.46}$$

where $\boldsymbol{\varphi}(x_1, x_2) = \mathbf{f}(x_1, x_2, 0)$. Conversely, given $\boldsymbol{\varphi}$ satisfying (5.46) we can define a unique $\mathbf{f}$ satisfying (5.45) by

$$\mathbf{f}(x_1, x_2, x_3) = \boldsymbol{\varphi}(x_1, x_2)e^{2\pi i x_3}.$$

The H_R-invariant inner product on X is $dx_1 dx_2$:

$$\langle \boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2 \rangle = \int_0^1 \int_0^1 \boldsymbol{\varphi}_1(x_1, x_2)\overline{\boldsymbol{\varphi}_2}(x_1, x_2)\,dx_1 dx_2, \tag{5.47}$$

and the operator $\tilde{\mathbf{T}}[\mathbf{y}] \equiv \tilde{\mathbf{T}}(A(y_1, y_2, y_3))$ acts on these functions by

$$(\tilde{\mathbf{T}}[\mathbf{y}]\boldsymbol{\varphi})(x_1, x_2) = \exp[2\pi i(y_3 + x_1 y_2)]\boldsymbol{\varphi}(x_1 + y_1, x_2 + y_2). \tag{5.48}$$

To recapitulate, we have defined a unitary rep $\tilde{\mathbf{T}}$ of H_R on the Hilbert space V' of all functions $\boldsymbol{\varphi}$ satisfying (5.46) and of finite norm with respect to the inner product (5.47). This is known as the **lattice representation** of H_R.

The lattice rep is equivalent to the irred Schrödinger rep $\mathbf{T}^1$, (5.5). To see this consider the periodizing operator (Weil-Brezin-Zak isomorphism)

$$\begin{aligned} \mathbf{P}\boldsymbol{\psi}(x_1, x_2, x_3) &= \sum_{n=-\infty}^{\infty} (\mathbf{T}^1[x_1, x_2, x_3]\boldsymbol{\psi})(n) \\ &= e^{2\pi i x_3} \sum_{n=-\infty}^{\infty} e^{2\pi i n x_2}\boldsymbol{\psi}(n + x_1) \end{aligned} \tag{5.49}$$

which is well defined for any $\boldsymbol{\psi} \in L_2(R)$ which belongs to the Schwartz space. It is straightforward to verify that $\mathbf{f} = \mathbf{P}\boldsymbol{\psi}$ satisfies the periodicity condition (5.45), hence $\mathbf{f}$ belongs to V. Now

$$\begin{aligned}&\langle \mathbf{P}\boldsymbol{\psi}(\cdot,\cdot,0), \mathbf{P}\boldsymbol{\psi}'(\cdot,\cdot,0)\rangle\\ &= \int_0^1 dx_1 \int_0^1 dx_2 \sum_{m,n=-\infty}^{\infty} e^{2\pi i(n-m)x_2}\boldsymbol{\psi}(n+x_1)\overline{\boldsymbol{\psi}'(m+x_2)}\\ &= \int_0^1 dx_1 \sum_{n=-\infty}^{\infty} \boldsymbol{\psi}(n+x_1)\overline{\boldsymbol{\psi}'(n+x_1)} = \int_{-\infty}^{\infty} \boldsymbol{\psi}(t_1)\overline{\boldsymbol{\psi}'}(t)\ dt\\ &=(\boldsymbol{\psi}, \boldsymbol{\psi}')\end{aligned}$$

so $\mathbf{P}$ can be extended to an inner product preserving mapping of $L_2(R)$ into V.

It is clear from (5.49) that if $\boldsymbol{\varphi}(x_1,x_2) = \mathbf{P}\boldsymbol{\psi}(x_1,x_2,0)$ then we can recover $\boldsymbol{\psi}(x_1)$ by integrating with respect to $x_2 : \boldsymbol{\psi}(x_1) = \int_0^1 \boldsymbol{\varphi}(x_1,y)dy$. Thus we define the mapping $\mathbf{P}^*$ of V' into $L_2(R)$ by

$$\mathbf{P}^*\boldsymbol{\varphi}(t) = \int_0^1 \boldsymbol{\varphi}(t,y)dy, \quad \boldsymbol{\varphi} \in V'. \tag{5.50}$$

Since $\boldsymbol{\varphi} \in V'$ we have

$$\mathbf{P}^*\boldsymbol{\varphi}(t+a) = \int_0^1 \boldsymbol{\varphi}(t,x)e^{-2\pi i a y}dy = \hat{\varphi}_{-a}(t)$$

for a an integer. (Here $\hat{\varphi}_n(t)$ is the nth Fourier coefficient of $\varphi(t,y)$.) The Parseval formula then yields

$$\int_0^1 |\boldsymbol{\varphi}(t,y)|^2 dy = \sum_{a=-\infty}^{\infty} |\mathbf{P}^*\boldsymbol{\varphi}(t+a)|^2$$

so

$$\begin{aligned}\langle \boldsymbol{\varphi}, \boldsymbol{\varphi}\rangle &= \int_0^1\int_0^1 |\boldsymbol{\varphi}(t,y)|^2 dt dy = \int_0^1 \sum_{a=-\infty}^{\infty} |\mathbf{P}^*\boldsymbol{\varphi}(t+a)|^2 dt\\ &= \int_{-\infty}^{\infty} |\mathbf{P}^*\boldsymbol{\varphi}(t)|^2 dt = (\mathbf{P}^*\boldsymbol{\varphi}, \mathbf{P}^*\boldsymbol{\varphi}).\end{aligned}$$

and $\mathbf{P}^*$ is an inner product preserving mapping of V' into $L_2(R)$. Moreover, it is easy to verify that

$$\langle \mathbf{P}\boldsymbol{\psi}, \boldsymbol{\varphi}\rangle = (\boldsymbol{\psi}, \mathbf{P}^*\boldsymbol{\varphi})$$

for $\boldsymbol{\psi} \in L_2(R)$, $\boldsymbol{\varphi} \in V'$, i.e., $\mathbf{P}^*$ is the adjoint of $\mathbf{P}$. Since $\mathbf{P}^*\mathbf{P} = \mathbf{E}$ on $L_2(R)$ it follows that $\mathbf{P}$ is a unitary operator mapping $L_2(R)$ **onto** V' and $\mathbf{P}^* = \mathbf{P}^{-1}$ is a unitary operator mapping V' **onto** $L_2(R)$.

Finally,

$$\begin{aligned}(\mathbf{P}\mathbf{T}^1[\mathbf{y}]\boldsymbol{\psi})(\mathbf{x}) &= e^{2\pi i(x_3+y_3+x_1y_2)} \sum_{n=-\infty}^{\infty} e^{2\pi i n(x_2+y_2)}\boldsymbol{\psi}(n+x_1+y_1)\\ &= (\tilde{\mathbf{T}}[\mathbf{y}]\mathbf{P}\boldsymbol{\psi})(\mathbf{x})\end{aligned}$$

so $\mathbf{P}\mathbf{T}^1[\mathbf{y}] = \tilde{\mathbf{T}}[\mathbf{y}]\mathbf{P}$ and the unitary reps $\mathbf{T}^1$ and $\tilde{\mathbf{T}}$ are equivalent.

5.8 Functions of positive type. Let G be a group. A complex function ρ on G is said to be of **positive type** provided for every finite set $g_1, \cdots, g_{k_0}$ of elements of G and every set of complex numbers $\lambda_1, \cdots, \lambda_{k_0}$ the inequality

$$\sum_{j,\ell} \rho(g_j^{-1} g_\ell) \bar{\lambda}_j \lambda_\ell \geqq 0 \tag{5.51}$$

holds. If $\mathbf{U}$ is a unitary rep of G on the Hilbert space H then the inner product $\rho(g) = \langle \mathbf{U}(g)\mathbf{f}, \mathbf{f} \rangle$ is of positive type on G for every $f \in H$. Indeed,

$$\begin{aligned} \sum_{j,\ell} \rho(g_j^{-1} g_\ell) \bar{\lambda}_j \lambda_\ell &= \sum_{j,\ell} \langle \mathbf{U}(g_j^{-1} g_\ell)\mathbf{f}, \mathbf{f} \rangle \bar{\lambda}_j \lambda_\ell \\ &= \sum_{j,\ell} \langle \mathbf{U}(g_\ell)\mathbf{f}, \mathbf{U}(g_j)\mathbf{f} \rangle \bar{\lambda}_j \lambda_\ell \\ &= \langle \sum_\ell \lambda_\ell \mathbf{U}(g_\ell)\mathbf{f}, \sum_\ell \lambda_\ell \mathbf{U}(g_\ell)\mathbf{f} \rangle \geqq 0. \end{aligned} \tag{5.52}$$

It follows from this construction, in the case where $G = H_R$, that narrow band ambiguity functions are of positive type on H_R.

It is an important result of abstract harmonic analysis that all functions of positive type on a group arise as diagonal matrix elements of unitary reps, exactly as in the construction (5.52). This result, whose proof we now sketch, sheds light on the structure of the set of ambiguity functions.

Note first that the inequality (5.51) implies that the $k_0 \times k_0$ matrix with elements $H_{j\ell} = \rho(g_j^{-1} g_\ell)$ is Hermitian ($H_{j\ell} = \bar{H}_{\ell j}$) and nonnegative. Thus the k_0 eigenvalues of this matrix are nonnegative; hence the determinant is also nonnegative.

Lemma 5.1. *Let ρ be a function of positive type on the group G. Then*
1) $\overline{\rho(g^{-1})} = \rho(g)$
2) $\rho(e) \geq |\rho(g)|$
for each $g \in G$.

Proof. For $k_0 = 1$, the inequality (5.51) yields $\rho(e) \geq 0$. Now take $k_0 = 2$, $g_1 = e$, $g_2 = g$. Then

$$(H_{j\ell}) = \begin{pmatrix} \rho(e) & \rho(g) \\ \rho(g^{-1}) & \rho(e) \end{pmatrix}.$$

Since $H_{12} = \bar{H}_{21}$ we have $\rho(g) = \overline{\rho(g^{-1})}$. Further, $\det H = \rho(e)^2 - \rho(g)\rho(g^{-1}) \geq 0$. □

Theorem 5.3. *Let ρ be a function of positive type on G. Then there is a unitary representation $\mathbf{U}$ of G on a Hilbert space H such that $\rho(g) = \langle \mathbf{U}(g)\mathbf{f}, \mathbf{f} \rangle$ for some $\mathbf{f} \in H$. Furthermore the span of the set $\{\mathbf{U}(g)\mathbf{f} : g \in G\}$ is dense in H.*

Proof. We will use the computation (5.52) as a motivation for the construction of H and $\mathbf{U}$. Let L be the subspace of the group ring R_G consisting of those elements x which take on only a finite number of nonzero values:

$$x = \sum_i x_i \cdot g_i.$$

(Alternatively we can consider x as a function $x : G \to \mathbb{C}$ which is zero except at a finite number of points g_i.) The inner product of two vectors in L is defined as

$$\langle y, x\rangle = \sum_{i,j} \rho(g_i^{-1} g_j)\bar{x}_i y_j$$

where we sum over all points g_ℓ such that either $x(g_\ell)$ or $y(g_\ell)$ is nonzero. It is evident that $\langle\cdot,\cdot\rangle$ is linear in its first argument and from Lemma 5.1 that $\langle y, x\rangle = \overline{\langle x, y\rangle}$ and $\langle x, x\rangle \geq 0$.

Thus, $\langle\cdot,\cdot\rangle$ satisfies all requirements for an inner product, except that we might have $\langle x, x\rangle = 0$ with $x \neq \theta$. It follows from this result that the Cauchy-Schwarz inequality is valid: $|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$.

We use a standard construction to convert $\langle\cdot,\cdot\rangle$ to a true inner product.

Let $N = \{x \in L : \langle x, x\rangle = 0\}$. Then N is a subspace of L. From the Cauchy-Schwarz inequality we have $\langle x, y\rangle = 0$ for $x \in N$, $y \in L$. Thus

$$\langle y_1 + x_1,\ y_2 + x_2\rangle = \langle y_1, y_2\rangle \tag{5.53}$$

for $y_j \in L, x_j \in N$. We can now define $\langle\cdot,\cdot\rangle$ on the factor space L/N whose elements are the sets $\mathbf{y} = y + N = \{y + x : x \in N\}$. The set $\boldsymbol{\theta} = \theta + N$ corresponds to the zero vector. From (5.53) we see that the definition

$$\langle \mathbf{y}_1, \mathbf{y}_2\rangle = \langle y_1 + N, y_2 + N\rangle = \langle y_1, y_2\rangle$$

is unambiguous. Furthermore $\langle \mathbf{y}, \mathbf{y}\rangle = 0$ only if $\mathbf{y} = \theta + N$, i.e., $\mathbf{y} = \boldsymbol{\theta}$. Thus $\langle\cdot,\cdot\rangle$ is a true inner product on L/N. Now by taking all Cauchy sequences in L/N we can complete L/N to a Hilbert space H.

Given $h \in G, x = \sum_i x_i \cdot g_i \in L$ we define the linear operator $\mathbf{U}(h) : L \to L$ by $\mathbf{U}(h)x = \sum_i x_i \cdot hg_i$. Then

$$\begin{aligned}\langle \mathbf{U}(h)y, \mathbf{U}(h)x\rangle &= \sum_{i,j} \rho(g_i^{-1}h^{-1}hg_j)\bar{x}_i x_j \\ &= \sum_{i,j} \rho(g_i^{-1}g_j)\bar{x}_i y_j = \langle y, x\rangle\end{aligned}$$

so $\mathbf{U}(h)$ is an isometry on L. Furthermore, $\mathbf{U}(h^{-1})[\mathbf{U}(h)x] = x$ for all $x \in L$, so $\mathbf{U}(h)$ is invertible, hence unitary on L. Also, $\mathbf{U}(h_1h_2)x = \mathbf{U}(h_1)[\mathbf{U}(h_2)x]$ so $\mathbf{U}$ is a unitary rep of G on L. Clearly, the operators $\mathbf{U}(h)$ extend uniquely to a unitary rep of G on H.

Let $f = 1 \cdot e \in L$. Then the vector $\{\mathbf{U}(g)f\}$ span L, for if $x = \sum_i x_i \cdot g_i$ we have $x = \sum_i x_i\mathbf{U}(g_i)f$. Moreover, $\langle \mathbf{U}(g)f, f\rangle = \rho(g)$. In the extension of L/N to H, f maps to $\mathbf{f} \in H$ such that the span of $\{\mathbf{U}(g)\mathbf{f}\}$ is dense in H and $\langle \mathbf{U}(g)\mathbf{f}, \mathbf{f}\rangle = \rho(g)$. □

If G is a linear Lie group, one can show that the function ρ is continuous on G if and only if $\mathbf{u}$ is a continuous rep of G, [G1], [N2].

The unitary rep $\mathbf{U}$ constructed in Theorem 5.3 is unique up to equivalence. Indeed, suppose there are unitary reps $\mathbf{U}_1, \mathbf{U}_2$ on Hilbert spaces H_1, H_2 such that

$\langle \mathbf{U}_1(g)\mathbf{f}_1, \mathbf{f}_1\rangle_1 = \langle \mathbf{U}_2(g)\mathbf{f}_2, \mathbf{f}_2\rangle_2$ where the spans of $\{\mathbf{U}_j(g)\mathbf{f}_j\}$ are dense in H_j. Define the map $\mathbf{S} : H_1 \to H_2$ by

$$\mathbf{S}(\sum_i \lambda_i \mathbf{U}_1(g_i)\mathbf{f}_1) = \sum_i \lambda_i \mathbf{U}_2(g_i)\mathbf{f}_2 \tag{5.54}$$

for all finite sums $\mathbf{f}'_1 = \sum_i \lambda_i \mathbf{U}_1(g_i)\mathbf{f}_1$. In particular $\mathbf{S}\mathbf{f}_1 = \mathbf{f}_2$. This mapping is well defined because if $\mathbf{f}'_1 = \boldsymbol{\theta}$ and $\mathbf{S}\mathbf{f}'_1 = \mathbf{f}'_2$ then

$$\begin{aligned}\langle \mathbf{f}'_2, \mathbf{f}'_2\rangle_2 &= \sum_{i,j} \langle \lambda_i \mathbf{U}_2(g_i)\mathbf{f}_2, \lambda_j \mathbf{U}_2(g_j)\mathbf{f}_2\rangle_2 \\ &= \sum_{i,j} \langle \mathbf{U}_2(g_j^{-1}g_i)\mathbf{f}_2, \mathbf{f}_2\rangle_2 \lambda_i \bar{\lambda}_j \\ &= \sum_{i,j} \langle \mathbf{U}_1(g_j^{-1}g_i)\mathbf{f}_1, \mathbf{f}_1\rangle_1 \lambda_i \bar{\lambda}_j = \langle \mathbf{f}'_1, \mathbf{f}'_1\rangle_1 \\ &= 0,\end{aligned}$$

so $\mathbf{f}'_2 = \boldsymbol{\theta}$. (Furthermore this same calculation shows that $\mathbf{S}$ is an isometry.) Thus, $\mathbf{S}$ extends to a unitary transformation from H_1 onto H_2. Finally $\mathbf{S}\mathbf{U}_1(g)\mathbf{f}'_1 = \mathbf{U}_2(g)\mathbf{f}'_2 = \mathbf{U}_2(g)\mathbf{S}\mathbf{f}'_1$ for all finite sums $\mathbf{f}'_1$, so $\mathbf{S}\mathbf{U}_1(g) = \mathbf{U}_2(g)\mathbf{S}$, and the reps $\mathbf{U}_1$ and $\mathbf{U}_2$ are equivalent.

Corollary 5.1. *Let* $\mathbf{U}$ *be a unitary irred rep of the group* G *on the Hilbert space* H *such that*

$$\langle \mathbf{U}(g)\mathbf{f}_1, \mathbf{f}_1\rangle = \langle \mathbf{U}(g)\mathbf{f}_2, \mathbf{f}_2\rangle$$

for all $g \in G$, *where* $\mathbf{f}_1, \mathbf{f}_2$ *are nonzero elements of* H. *Then* $\mathbf{f}_2 = \lambda \mathbf{f}_1$ *for some* $\lambda \in \mathbb{C}$ *with* $|\lambda| = 1$.

Proof. Since $\mathbf{U}$ is irred, the spans of $\{\mathbf{U}(g)\mathbf{f}_j\}$ are dense in H. From the construction (5.54) with $H_1 \equiv H_2$ and $\mathbf{U}_1(g) \equiv \mathbf{U}_2(g)$ we see that there is a unitary operator $\mathbf{S} : H \to H$ such that $\mathbf{S}\mathbf{U}(g) = \mathbf{U}(g)\mathbf{S}$ for all $g \in G$ and $\mathbf{S}\mathbf{f}_1 = \mathbf{f}_2$. Since $\mathbf{U}$ is irred it follows from Theorem 5.1 that $\mathbf{S} = \lambda \mathbf{E}$, where $|\lambda| = 1$ since $\mathbf{S}$ is unitary. Thus, $\lambda \mathbf{f}_1 = \mathbf{f}_2$. □

Note that the corollary implies that two signals correspond to the same ambiguity function if and only if they differ by a constant factor of absolute value one.

Next we will characterize those functions of positive type on a topological group G that correspond to **irred** unitary reps of G. Consider functions ρ_1, ρ_2 of positive type on G. We say that ρ_1 **dominates** ρ_2 if $\rho_1 - \rho_2$ is of positive type on G (so that $\rho_1 = \rho_2 + (\rho_1 - \rho_2)$ is a sum of two functions of positive type). A function ρ of positive type on G is **indecomposable** if the only functions of positive type dominated by ρ are scalar multiples of ρ. (Clearly, the multiples must be of the form $a\rho$ where $0 < a < 1$.)

Theorem 5.4. *The function* ρ *of positive type on the topological group* G *is indecomposable if and only if*

$$\rho(g) = \langle \mathbf{U}(g)\mathbf{f}, \mathbf{f}\rangle \tag{5.55}$$

for all $g \in G$ *where* $\mathbf{U}$ *is a unitary* **irred** *rep of* G *on some Hilbert space* H *and* $\mathbf{f} \in H$.

Proof. Suppose ρ is indecomposable. By Theorem 5.3 there is a unitary rep $\mathbf{U}$ of G on the Hilbert space H and a vector $\mathbf{f} \in H$ such that the span of $\{\mathbf{U}(g)\mathbf{f}\}$ is dense in H, and $\rho(g) = \langle \mathbf{U}(g)\mathbf{f}, \mathbf{f}\rangle$. Let K be a nonzero subspace of H which is invariant under $\mathbf{U} : \mathbf{U}(g)K \subseteq K$ for all $g \in G$. Then $K^{\perp}$ is also invariant under $\mathbf{U}$ (see Theorem 4.3), and we have the unique decomposition $\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2$ with $\mathbf{f}_1 \in K, \mathbf{f}_2 \in K^{\perp}$. It follows easily that $\rho(g) = \langle \mathbf{U}(g)\mathbf{f}, \mathbf{f}\rangle = \langle \mathbf{U}(g)\mathbf{f}_1, \mathbf{f}_1\rangle + \langle \mathbf{U}(g)\mathbf{f}_2, \mathbf{f}_2\rangle = \rho_1(g) + \rho_2(g)$. Since $\{\mathbf{U}(g)\mathbf{f}_1\}$ is dense in K, $\rho_1(g) = \langle \mathbf{U}(g)\mathbf{f}_1, \mathbf{f}_1\rangle$ is a function of positive type and ρ dominates ρ_1. Hence $\rho_1 = a\rho$ for some constant $a \neq 0$, so $\langle \mathbf{U}(g)\mathbf{f}_1, \mathbf{f}_1\rangle = \langle \mathbf{U}(g)\mathbf{f}, \mathbf{f}_1\rangle = \langle \mathbf{U}(g)\mathbf{f}, a\mathbf{f}\rangle$ or,

$$\langle \mathbf{U}(g)\mathbf{f}, \mathbf{f}_1 - a\mathbf{f}\rangle = 0 \tag{5.56}$$

for all $g \in G$. It follows that $\mathbf{f}_1 = a\mathbf{f}$ so $\mathbf{f} \in K$, hence $K = H$ and $\mathbf{U}$ is irred.

Conversely, suppose there is a unitary irred rep $\mathbf{U}$ of G on H such that (5.55) holds for some $\mathbf{f} \in H$. Let ρ_1 be a function of positive type on G that is dominated by ρ. Without loss of generality we can assume that $\mathbf{U}$ is obtained from ρ and the space L/N according to the construction given in the proof of Theorem 5.3. On this same space we can construct the inner product $\langle\cdot,\cdot\rangle_1$ associated with the function ρ_1. Since ρ_1 is dominated by ρ, the Cauchy-Schwarz inequality implies $|\langle x, y\rangle_1|^2 \leq \langle x, x\rangle_1\langle y, y\rangle_1 \leq ||x||^2||y||^2$ for all $x, y \in L/N$. Thus $\langle\cdot,\cdot\rangle_1$ extends to a positive definite Hermitian form on H such that

$$|\langle \mathbf{k}_1, \mathbf{k}_2\rangle_1|^2 \leq ||\mathbf{k}_1||_1^2||\mathbf{k}_2||^2 \leqq ||\mathbf{k}_1||^2||\mathbf{k}_2||^2. \tag{5.57}$$

This means that there exists a bounded self-adjoint operator $\mathbf{A}$ on H such that

$$\langle \mathbf{k}_1, \mathbf{k}_2\rangle_1 = \langle \mathbf{k}_1, \mathbf{A}\mathbf{k}_2\rangle \tag{5.58}$$

for all $\mathbf{k}_1, \mathbf{k}_2 \in H$. (Indeed, it follows from (5.57) that for fixed $\mathbf{k}_2$, $\langle \mathbf{k}_1, \mathbf{k}_2\rangle_1$ is a bounded linear functional H. By the Riesz representation theorem [RN] there exists a vector $\mathbf{s_k} \in H$ for all $\mathbf{k} \in H$ such that $\langle \mathbf{k}_1, \mathbf{k}\rangle_1 = \langle \mathbf{k}_1, \mathbf{s_k}\rangle$. Clearly, the map $\mathbf{k} \to \mathbf{s_k}$ is linear, so there exists a linear operator $\mathbf{A} : H \to H$ such that $\mathbf{s_k} = \mathbf{A}\mathbf{k}$. Since $|\langle \mathbf{k}_1, \mathbf{A}\mathbf{k}_2\rangle|^2 = |\langle \mathbf{k}_1, \mathbf{k}_2\rangle_1|^2 \leq ||\mathbf{k}_1||^2||\mathbf{k}_2||^2$, we have in the case $\mathbf{k}_1 = \mathbf{A}\mathbf{k}_2$ the inequality $||\mathbf{A}\mathbf{k}_2||^4 \leq ||\mathbf{A}\mathbf{k}_2||^2||\mathbf{k}_2||^2$ or $||\mathbf{A}\mathbf{k}_2||^2 \leq ||\mathbf{k}_2||^2$. Thus, $\mathbf{A}$ is a bounded operator. Furthermore, $\langle \mathbf{k}_1, \mathbf{A}\mathbf{k}_2\rangle = \langle \mathbf{k}_1, \mathbf{k}_2\rangle_1 = \overline{\langle \mathbf{k}_2, \mathbf{k}_1\rangle_1} = \overline{\langle \mathbf{k}_2, \mathbf{A}\mathbf{k}_1\rangle} = \langle \mathbf{A}\mathbf{k}_1, \mathbf{k}_2\rangle$, so $\mathbf{A}$ is self-adjoint. Since $\langle \mathbf{k}, \mathbf{A}\mathbf{k}\rangle = \langle \mathbf{k}, \mathbf{k}\rangle_1 \geq 0$, $\mathbf{A}$ is nonnegative.)

By construction of $\langle\cdot,\cdot\rangle_1$ through the completion of L/N we have $\langle \mathbf{U}(g)\mathbf{k}_1, \mathbf{U}(g)\mathbf{k}_2\rangle_1 = \langle \mathbf{k}_1, \mathbf{k}_2\rangle$ for all $g \in G$. Hence, $\langle \mathbf{U}(g)\mathbf{k}_1, \mathbf{A}\mathbf{U}(g)\mathbf{k}_2\rangle = \langle \mathbf{k}_1, \mathbf{A}\mathbf{k}_2\rangle$ and $\mathbf{U}(g)^{-1}\mathbf{A}\mathbf{U}(g) = \mathbf{A}$, which implies $\mathbf{A}\mathbf{U}(g) = \mathbf{U}(g)\mathbf{A}$ for all $g \in G$. Since $\mathbf{U}$ is irreducible, we have $\mathbf{A} = a\mathbf{E}$ for some positive constant a. Thus $\rho_1(g) = \langle \mathbf{U}(g)\mathbf{f}, \mathbf{f}\rangle_1 = \langle \mathbf{U}(g)\mathbf{f}, \mathbf{A}\mathbf{f}\rangle = a\langle \mathbf{U}(g)\mathbf{f}, \mathbf{f}\rangle = a\rho(g)$, so ρ is indecomposable. □

Corollary 5.2. *Let* $\mathbf{U}$ *be an irred unitary rep of* G *on* H, *and suppose* $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2$ *are nonzero elements of* H *such that*

$$\langle \mathbf{U}(g)\mathbf{f}_0, \mathbf{f}_0\rangle = \langle \mathbf{U}(g)\mathbf{f}_1, \mathbf{f}_1\rangle + \langle \mathbf{U}(g)\mathbf{f}_2, \mathbf{f}_2\rangle \tag{5.59}$$

for all $g \in G$. *Then* $\mathbf{f}_1 = \kappa \mathbf{f}_2$, $\kappa \in \mathcal{C}$.

Proof. The functions of positive type $\rho_j(g) = \langle \mathbf{U}(g)\mathbf{f}_j, \mathbf{f}_j\rangle$ satisfy $\rho_0(g) = \rho_1(g) + \rho_2(g)$, so ρ_0 dominates ρ_1 and ρ_2. Since $\mathbf{U}$ is irred, ρ is indecomposable. Thus $\rho_0 = c_1^2\rho_1 = c_2^2\rho_2$ where c_1, c_2 are positive constants. Setting $\mathbf{f}'_\ell = c_\ell \mathbf{f}_\ell$, $\ell = 1,2$ we have

$$\langle \mathbf{U}(g)\mathbf{f}_0, \mathbf{f}_0\rangle = \langle \mathbf{U}(g)\mathbf{f}'_1\mathbf{f}'_1\rangle = \langle \mathbf{U}(g)\mathbf{f}'_2, \mathbf{f}'_2\rangle.$$

By Corollary 5.1, $\mathbf{f}'_1 = \lambda \mathbf{f}'_2$. Hence $\mathbf{f}_1 = \kappa \mathbf{f}_2$ for $\kappa = \lambda c_2/c_1$. □

Note that Corollary 5.2 implies that the sum of two ambiguity functions corresponding to nonzero signals s_1, s_2 is again an ambiguity function if and only if $s_1 = \lambda s_2$.

5.9 Exercises.

5.1 Verify explicitly equation (5.10).

5.2 Show that if $\{f_n,\ n = 0, 1, 2, \cdots\}$ is an ON basis for $L_2(R)$ then the matrix elements $\{< \mathbf{T}^\lambda(x_1, x_2)f_n, f_m >\}$ form an ON basis for $L_2(R^2)$. Hint: From exercise 5.1, the matrix elements form an ON set. Hence to show that they form a basis it is enough to prove that if

$$\int\int_{-\infty}^{\infty} dx_1 dx_2 < \mathbf{T}^\lambda(x_1, x_2)f_n, f_m > \overline{g(x_1, x_2)} = 0$$

for $g \in L_2(R^2)$ and all n, m, then $g = 0$ almost everywhere. This shows that the ON set is also dense in $L_2(R^2)$, hence is a basis.

5.3 Set $\mathbf{B}(f, g) =< \mathbf{T}^\lambda(x_1, x_2)f, g >$ for $f, g \in L_2(R)$ and $\mathbf{B}(f) = \mathbf{B}(f, f)$. Show that

$$\mathbf{B}(f+g) = \mathbf{B}(f) + \mathbf{B}(f, g) + \mathbf{B}(g, f) + \mathbf{B}(g),$$
$$\mathbf{B}(f+ig) = \mathbf{B}(f) + i\mathbf{B}(g, f) - i\mathbf{B}(f, g) + \mathbf{B}(g).$$

5.4 Prove that the ambiguity functions $< \mathbf{T}^\lambda(x_1, x_2)f, f >$ for all $f \in L_2(R)$ span a dense subspace of $L_2(R^2)$.

5.5 Suppose $f \in L_2(R)$ such that $f_{\mathbf{P}} \neq 0$ almost everywhere. Prove that the set $\{e^{2\pi i(m_1x_1+m_2x_2)} f_{\mathbf{P}}/|f_{\mathbf{P}}|,\ m_1, m_2 = \pm 1, \pm 2, \cdots\}$ is an ON basis for the lattice Hilbert space V'. Find an explicit expression for the corresponding ON basis of $L_2(R)$ obtained from the mapping $\mathbf{P}^{-1}$.

5.6 Show that the rep $\mathbf{T}^\lambda$ of H_R is *continuous* in the sense that

$$||\mathbf{T}^\lambda(x_1, x_2, x_3)f - f|| \to 0, \quad \text{as } (x_1, x_2, x_3) \to (0, 0, 0)$$

for each $f \in L_2(R)$.

5.7 Show that the ambiguity and cross-ambiguity functions $< \mathbf{T}^\lambda(x_1, x_2)f, g >$ are continuous functions of (x_1, x_2).

5.8 Construct the unitary rep $\tilde{\mathbf{T}}_n$ of H_R induced by the one-dimensional rep $\tilde{\mathbf{T}}_0^n(a_1, a_2, y_3) = e^{2\pi i n y_3}$ of the subgroup H^1, where n is an integer (not necessarily positive). Determine the action of H_R on the rep space, in analogy with (5.48). Under what conditions on n is $\tilde{\mathbf{T}}_n$ irred? Show that the rep space contains $|n|$ linearly independent ground state wave functions.

§6. REPRESENTATIONS OF THE AFFINE GROUP

6.1 Induced irreducible representations of G_A. Recall that the affine group G_A is the matrix group with elements

$$A(a,b) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \quad a > 0 \tag{6.1}$$

and multiplication rule

$$(a,b)(a',b') = (aa', ab' + b).$$

Even though this is a 2×2 matrix group with a very simple structure, there are some difficulties in developing its rep theory in accordance with the general results presented in Chapters 2-4. An indication of the complications appeared already in Chapter 4 where we showed that G_A is not unimodular, i.e., $d_\ell A \neq d_r A$. Indeed

$$d_\ell A = \frac{dadb}{a^2}, \quad d_r A = \frac{dadb}{a}. \tag{6.2}$$

We begin by deriving the irred unitary reps of G_A. One family of such reps is evident: $\chi_\rho[a,b] = a^{i\rho}$, ρ real. We use the method of induced reps to derive several forms of the remaining irred unitary reps.

Consider the subgroup $H_1 \cong R$ of elements of the form $A(1,b)$, $b \in R$. The unitary irred reps of H_1 take the form $\xi_\lambda(b) = e^{i\lambda b}$. We now construct the unitary rep of G_A induced by the rep ξ_λ of H_1. The rep is defined on a space of functions $\mathbf{f}(a,b)$ on G_A such that

$$\mathbf{f}(BA) = \xi_\lambda(B)\mathbf{f}(A), \quad B \in H_1, \ A \in G_A,$$

i.e.,

$$\mathbf{f}(a, b+b') = e^{i\lambda b'}\mathbf{f}(a,b) \tag{6.3}$$

Thus $\mathbf{f}(a,b) = e^{i\lambda b}\mathbf{f}(a,0) = e^{i\lambda b}\boldsymbol{\varphi}(a)$ where $\boldsymbol{\varphi}$ is defined on the coset space $X_1 \cong H_1 \backslash G_A \cong H_2$ and H_2 is the subgroup of elements $A(a,0), a > 0$. The action of G_A in the induced rep is given by

$$[\mathbf{R}_\lambda(A)\mathbf{f}](A') = \mathbf{f}(A'A), \quad A, A' \in G_A$$

or, restricted to the functions $\boldsymbol{\varphi}$:

$$(\mathbf{R}_\lambda[a,b]\boldsymbol{\varphi})(x) = e^{i\lambda x b}\boldsymbol{\varphi}(ax) \tag{6.4}$$

where $A' = A'(x,y)$. The right-invariant measure on X_1 is $d\mu(x) = dx/x$, so the inner product $\langle \cdot, \cdot \rangle_1$ with respect to which the operators $\mathbf{R}_\lambda[a,b]$ are unitary, is

$$\langle \boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2 \rangle_1 = \int_0^\infty \varphi_1(x)\bar{\varphi}_2(x)\frac{dx}{x}. \tag{6.5}$$

The elements of the Hilbert space H_1 with the inner product are Lebesgue measurable functions $\boldsymbol{\varphi}(x)$ such that $||\boldsymbol{\varphi}||_1^2 = \langle \boldsymbol{\varphi}, \boldsymbol{\varphi} \rangle_1 < \infty$. The rep $\mathbf{R}_0$ is reducible. However, we have

Theorem 6.1. *For $\lambda \neq 0$ the representation $\mathbf{R}_\lambda$ of G_A is irreducible.*

Proof. This demonstration is very similar to that of Theorem 5.2. For completeness we repeat the basic ideas of the proof, omitting some of the technical details. Our aim will be to show that if $\mathbf{L}$ is a bounded operator on H_1 which commutes with the operators $\mathbf{R}_\lambda(A)$ for all $A \in G_A$ then $\mathbf{L} = \kappa\mathbf{E}$ for some constant κ, where $\mathbf{E}$ is the identity operator. (It follows from this that $\mathbf{R}_\lambda$ is irred, for if M were a proper closed subspace of H_1, invariant under $\mathbf{R}_\lambda$, then the self-adjoint projection operator $\mathbf{P}$ on M would commute with the operators $\mathbf{R}_\lambda(A)$. This is impossible since $\mathbf{P}$ could not be a scalar multiple of $\mathbf{E}$.)

Suppose the bounded operator $\mathbf{L}$ satisfies

$$\mathbf{L}\mathbf{R}_\lambda[a,b] = \mathbf{R}_\lambda[a,b]\mathbf{L}$$

for all real a, b with $a > 0$. First consider the case $a = 1$: $\mathbf{L}$ commutes with the operation of multiplication by the function $e^{i\lambda bx}$. Clearly $\mathbf{L}$ must also commute with multiplication by finite sums of the form $\sum_{b_j} c_j e^{i\lambda b_j x}$ and, by using the well-known fact that trigonometric polynomials are dense in the space of measurable functions, $\mathbf{L}$ must commute with multiplication by any bounded function $f(x)$ on $(0,\infty)$. Now let Q be a bounded closed internal in $(0,\infty)$ and let $\chi_Q \in H_1$ be the **characteristic function** of Q:

$$\chi_Q(x) = \begin{cases} 1 & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}.$$

Let $f_Q \in H_1$ be the function $f_Q = \mathbf{L}\chi_Q$. Since $\chi_Q^2 = \chi_Q$ we have $f_Q(x) = \mathbf{L}\chi_Q(x) = \mathbf{L}\chi_Q^2(x) = \chi_Q(x)L\chi_Q(x) = \chi_Q(x)f_Q(x)$ so f_Q is nonzero only for $x \in Q$. Furthermore, if Q' is a closed interval with $Q' \subseteq Q$ and $f_{Q'} = \mathbf{L}\chi_{Q'}$ then $f_{Q'}(x) = \mathbf{L}\chi_{Q'}\chi_q(x) = \chi_{Q'}(x)\mathbf{L}\chi_Q(x) = \chi_{Q'}(x)F_q(x)$ so $f_{Q'}(x) = f_Q(x)$ for $x \in Q'$ and $f_{Q'}(x) = 0$ for $x \notin Q'$. It follows that there is a unique function $f(x)$ such that $\chi_{\tilde{Q}} f \in H_1$ and $\chi_{\tilde{Q}}(x)f(x) = L\chi_{\tilde{Q}}(x)$ for any closed bounded interval $\tilde{Q}$ in $(0,\infty)$. Now let $\boldsymbol{\varphi}$ be a C^∞ function which is zero in the exterior of $\tilde{Q}$. Then $\mathbf{L}\boldsymbol{\varphi}(x) = \mathbf{L}(\boldsymbol{\varphi}\chi_{\tilde{Q}}(x)) = \boldsymbol{\varphi}(x)\mathbf{L}\chi_{\tilde{Q}}(x) = \boldsymbol{\varphi}(x)f(x)\chi_{\tilde{Q}}(x) = f(x)\boldsymbol{\varphi}(x)$, so $\mathbf{L}$ acts on $\boldsymbol{\varphi}$ by multiplication by the function $f(x)$. Since as $\tilde{Q}$ runs over all finite subintervals of $(0,\infty)$ the functions $\boldsymbol{\varphi}$ are dense in H_1, it follows that $\mathbf{L} = f(x)\mathbf{E}$.

Now we use the hypothesis that $\mathbf{L}\mathbf{R}_\lambda[a,0]\boldsymbol{\varphi}(x) = \mathbf{R}_\lambda[a,0]\mathbf{L}\boldsymbol{\varphi}(x)$ for all $a > 0$ and $\boldsymbol{\varphi} \in H_1 : f(x)\varphi(ax) = f(ax)\varphi(ax)$. Thus $f(x) = f(ax)$ almost everywhere, which implies that $f(x)$ is a constant. □

Lemma 6.1. *For $\tau > 0$ the reps $\mathbf{R}_\lambda$ and $\mathbf{R}_{\tau\lambda}$ are equivalent.*

Proof. Let $\mathbf{S}_\tau : H_0 \to H_0$ be the linear unitary operator $\mathbf{S}_\tau\boldsymbol{\varphi}(x) = \boldsymbol{\varphi}(\tau x)$. Then $\mathbf{S}_\tau^{-1} = \mathbf{S}_{\tau^{-1}}$ and $\mathbf{R}_{\tau\lambda}[a,b]\mathbf{S}_\tau = \mathbf{S}_\tau\mathbf{R}_\lambda[a,b]$. □

It follows that there are just two distinct irred reps in the family $\mathbf{R}_\lambda$. We choose these reps in the normalized form $\lambda = \pm 1$. Thus we have constructed the following irred unitary reps of $G_A : \chi_\rho$, $(-\infty < \rho < \infty)$; $\mathbf{R}_+$ and $\mathbf{R}_-$ where

$$\begin{aligned} \chi_\rho[a,b] &= a^{i\rho} \\ \mathbf{R}_+[a,b]\,\boldsymbol{\varphi}(x) &= e^{ixb}\boldsymbol{\varphi}(ax), \qquad (6.6) \\ \mathbf{R}_-[a,b]\boldsymbol{\varphi}(x) &= e^{-ixb}\boldsymbol{\varphi}(ax), \end{aligned}$$

and $\varphi \in H_1$. It can be shown [K5] that these are the only unitary irred reps of G_A.

6.2 The wideband cross-ambiguity functions. Another important class of unitary reps of G_A can be induced from the subgroup $H_2 \cong R^+$ of elements of the form $A(a,0), a > 0$. Consider the irred reps η_σ of H_2: $\eta_\sigma(a) = a^\sigma$, σ complex. We construct the rep $\mathbf{L}'_\sigma$ of G_A induced by η_σ. It is defined on a space of functions $\mathbf{f}(a,b)$ on G_A such that

$$\mathbf{f}(BA) = \eta_\sigma(B)\mathbf{f}(A), \quad B \in H_2,\ A \in G_A,$$

i.e.,

$$\mathbf{f}(a'a, a'b) = (a')^\sigma \mathbf{f}(a,b). \tag{6.7}$$

Thus $\mathbf{f}(a,b) = a^\sigma \mathbf{f}(1, b/a) = a^\sigma \varphi(b/a)$ where φ is defined on the coset space $X_2 \cong H_2 \setminus G_A \cong H_1$. The action of G_A in the induced rep is given by

$$[\mathbf{L}'_\sigma(A)\mathbf{f}](A') = \mathbf{f}(A'A), \quad A, A' \in G_A,$$

or, restricted to the functions φ:

$$(\mathbf{L}'_\sigma[a,b]\varphi)(t) = a^\sigma \varphi\left(\frac{t+b}{a}\right). \tag{6.8}$$

There is no right-invariant measure on X_2. However, $d\rho(t) = dt$ goes to a multiple of itself, so an inner product $\langle\cdot,\cdot\rangle_2$ with respect to which the operators $\mathbf{L}'_\sigma[a,b]$ are unitary is

$$\langle\varphi_1, \varphi_2\rangle_2 = \int_{-\infty}^{\infty} \varphi_1(t)\overline{\varphi_2(t)}dt, \tag{6.9}$$

provided $\sigma = i\mu - \frac{1}{2}$ where μ is real. The elements of the Hilbert space $H_2 = L_2(R)$ with this inner product are Lebesgue measurable functions $\varphi(t)$ such that $||\varphi||_2^2 = \langle\varphi,\varphi\rangle_2 < \infty$. Since we will be restricting to the case of unitary reps, we introduce the notation $\mathbf{L}_\mu \equiv \mathbf{L}'_{i\mu-\frac{1}{2}}$:

$$(\mathbf{L}_\mu[a,b]\varphi)(t) = a^{i\mu-\frac{1}{2}}\varphi\left(\frac{t+b}{a}\right), \quad \varphi \in L_2(R). \tag{6.10}$$

The rep $\mathbf{L}_\mu$ is reducible. Indeed let us consider the Fourier transform $\mathbf{F}$ as a unitary map from $L_2(R)$ to $L_2(R)$:

$$\hat{\varphi}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \varphi(t)e^{-ity}dt = \mathbf{F}\varphi(y) \tag{6.11}$$

Then

$$\varphi(t) = \mathbf{F}^{-1}\hat{\varphi}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\varphi}(y)e^{ity}dy$$

and the action $\hat{\mathbf{L}}_\mu$ of G_A on the function $\hat{\varphi}$ corresponding to the action $\mathbf{L}_\mu$ on $\boldsymbol{\varphi}$ is

$$\begin{aligned}\hat{\mathbf{L}}_\mu[a,b]\hat{\boldsymbol{\varphi}}(y) &= \mathbf{F}(\mathbf{L}_\mu[a,b]\boldsymbol{\varphi})(y)\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} a^{i\mu-\frac{1}{2}}\boldsymbol{\varphi}\left(\frac{t+b}{a}\right)e^{-ity}\,dt\\ &= a^{i\mu+1/2}e^{iby}\hat{\boldsymbol{\varphi}}(ay).\end{aligned} \tag{6.12}$$

Now consider the map $\mathbf{S}_\mu : L_2(R)\to H_1^+\oplus H_1^-$ defined by

$$\mathbf{S}_\mu\hat{\boldsymbol{\varphi}}(y) = |y|^{i\mu+1/2}\hat{\boldsymbol{\varphi}}(y);\quad y\neq 0.$$

and let

$$\begin{aligned}\hat{\boldsymbol{\varphi}}^+(y) &= \mathbf{S}_\mu\hat{\boldsymbol{\varphi}}(y)\in H_1^+ \text{ for } y>0\\ \hat{\boldsymbol{\varphi}}^-(z) &= \mathbf{S}_\mu\hat{\boldsymbol{\varphi}}(y)\in H_1^- \text{ for } z=-y>0.\end{aligned}$$

Then the action induced on $\hat{\boldsymbol{\varphi}}^+\in H_1^+$ by $\mathbf{L}_\mu[a,b]$ is

$$\mathbf{R}_+[a,b]\hat{\boldsymbol{\varphi}}^+(y) = e^{iby}\hat{\boldsymbol{\varphi}}^+(ay)$$

and the action induced on $\hat{\boldsymbol{\varphi}}^-\in H_1^-$ is

$$\mathbf{R}_-[a,b]\hat{\boldsymbol{\varphi}}^-(z) = e^{-ibz}\hat{\boldsymbol{\varphi}}^-(az).$$

(Here, the spaces H_1^+, H_1^- consist of functions $\boldsymbol{\psi}^+(y), \boldsymbol{\psi}^-(z)$ on the positive real line with weight functions dy/y, dz/z, respectively. The operator $\mathbf{S}_\mu$ is responsible for the change in weight function.) As the reader can easily verify, we have the "Plancherel formula"

$$\langle\mathbf{L}_\mu[a,b]\boldsymbol{\varphi},\boldsymbol{\psi}\rangle_2 = \langle\mathbf{R}_+[a,b]\boldsymbol{\varphi}^+,\boldsymbol{\psi}^+\rangle_1 + \langle\mathbf{R}_-[a,b]\boldsymbol{\varphi}^-,\boldsymbol{\psi}^-\rangle_1 \tag{6.13}$$

where

$$\begin{aligned}\boldsymbol{\psi}^+(y) &= \mathbf{S}_\mu\mathbf{F}\boldsymbol{\psi}(y),\quad y>0,\\ \boldsymbol{\psi}^-(z) &= \mathbf{S}_\mu\mathbf{F}\boldsymbol{\psi}(y),\quad y=-z<0.\end{aligned} \tag{6.14}$$

Thus the rep $\mathbf{L}_\mu$ decomposes as the direct sum of the irred reps $\mathbf{R}_+$ and $\mathbf{R}_-$; the Fourier component $\hat{\boldsymbol{\varphi}}(y)$ of $\boldsymbol{\varphi}$ corresponds to $\mathbf{R}_+$ for positive y and to $\mathbf{R}_-$ for negative y. (If, however, we restrict the rep $\mathbf{L}^\mu$ to those $\boldsymbol{\varphi}\in L_2(R)$ whose Fourier transform $\hat{\varphi}(y)$ has support on the positive reals, then $\mathbf{L}_\mu$ is irreducible and equivalent to $\mathbf{R}_-$.)

Note that the matrix element $\langle\mathbf{L}_0[a,b]s_n, s_m\rangle_2$ coincides with the wideband cross-ambiguity function (2.19) with $y=a^{-1}, x=b/a$. Formula (6.13) suggests that, for computational simplicity, in selecting a basis for $L_2(R)$ in which to determine the cross-ambiguity function, one should choose the union of a basis on the subspace transforming according to $\mathbf{R}_+$ and a basis on the subspace transforming according to $\mathbf{R}_-$.

6.3 Decomposition of the regular representation. Even though G_A has a very simple structure, the decomposition of the regular rep of G_A and expansion formulas for functions on G_A in terms of the matrix elements of irred reps are not trivial consequences of the general theory worked out in the earlier chapters. In particular, G_A is not unimodular, i.e., $d_r A \neq d_\ell A$. (The machinery developed in Chapter 4 for averaging over a group assumed that the group is unimodular.) Choosing the measure $d_r A$ to be definite and assuming that the functions $\boldsymbol{\varphi}_j(x) \in H_1^+$ are C^∞ with compact support in $(0, \infty)$ to avoid convergence problems, we can verify directly that

$$\int_0^\infty \int_{-\infty}^\infty \frac{dadb}{a} \langle \mathbf{R}_+[a,b]\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2\rangle_1 \overline{\langle \mathbf{R}_-[a,b]\boldsymbol{\varphi}_3, \boldsymbol{\varphi}_4\rangle_1} = 0, \tag{6.15}$$

in accordance with the earlier theory, but

$$\begin{aligned} &\int_0^\infty \int_{-\infty}^\infty \frac{dadb}{a} \langle \mathbf{R}_\pm[a,b]\varphi_1, \boldsymbol{\varphi}_2\rangle_1 \overline{\langle \mathbf{R}_\pm[a,b]\varphi_3, \varphi_4\rangle_1} \\ &\qquad = 2\pi \langle \boldsymbol{\varphi}_1, \boldsymbol{\varphi}_3\rangle_1 \langle \boldsymbol{\varphi}_4, \boldsymbol{\varphi}_2'\rangle_1 \end{aligned} \tag{6.16}$$

where $\boldsymbol{\varphi}_2'(t) = \boldsymbol{\varphi}_2(t)/t$. To investigate the problem in more detail we will decompose explicitly the right regular rep of G_A into irred reps of G_A, [V].

Recall that the right regular rep of G_A is defined on the Hilbert space H_R of measurable functions $f(A(a,b)) \equiv f(a,b)$, square integrable with respect to the measure $d_r A = dadb/a$. The inner product is

$$\langle f_1, f_2\rangle = \int_0^\infty \int_{-\infty}^\infty \frac{dadb}{a} f_1(a,b)\overline{f_2(a,b)}, \tag{6.17}$$

and G_A acts on this space in terms of the unitary operators $\mathbf{R}(A')$:

$$\mathbf{R}(A')f(A) = f(AA'). \tag{6.18}$$

To decompose H_R into irreducible components we project out subspaces of functions which transform irreducibly under the left action of the subgroup $H_2 = \{C(c) = A(c,0)\}$. (Since the left action of G_A commutes with the right action, it follows that these subspaces will be invariant under the operators $\mathbf{R}(A')$.) The function $\chi_\mu(C(c)) = c^{i\mu+1/2}$, $\mu \in R$, defines a one-dimensional rep of H_2. Now consider the map $f \to f^\mu$ where $f \in H_R$, given by

$$f^\mu(A) = \int_{H_2} f(CA)\overline{\chi_\mu(C)}dC \tag{6.19}$$

where $dC(c) = dc/c$ is the two-sided invariant measure on H_2. Note that f^μ satisfies $f^\mu(C'A) = (c')^{i\mu-1/2} f^\mu(A)$. In terms of coordinates we have

$$\begin{aligned} f^\mu(a,b) &= \int_0^\infty f(\tau a, \tau b)\tau^{-i\mu-1/2}d\tau, \\ f^\mu(ca,cb) &= c^{i\mu-1/2}f^\mu(a,b). \end{aligned} \tag{6.20}$$

Note also that the map $f \to f^\mu$ is invertible:

$$f(\tau a, \tau b) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^\mu(a,b)\tau^{i\mu - 1/2} d\mu.$$

(This is just the inverse formula for the Mellin transform, a variant of the Fourier transform, [EMOT2], [V].)

Now (6.20) agrees with (6.7) with $\sigma = i\mu - 1/2$, so the action of G_A on the functions f^μ induced by the operator $\mathbf{R}(A)$ is just $\mathbf{L}'_{i\mu-1/2}$, or in terms of the functions $\boldsymbol{\varphi}(t)$ where

$$f^\mu(a,b) = a^{i\mu-1/2}\boldsymbol{\varphi}^\mu\left(\frac{b}{a}\right), \tag{6.21}$$

it reads

$$(\mathbf{L}_\mu[a,b]\boldsymbol{\varphi}^\mu)(t) = a^{i\mu-1/2}\boldsymbol{\varphi}^\mu\left(\frac{t+b}{a}\right), \tag{6.22}$$

in agreement with (6.10). Furthermore, we have the decomposition formula

$$\int_0^\infty \int_{-\infty}^{\infty} \frac{da db}{a} f_1(a,b)\overline{f_2(a,b)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\mu \int_{-\infty}^{\infty} \varphi_1^\mu(t)\overline{\boldsymbol{\varphi}_2^\mu(t)}dt \tag{6.23}$$

where the $\boldsymbol{\varphi}_j^\mu \in L_2(R)$ are related to $f_j \in H_R$ via (6.20) and (6.21). On each space of square integrable functions $\boldsymbol{\varphi}^\mu$, $-\infty < \mu < \infty$, the rep $\mathbf{L}_\mu$ decomposes into the direct sum of the irred reps $\mathbf{R}_+$ and $\mathbf{R}_-$, (6.12) and (6.13). Thus the right regular rep $\mathbf{R}$ decomposes into a **direct integral** (rather than a direct sum) of a continuous number of copies of the irred reps $\mathbf{R}_+$ and $\mathbf{R}_-$. (The one-dimensional unitary reps $\chi_\rho[a,b] = a^{i\rho}$ do not appear in the decomposition of $\mathbf{R}$.)

As we have shown

$$f(a,b) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty} d\mu \int_{-\infty}^{\infty} dy\ e^{iby}|y|^{-i\mu-1/2} \times \{\varphi^{\mu^+}(ay)\chi(y) + \varphi^{\mu^-}(-ay)\chi(-y)\}dy \tag{6.24}$$

where

$$\chi(y) = \begin{cases} 1 & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}.$$

We can express these results in another form by choosing an explicit ON basis $\{\overline{S_n^\mp}(y)\}$ for $H_1^\pm$ in the reps $\mathbf{R}_\pm$. Then

$$\varphi^{\mu\pm}(y) = \sum_{n=0}^{\infty} \kappa_n^{\mu\pm}\overline{S_n^\mp}(y) \tag{6.25}$$

where

$$\kappa_n^{\mu\pm} = \int_0^\infty \varphi^{\mu\pm}(y)S_n^\mp(y)\frac{dy}{y}. \tag{6.26}$$

Substituting (6.25) and (6.26) into (6.24), using (6.20) and (6.14) to express the expansion in terms of $\{S_n^{\mp}\}$ and f alone, and making appropriate interchanges of integration and summation orders, we arrive at the formula

$$
\begin{aligned}
f(a,b) &= \frac{1}{2\pi}\sum_{n=0}^{\infty}(\langle F^-(f)t\circ S_n^-(t),\ \mathbf{R}_-[a,b]S_n^-(t)\rangle_1 \\
&+ \langle F^+(f)t\circ S_n^+(t), \mathbf{R}_+[a,b]S_n^+(t)\rangle_1) \\
&= \frac{1}{2\pi}\sum_{\pm}\mathrm{tr}(\mathbf{R}_{\pm}^*[a,b]F^{\pm}(f)\circ t).
\end{aligned} \tag{6.27}
$$

Here $[t\circ S_n](t) = tS_n(t)$,

$$\langle S, T\rangle_1 = \int_0^{\infty} S(t)\bar{T}(t)\frac{dt}{t}$$

and $F^{\pm}(f)$ is the operator

$$F^{\pm}(f) = \int_{A\in G_A} f(A)\mathbf{R}_{\pm}(A)d_{\ell}A, \quad d_{\ell}A = \frac{dadb}{a^2}. \tag{6.28}$$

(Since the bases $\{S_n^{\pm}\}$ are the same for all μ, dependence on this parameter disappears from the final result.)

Similarly, the Parseval formula

$$\int_0^{\infty}\int_{-\infty}^{\infty}\frac{dadb}{a}|f(a,b)|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}d\mu\sum_{n=0}^{\infty}(|\kappa_n^{\mu+}|^2 + |\kappa_n^{\mu-}|^2)$$

yields

$$\int_0^{\infty}\int_{-\infty}^{\infty}\frac{dadb}{a}|f(a,b)|^2 = \frac{1}{2\pi}\sum_{\pm}\mathrm{tr}([F^{\pm}(\delta f)\circ\sqrt{t}]^*[F^{\pm}(\delta f)\circ\sqrt{t}]) \tag{6.29}$$

where $\delta f(a,b) = a^{1/2}f(a,b)$. (The simple derivation of (6.27) and (6.29) given here is motivated by Vilenkin's treatment of the affine group, [V]. It is not our purpose here to give a rigorous derivation with precise convergence criteria. Rather, we want to demonstrate simply how group theory concepts lead to the correct expansion formulas. A rigorous, but much more complicated, derivation was given by Khalil [K4], and the relevance of these expansions to the radar ambiguity function was pointed out by Naparst [N3].)

The expansion (6.27) has been expressed in terms of the inner product on the Hilbert space H_1 and the operators $\mathbf{R}_{\pm}$. It is enlightening to re-express it in terms of the operators $\mathbf{L}_0$, (6.10) and the Hilbert space $L_2(R)$. For this we set $\{S_n^{\pm}(t) = \sqrt{t}S_n^{'\pm}(t)\}$ where $\{S_n^{'+}\}$ and $\{S_n^{'-}\}$ are each ON bases for $L_2(R)$. Now let

$$s_n^+(\tau) = \mathbf{F}^{-1}S_n^{'+}(\tau) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{it\tau}S_n^{'+}(t)dt$$

and

$$s_n^-(\tau) = \mathbf{F}^{-1}S_n^{'-}(\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{it\tau}S_n^{'-}(-t)dt,$$

i.e., $s_n^\pm$ are the inverse Fourier transforms of $S_n^{'\pm}$. (Recall from the discussion following (6.12) that $S_n^{'+}$ corresponds to a Fourier transform with support on the positive t axis and $S_n^{'-}$ corresponds to a Fourier transform with support on the negative t axis.) Further, let $\tilde{s}_n^\pm(\tau) = \mathbf{F}^{-1}[t \circ S_n^{'\pm}(t)](\tau)$ with the same support conventions as for $s_n^\pm$. Then we can write (6.27) in the form

$$f(a,b) = \sum_{n=0}^{\infty}(\langle F'(f)\tilde{s}_n^-,\ \mathbf{L}_0[a,b]s_n^-\rangle_2 + \langle F'(f)\tilde{s}_n^+,\ \mathbf{L}_0[a,b]s_n^+\rangle_2) \tag{6.30}$$

where $\langle\cdot,\cdot\rangle_2$ is the usual $L_2(R)$ inner product, and

$$\mathbf{L}_0[a,b]s_n^\pm(\tau) = a^{-1/2}s_n^\pm\left(\frac{\tau+b}{a}\right),$$

while

$$\begin{aligned} F'(f)\tilde{s}_n(\tau) &= \int_{G_A} f(a',b')\mathbf{L}_0[a',b']\tilde{s}_n(\tau)d_\ell A \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} f(a',b')\frac{1}{\sqrt{a'}}\tilde{s}_n\left(\frac{\tau+b'}{a'}\right)\frac{da'db'}{(a')^2} \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} D(x,y)\sqrt{y}\tilde{s}_n(y[\tau+x])dydx \\ &= \tilde{e}_n(\tau), \end{aligned} \tag{6.31}$$

where $D(x,y) \equiv f(a',b')$ with $y = 1/a', x = b'$. Comparing (6.31) with (2.13) we see that $F'(f)\tilde{s}_n(\tau) = \tilde{e}_n(\tau)$ is just the echo generated at time τ from the signal $\tilde{s}_n(\tau)$ and the target position-velocity distribution $D(x,y)$. Each inner product on the right-hand side of (6.30) is the correlation function between the echo $\tilde{e}_n^\pm$ and a test signal $\mathbf{L}_0[a,b]s_n^\pm$:

$$D(X,Y) = \sum_{n=0}^{\infty}(\langle\tilde{e}_n^-,\mathbf{L}_0[a,b]s_n^-\rangle_2 + \langle\tilde{e}_n^+,\mathbf{L}_0[a,b]s_n^+\rangle_2) \tag{6.32}$$

where $X = b, Y = a^{-1}$. Note that (6.32) provides a scheme for determining the distribution $D(X,Y)$ experimentally: we can send out signals $\tilde{s}_n^\pm$, measure the echos $\tilde{e}_n^\pm$ and then cross-correlate these echos with the test signals $\mathbf{L}_0[a,b]s_n^\pm$ to construct D.

6.4 Exercises.

6.1 Show that the representation $\mathbf{R}_0$ of the affine group G_A is reducible.

6.2 Verify directly equation (6.15).

6.3 Verify directly equation (6.16).

6.4 Show that the rep $\mathbf{R}_\lambda$ of G_A is *continuous* in the sense that

$$||\mathbf{R}_\lambda(a,b)\boldsymbol{\varphi} - \boldsymbol{\varphi}|| \to 0, \quad \text{as } (a,b)\to(1,0)$$

for each $\boldsymbol{\varphi} \in H_1$.

6.5 Show that the ambiguity and cross-ambiguity functions $< \mathbf{L}_0[a,b]\boldsymbol{\varphi},\boldsymbol{\psi} >$ are continuous functions of (a,b).

§7. WEYL-HEISENBERG FRAMES

7.1 Windowed Fourier transforms. In this and the next chapter we introduce and study two procedures for the analysis of time-dependent signals, locally in both frequency and time. The first procedure, the "windowed Fourier transform" is associated with the Heisenberg group while the second, the "wavelet transform" is associated with the affine group.

Let $g \in L_2(R)$ with $||g|| = 1$ and define the time-frequency translation of g by

$$g^{[x_1,x_2]}(t) = e^{2\pi i t x_2} g(t + x_1) = \mathbf{T}^1[x_1, x_2, 0]g(t) \tag{7.1}$$

where $\mathbf{T}^1$ is the unitary irred rep (5.5) of the Heisenberg group H_R with $\lambda = 1$. Now suppose g is centered about the point (t_0, ω_0) in phase (time-frequency) space, i.e., suppose

$$\int_{-\infty}^{\infty} t|g(t)|^2 dt = t_0, \qquad \int_{-\infty}^{\infty} \omega|\hat{g}(\omega)|^2 d\omega = \omega_0$$

where $\hat{g}(\omega) = \int_{-\infty}^{\infty} g(t)e^{-2\pi i\omega t}dt$ is the Fourier transform of $g(t)$. Then

$$\int_{-\infty}^{\infty} t|g^{[x_1,x_2]}(t)|^2 dt = t_0 - x_1, \qquad \int_{-\infty}^{\infty} \omega|\hat{g}^{[x_1,x_2]}(t)|^2 d\omega = \omega_0 + x_2$$

so $g^{[x_1,x_2]}$ is centered about $(t_0 - x_1, \omega_0 + x_2)$ in phase space. To analyze an arbitrary function $f(t)$ in $L_2(R)$ we compute the inner product

$$F(x_1, x_2) = \langle f, g^{[x_1,x_2]}\rangle = \int_{-\infty}^{\infty} f(t)\bar{g}^{[x_1,x_2]}(t)dt$$

with the idea that $F(x_1, x_2)$ is sampling the behavior of f in a neighborhood of the point $(t_0 - x_1, \omega_0 + x_2)$ in phase space. As x_1, x_2 range over all real numbers the samples $F(x_1, x_2)$ give us enough information to reconstruct $f(t)$. Indeed, since $\mathbf{T}^1$ is an irred rep of H_R the functions $\mathbf{T}^1[x_1, x_2, 0]g = g^{[x_1,x_2]}$ are dense in $L_2(R)$ as $[x_1, x_2]$ runs over R^2. Furthermore, $f \in L_2(R)$ is uniquely determined by the inner products $\langle f, g^{[x_1,x_2]}\rangle, -\infty < x_1, x_2 < \infty$. (Suppose $\langle f_1, g^{[x_1,x_2]}\rangle = \langle f_2, g^{[x_1,x_2]}\rangle$ for $f_1, f_2 \in L_2(R)$ and all x_1, x_2. Then with $f = f_1 - f_2$ we have $\langle f, g^{[x_1,x_2]}\rangle \equiv 0$, so f is orthogonal to the closed subspace of $L_2(R)$ generated by the $g^{[x_1,x_2]}$. Since $\mathbf{T}^1$ is irreducible this closed subspace is $L_2(R)$. Hence $f = 0$ and $f_1 = f_2$.)

However, the set of basis states $g^{[x_1,x_2]}$ is overcomplete: the coefficients $\langle f, g^{[x_1,x_2]}\rangle$ are not independent of one another, i.e., in general there is no $f \in L_2(R)$ such that $\langle f, g^{[x_1,x_2]}\rangle = F(x_1, x_2)$ for an arbitrary $F \in L_2(R^2)$. The $g^{[x_1,x_2]}$ are examples of **coherent states**, continuous overcomplete Hilbert space bases which are of interest in quantum optics, quantum field theory, group representation theory, etc., [KS].

As an important example we consider the case $g = \boldsymbol{\psi}_0(t) = \pi^{-1/4}e^{-t^2/2}$, (5.26), the ground state. (Recall that $\mathbf{a}\boldsymbol{\psi} = 0$ where $\mathbf{a}$ is the annihilation operator for bosons (5.18). This property uniquely determines the ground state.) Since $\boldsymbol{\psi}_0$ is essentially its own Fourier transform, (5.43), we see that $g = \boldsymbol{\psi}_0$ is centered about $(t_0, \omega_0) = (0, 0)$ in phase space. Thus

$$g^{[x_1,x_2]}(t) = \pi^{-1/4}e^{2\pi i t x_2}e^{-(t+x_1)^2/2} \tag{7.2}$$

is centered about $(-x_1, x_2)$. It is very instructive to map these vectors in $L_2(R)$ with inner product $\langle\cdot,\cdot\rangle$ to the Bargmann-Segal Hilbert space F with inner product $(\cdot,\cdot)$, (5.31), via the unitary operator $\mathbf{A}$, (5.36). In F the ground state is $\mathbf{j}_0(z) = 1$. Thus the corresponding coherent states are

$$\begin{aligned} \mathbf{A}g^{[x_1,x_2]}(z) &= \mathbf{T}^1[x_1,x_2]\,\mathbf{j}_0(z) = \mathbf{j}_0^{[x_1,x_2]}(z) \\ &= \exp\left[-(x_1^2+x_2^2)/4 + \frac{(x_1-ix_2)}{\sqrt{2}}z - \frac{ix_1x_2}{2}\right] \\ &= \exp[-(x_1^2+x_2^2)/4 - ix_1x_2/2]\mathbf{e}_{(x_1+ix_2)/\sqrt{2}}(z) \end{aligned} \tag{7.3}$$

where $\mathbf{e}_b(z) = \exp(\bar{b}z) \in F$ is the "delta function" with the property $(\mathbf{f}, \mathbf{e}_b) = \mathbf{f}(b)$ for each $\mathbf{f} \in F$. Clearly

$$\begin{aligned} \langle g^{[x_1,x_2]}, g^{[y_1,y_2]}\rangle &= (\mathbf{j}_0^{[x_1,x_2]}, \mathbf{j}_0^{[y_1,y_2]}) \\ &= \exp[-(x_1^2+x_2^2+y_1^2+y_2^2)/4 - ix_1x_2 + iy_1y_2]\times \\ &\quad \exp\left[\frac{(y_1+iy_2)(x_1-ix_2)}{2}\right] \end{aligned} \tag{7.4}$$

so the $g^{[x_1x_2]}$ are not mutually orthogonal. Moreover, given $f \in L_2(R)$ with $\mathbf{f} = \mathbf{A}f \in F$ we have

$$\langle f, g^{[x_1,x_2]}\rangle = (\mathbf{f}, \mathbf{j}_0^{[x_1,x_2]}) = \exp[-(x_1^2+x_2^2)/4 + ix_1x_2]\mathbf{f}\left(\frac{x_1+ix_2}{\sqrt{2}}\right). \tag{7.5}$$

Expression (7.5) displays clearly the overcompleteness of the coherent states. Since $\mathbf{f}$ is an entire function, it is uniquely determined by its values in an open set of the complex plane (or a line segment or even on a discrete set of points in $\mathcal{C}$ which have a limit point). Thus the values $\langle f, g^{[x_1x_2]}\rangle$ cannot be prescribed arbitrarily. However, from the "delta function" property

$$\mathbf{f}(b) = (\mathbf{f}, \mathbf{e}_b)$$

we can easily expand $\mathbf{f} \in F$ as a double integral over the coherent states $\mathbf{j}_0^{[x_1,x_2]}$, hence we can expand $f = \mathbf{A}^{-1}\mathbf{f} \in L_2(R)$ as the corresponding double integral over the coherent states $g^{[x_1,x_2]}$.

There are two features of the foregoing discussion that are worth special emphasis. First there is the great flexibility in the coherent function approach due to the fact that the function $g \in L_2(R)$ can be chosen to fit the problem at hand. Second is the fact that coherent states are always overcomplete. Thus it isn't necessary to compute the inner products $\langle f, g^{[x_1,x_2]}\rangle = F(x_1,x_2)$ for every point in phase space. In the windowed Fourier approach one typically samples F at the lattice points $(x_1,x_2) = (ma, nb)$ where a, b are fixed positive numbers and m, n range over the integers. Here, a, b and $g(t)$ must be chosen so that the map $f \to \{F(ma,nb)\}$ is one-to-one; then f can be recovered from the lattice point values $F(ma,nb)$.

7.2 The Weil-Brezin-Zak transform. The Weil-Brezin transform (earlier used in radar theory by Zak, so also called the Zak transform) (5.49) between the Schrödinger rep $\mathbf{T}^1$ of H_R on $L_2(R)$ and the lattice rep $\tilde{\mathbf{T}}$, (5.46)-(5.48), is very useful in studying the lattice sampling problem, particularly in the case $a = b = 1$. Restricting to this case for the time being, we let $\psi \in L_2(R)$. Then

$$\psi_{\mathbf{P}}(x_1, x_2) = \mathbf{P}\psi(x_1, x_2, 0) = \sum_{k=-\infty}^{\infty} e^{2\pi i k x_2}\psi(x_1 + k) \tag{7.6}$$

satisfies

$$\psi_{\mathbf{P}}(k_1 + x_1, k_2 + x_2) = e^{-2\pi i k_1 x_2}\psi_{\mathbf{P}}(x_1, x_2)$$

for integers k_1, k_2. (Here (7.6) is meaningful if ψ belongs to, say, the Schwartz class. Otherwise $\mathbf{P}\psi = \lim_{n\to s} \mathbf{P}\psi_n$ where $\psi = \lim_{n=s}\psi_n$ and the ψ_n are Schwartz class functions. The limit is taken with respect to the Hilbert space norm.) Furthermore

$$\begin{aligned}\left[\mathbf{T}^1[y_1, y_2, 0]\psi\right]_{\mathbf{P}}(x_1, x_2, 0) &= \tilde{\mathbf{T}}[y_1, y_2, 0]\psi_{\mathbf{P}}(x_1, x_2)\\ &= \exp[2\pi i x_1 y_2]\psi_{\mathbf{P}}(x_1 + y_1, x_2 + y_2).\end{aligned}$$

Hence if $\psi = g^{[m,n]} = \mathbf{T}^1[m, n]g$ we have

$$g_{\mathbf{P}}^{[m,n]}(x_1, x_2) = \exp[2\pi i(x_1 n - x_2 m)]g_{\mathbf{P}}(x_1, x_2). \tag{7.7}$$

Thus in the lattice rep, the functions $g_{\mathbf{P}}^{[m,n]}$ differ from $g_{\mathbf{P}}$ simply by the multiplicative factor $e^{2\pi i(x_1 n - x_2 m)} = \mathbf{E}_{n,m}(x_1, x_2)$, and as n, m range over the integers the $\mathbf{E}_{n,m}$ form an *ON* basis for the Hilbert space of the lattice rep:

$$(\varphi_1, \varphi_2) = \int_0^1\int_0^1 \varphi_1(x_1, x_2)\overline{\varphi_2(x_1, x_2)}dx_1 dx_2. \tag{7.8}$$

Theorem 7.1. *For* $(a, b) = (1, 1)$ *and* $g \in L_2(R)$ *the transforms* $\{g^{[m,n]} : m, n = 0 \pm 1, \pm 2, \cdots\}$ *span* $L_2(R)$ *if and only if* $\mathbf{P}g(x_1, x_2, 0) = g_{\mathbf{P}}(x_1, x_2) \neq 0$ a.e..

Proof. Let M be the closed linear subspace of $L_2(R)$ spanned by the $\{g^{[m,n]}\}$. Clearly $M = L_2(R)$ iff $f = 0$ a.e. is the only solution of $\langle f, g^{[m,n]}\rangle = 0$ for all integers m and n. Applying the Weyl-Brezin -Zak isomorphism $\mathbf{P}$ we have

$$\begin{aligned}\langle f, g^{[m,n]}\rangle &= (\mathbf{P}f, \mathbf{E}_{n,m}\mathbf{P}g)\\ &= ([\mathbf{P}f][\overline{\mathbf{P}g}], \mathbf{E}_{n,m}) = (f_{\mathbf{P}}\bar{g}_{\mathbf{P}}, \mathbf{E}_{n,m}).\end{aligned} \tag{7.9}$$

Since the functions $\mathbf{E}_{n,m}$ form an *ON* basis for the Hilbert space (7.8) it follows that $\langle f, g^{[m,n]}\rangle = 0$ for all integers m, n iff $f_{\mathbf{P}}(x_1, x_2)g_{\mathbf{P}}(x_1, x_2) = 0$, a.e.. If $g_{\mathbf{P}} \neq 0$, a.e. then $f_{\mathbf{P}} = f = 0$ and $M = L_2(R)$. If $g_{\mathbf{P}} = 0$ on a set S of positive measure on the unit square, then the characteristic function $\chi_S = \mathbf{P}f = f_{\mathbf{P}}$ satisfies $f_{\mathbf{P}}g_{\mathbf{P}} = \chi_S g_{\mathbf{P}} = 0$ a.e., hence $\langle f, g^{[m,n]}\rangle = 0$ and $M \neq L_2(R)$. □

In the case $g(t) = \pi^{-1/4} e^{-t^2/2}$ one finds that

$$g_{\mathbf{P}}(x_1, x_2) = \pi^{-1/4} \sum_{k=-\infty}^{\infty} e^{2\pi i k x_2 - (x_1+k)^2/2}. \tag{7.10}$$

As is well-known, [EMOT1], [WW], the series (7.10) defines a Jacobi Theta function. Using complex variable techniques it can be shown that this function vanishes at the single point $(\frac{1}{2}, \frac{1}{2})$ in the square $0 \le x_1 < 1$, $0 \le x_2 < 1$, [WW]. Thus $g_{\mathbf{P}} \neq 0$ a.e. and the functions $\{g^{[m,n]}\}$ span $L_2(R)$. (However, the expansion of an $L_2(R)$ function in terms of this set is not unique and the $\{g^{[m,n]}\}$ do not form a frame in the sense of §7.4.)

Corollary 7.1. *For $(a,b) = (1,1)$ and $g \in L_2(R)$ the transforms $\{g^{[m,n]} : m, n = 0, \pm 1, \cdots\}$ form an ON basis for $L_2(R)$ iff $|g_{\mathbf{P}}(x_1, x_2)| = 1$, a.e.*

Proof. We have

$$\begin{aligned} \delta_{mm'}\delta_{nn'} &= \langle g^{[m,n]}, g^{[m',n']} \rangle = (E_{n,m} g_{\mathbf{P}}, E_{n',m'} g_{\mathbf{P}}) \\ &= (|g_{\mathbf{P}}|^2, E_{n'-n,m'-m}) \end{aligned}$$

iff $|g_{\mathbf{P}}|^2 = 1$, a.e. □

As an example, let $g = \chi_{[0,1)}$ where

$$\chi_{[0,1)}(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Then it is easy to see that $|g_{\mathbf{P}}(x_1, x_2)| \equiv 1$. Thus $\{g^{[m,n]}\}$ is an ON basis for $L_2(R)$.

Theorem 7.2. *For $(a,b) = (1,1)$ and $g \in L_2(R)$, suppose there are constants A, B such that*

$$0 < A \le |g_{\mathbf{P}}(x_1, x_2)|^2 \le B < \infty$$

almost everywhere in the square $0 \le x_1, x_2 < 1$. Then $\{g^{[m,n]}\}$ is a basis for $L_2(R)$, i.e., each $f \in L_2(R)$ can be expanded **uniquely** *in the form $f = \sum_{m,n} a_{mn} g^{[m,n]}$. Indeed,*

$$a_{mn} = \left(f_{\mathbf{P}}, g_{\mathbf{P}}^{[m,n]} / |g_{\mathbf{P}}|^2 \right) = (f_{\mathbf{P}}/g_{\mathbf{P}}, E_{n,m})$$

Proof. By hypothesis $|g_{\mathbf{P}}|^{-1}$ is a bounded function on the domain $0 \le x_1, x_2 < 1$. Hence $f_{\mathbf{P}}/g_{\mathbf{P}}$ is square integrable on this domain and, from the periodicity properties of elements in the lattice Hilbert space, $\frac{f_{\mathbf{P}}}{g_{\mathbf{P}}}(x_1+n, x_2+m) = \frac{f_{\mathbf{P}}}{g_{\mathbf{P}}}(x_1, x_2)$. It follows that

$$\frac{f_{\mathbf{P}}}{g_{\mathbf{P}}} = \sum a_{mn} E_{n,m}$$

where $a_{mn} = (f_{\mathbf{P}}/g_{\mathbf{P}}, E_{n,m})$, so $f_{\mathbf{P}} = \sum a_{mn} E_{n,m} g_{\mathbf{P}}$. This last expression implies $f = \sum a_{mn} g^{[m,n]}$. Conversely, given $f = \sum a_{mn} g^{[m,n]}$ we can reverse the steps in the preceding argument to obtain $a_{mn} = (f_{\mathbf{P}}/g_{\mathbf{P}}, E_{n,m})$. □

7.3 Windowed transforms and ambiguity functions. We can relate these expansions to radar cross-ambiguity functions as follows. The expansion $f = \sum a_{mn} g^{[n,n]}$ is equivalent to the lattice Hilbert space expansion $f_{\mathbf{P}} = \sum a_{mn} E_{n,m} g_{\mathbf{P}}$ or

$$f_{\mathbf{P}} \bar{g}_{\mathbf{P}} = \sum (a_{mn} E_{n,m}) |g_{\mathbf{P}}|^2 \tag{7.11}$$

Now if $g_{\mathbf{P}}$ is a bounded function then $f_{\mathbf{P}} \bar{g}_{\mathbf{P}}(x_1, x_2)$ and $|g_{\mathbf{P}}|^2$ both belong to the lattice Hilbert space and are periodic functions in x_1 and x_2 with period 1. Hence,

$$f_{\mathbf{P}} \bar{g}_{\mathbf{P}} = \sum b_{mn} E_{n,m}$$
$$|g_{\mathbf{P}}|^2 = \sum c_{mn} E_{n,m}$$

with

$$b_{mn} = (f_{\mathbf{P}} \bar{g}_{\mathbf{P}}, E_{n,m}) = (f_{\mathbf{P}}, g_{\mathbf{P}} E_{n,m}) = \langle f, g^{[m,n]} \rangle = \langle f, \mathbf{T}^1[m,n] g \rangle,$$
$$c_{mn} = (g_{\mathbf{P}} \bar{g}_{\mathbf{P}}, E_{n,m}) = \langle g, g^{[m,n]} \rangle = \langle g, \mathbf{T}^1[m,n] g \rangle.$$

Thus (7.11) gives the Fourier series expansion for $f_{\mathbf{P}} \bar{g}_{\mathbf{P}}$ as the product of two other Fourier series expansions. (We consider the functions f, g, hence $f_{\mathbf{P}}, g_{\mathbf{P}}$ as known.) The Fourier coefficients in the expansions of $f_{\mathbf{P}} \bar{g}_{\mathbf{P}}$ and $|g_{\mathbf{P}}|^2$ are cross-ambiguity functions. If $|g_{\mathbf{P}}|^2$ never vanishes we can solve for the a_{mn} directly:

$$\sum a_{mn} E_{n,m} = (\sum b_{mn} E_{n,m})(\sum c'_{mn} E_{n,m}).$$

where the c'_{mn} are the Fourier coefficients of $|g_{\mathbf{P}}|^{-2}$. However, if $|g_{\mathbf{P}}|^2$ vanishes at some point then the best we can do is obtain the convolution equations $b = a * c$, i.e.,

$$b_{mn} = \sum_{\substack{k+k'=m \\ \ell+\ell'=n}} a_{kl} c_{\ell'_2 \ell'}. \tag{7.12}$$

(Auslander and Tolimieri [AT5] have shown how to approximate the coefficients $a_{k\ell}$ even in the cases where $|g_{\mathbf{P}}|^2$ vanishes at some points. The basic idea is to truncate $\sum a_{mn} E_{n,m}$ to a finite number of nonzero terms and to sample equation (7.11), making sure that $|g_{\mathbf{P}}|(x_1, x_2)$ is nonzero at each sample point. The a_{mn} can then be computed by using the inverse finite Fourier transform (3.37).)

The problem of $|g_{\mathbf{P}}|$ vanishing at a point is not confined to an isolated example, such as (7.10). Indeed it can be shown that if $g_{\mathbf{P}}$ is an everywhere continuous function in the lattice Hilbert space then it must vanish at at least one point, [HW].

7.4 Frames. To understand the nature of the complete sets $\{g^{[m,n]}\}$ it is useful to broaden our perspective and introduce the idea of a **frame** in an arbitrary Hilbert space H. In this more general point of view we are given a sequence $\{\mathbf{f}_n\}$ of elements of H and we want to find conditions on $\{\mathbf{f}_n\}$ so that we can recover an arbitrary

$\mathbf{f} \in H$ from the inner products $\langle \mathbf{f}, \mathbf{f}_n \rangle$ on H. Let $L_2(Z)$ be the Hilbert space of countable sequences $\{\xi_n\}$ with inner product $(\xi, \eta) = \sum_n \xi_n \bar{\eta}_n$. (A sequence $\{\xi_n\}$ belongs to $L_2(Z)$ provided $\sum_n \xi_n \bar{\xi}_n < \infty$.) Now let $\mathbf{T} : H \to L_2(Z)$ be the linear mapping defined by

$$(\mathbf{Tf})_n = \langle \mathbf{f}, \mathbf{f}_n \rangle. \tag{7.13}$$

We require that $\mathbf{T}$ is a bounded operator from H to $L_2(Z)$, i.e., that there is a finite $B > 0$ such that $\sum_n |\langle \mathbf{f}, \mathbf{f}_n \rangle|^2 \le B||\mathbf{f}||^2$. In order to recover $\mathbf{f}$ from the $\langle \mathbf{f}, \mathbf{f}_n \rangle$ we want $\mathbf{T}$ to be invertible with $\mathbf{T}^{-1} : R_{\mathbf{T}} \to H$ where $R_{\mathbf{T}}$ is the range $\mathbf{T}H$ of $\mathbf{T}$ in $L_2(Z)$. Moreover, for numerical stability in the computation of $\mathbf{f}$ from the $\langle \mathbf{f}, \mathbf{f}_n \rangle$ we want $\mathbf{T}^{-1}$ to be bounded. (In other words we want to require that a "small" change in the data $\langle \mathbf{f}, \mathbf{f}_n \rangle$ leads to a "small" change in $\mathbf{f}$.) This means that there is a finite $A > 0$ such that $\sum_n |\langle \mathbf{f}, \mathbf{f}_n \rangle|^2 \ge A||\mathbf{f}||^2$. (Note that $\mathbf{T}^{-1}\xi = \mathbf{f}$ if $\xi_n = \langle \mathbf{f}, \mathbf{f}_n \rangle$.) If these conditions are satisfied, i.e., if there exist positive constants A, B such that

$$A||\mathbf{f}||^2 \le \sum_n |\langle \mathbf{f}, \mathbf{f}_n \rangle|^2 \le B||\mathbf{f}||^2 \tag{7.14}$$

for all $\mathbf{f} \in H$, we say that the sequence $\{\mathbf{f}_n\}$ is a **frame** for H and that A and B are **frame bounds**.

The **adjoint** $\mathbf{T}^*$ of $\mathbf{T}$ is the linear mapping $\mathbf{T}^* : L_2(Z) \to H$ defined by

$$\langle \mathbf{T}^*\xi, \mathbf{f} \rangle = (\xi, \mathbf{Tf})$$

for all $\xi \in L_2(Z)$, $\mathbf{f} \in H$. A simple computation yields

$$\mathbf{T}^*\xi = \sum_n \xi_n \mathbf{f}_n. \tag{7.15}$$

(Since $\mathbf{T}$ is bounded, so is $\mathbf{T}^*$ and the right-hand side of (7.15) is well-defined for all $\xi \in L_2(Z)$.) Now the bounded self-adjoint operator $\mathbf{S} = \mathbf{T}^*\mathbf{T} : H \to H$ is given by

$$\mathbf{Sf} = \mathbf{T}^*\mathbf{Tf} = \sum_n \langle \mathbf{f}, \mathbf{f}_n \rangle \mathbf{f}_n, \tag{7.16}$$

and we can rewrite the defining inequality (7.14) for the frame as

$$A||\mathbf{f}||^2 \le \langle \mathbf{T}^*\mathbf{Tf}, \mathbf{f} \rangle \le B||\mathbf{f}||^2. \tag{7.17}$$

Since $A > 0$, if $\mathbf{T}^*\mathbf{Tf} = \boldsymbol{\theta}$ then $\mathbf{f} = \boldsymbol{\theta}$, so $\mathbf{S}$ is one-to-one, hence invertible. Furthermore, the range $\mathbf{S}H$ of $\mathbf{S}$ is H. Indeed, if $\mathbf{S}H$ is a proper subspace of H then we can find a nonzero vector $\mathbf{g}$ in $(\mathbf{S}H)^\perp$: $\langle \mathbf{Sf}, \mathbf{g} \rangle = 0$ for all $\mathbf{f} \in H$. However, $\langle \mathbf{Sf}, \mathbf{g} \rangle = \langle \mathbf{T}^*\mathbf{Tf}, \mathbf{g} \rangle = (\mathbf{Tf}, \mathbf{Tg}) = \sum_n \langle \mathbf{f}, \mathbf{f}_n \rangle \langle \mathbf{f}_n, \mathbf{g} \rangle$. Setting $\mathbf{f} = \mathbf{g}$ we obtain

$$\sum_n |\langle \mathbf{g}, \mathbf{f}_n \rangle|^2 = 0.$$

By (7.14) we have $\mathbf{g} = \boldsymbol{\theta}$, a contradiction. thus $\mathbf{S}H = H$ and the inverse operator $\mathbf{S}^{-1}$ exists and has domain H.

Since $\mathbf{S}\mathbf{S}^{-1}\mathbf{f} = \mathbf{S}^{-1}\mathbf{S}\mathbf{f} = \mathbf{f}$ for all $\mathbf{f} \in H$, we immediately obtain two expansions for $\mathbf{f}$ from (7.16):

$$(7.18) \qquad \begin{aligned} &a)\ \mathbf{f} = \sum_n \langle \mathbf{S}^{-1}\mathbf{f}, \mathbf{f}_n\rangle \mathbf{f}_n = \sum_n \langle \mathbf{f}, \mathbf{S}^{-1}\mathbf{f}_n\rangle \mathbf{f}_n \\ &b)\ \mathbf{f} = \sum_n \langle \mathbf{f}, \mathbf{f}_n\rangle \mathbf{S}^{-1}\mathbf{f}_n. \end{aligned}$$

(The second equality in (7.18a) follows from the identity $\langle \mathbf{S}^{-1}\mathbf{f}, \mathbf{f}_n\rangle = \langle \mathbf{f}, \mathbf{S}^{-1}\mathbf{f}_n\rangle$, which holds since $\mathbf{S}^{-1}$ is self-adjoint.)

Recall that for a **positive** operator $\mathbf{S}$, i.e., an operator such that $\langle \mathbf{S}\mathbf{f}, \mathbf{f}\rangle \geq 0$ for all $\mathbf{f} \in H$ the inequalities

$$A||\mathbf{f}||^2 \leq \langle \mathbf{S}\mathbf{f}, \mathbf{f}\rangle \leq B||\mathbf{f}||^2$$

for $A, B > 0$ are equivalent to the inequalities

$$(7.19) \qquad A||\mathbf{f}|| \leq ||\mathbf{S}\mathbf{f}|| \leq B||\mathbf{f}||,$$

see [RN] or [DS2].

An examination of (7.18a) and (7.18b) suggests that if the $\{\mathbf{f}_n\}$ form a frame then so do the $\{\mathbf{S}^{-1}\mathbf{f}_n\}$.

Theorem 7.3. *Suppose $\{\mathbf{f}_n\}$ is a frame with frame bounds A, B and let $\mathbf{S} = \mathbf{T}^*\mathbf{T}$. Then $\{\mathbf{S}^{-1}\mathbf{f}_n\}$ is also a frame, called the* **dual frame** *of $\{\mathbf{f}_n\}$, with frame bounds B^{-1}, A^{-1}.*

Proof. Setting $\mathbf{f} = \mathbf{S}^{-1}\mathbf{g}$ in (7.19) we have $B^{-1}||\mathbf{g}|| \leq ||\mathbf{S}^{-1}\mathbf{g}|| \leq A^{-1}||\mathbf{g}||$. Since $\mathbf{S}^{-1}$ is self-adjoint, this implies $B^{-1}||\mathbf{g}||^2 \leq \langle \mathbf{S}^{-1}\mathbf{g}, \mathbf{g}\rangle \leq A^{-1}||\mathbf{g}||^2$. From (7.18b) we have $\mathbf{S}^{-1}\mathbf{g} = \sum_n \langle \mathbf{S}^{-1}\mathbf{g}, \mathbf{f}_n\rangle \mathbf{S}^{-1}\mathbf{f}_n$ so $\langle \mathbf{S}^{-1}\mathbf{g}, \mathbf{g}\rangle = \sum_n \langle \mathbf{S}^{-1}\mathbf{g}, \mathbf{f}_n\rangle \langle \mathbf{S}^{-1}\mathbf{f}_n, \mathbf{g}\rangle = \sum_n |\langle \mathbf{g}, \mathbf{S}^{-1}\mathbf{f}_n\rangle|^2$. Hence $\{\mathbf{S}^{-1}\mathbf{f}_n\}$ is a frame with frame bounds B^{-1}, A^{-1}. □

We say that $\{\mathbf{f}_n\}$ is a **tight frame** if $A = B$.

Corollary 7.2. *If $\{\mathbf{f}_n\}$ is a tight frame then every $\mathbf{f} \in H$ can be expanded in the form*

$$\mathbf{f} = A^{-1} \sum_n \langle \mathbf{f}, \mathbf{f}_n\rangle \mathbf{f}_n.$$

Proof. Since $\{\mathbf{f}_n\}$ is a tight frame we have $A||\mathbf{f}||^2 = \langle \mathbf{S}\mathbf{f}, \mathbf{f}\rangle$ or $\langle (\mathbf{S} - A\mathbf{E})\mathbf{f}, \mathbf{f}\rangle = 0$ where $\mathbf{E}$ is the identity operator $\mathbf{E}\mathbf{f} = \mathbf{f}$. Since $\mathbf{S} - A\mathbf{E}$ is a self-adjoint operator we have $||(\mathbf{S} - A\mathbf{E})\mathbf{f}|| = 0$ for all $\mathbf{f} \in H$. Thus $\mathbf{S} = A\mathbf{E}$. However, from (7.18), $\mathbf{S}\mathbf{f} = \sum_n \langle \mathbf{f}, \mathbf{f}_n\rangle \mathbf{f}_n$. □

7.5 Frames of $W - H$ type. We can now relate frames with the Heisenberg group lattice construction (7.6).

Theorem 7.4. *For $(a,b)=(1,1)$ and $g\in L_2(R)$, we have*

$$0<A\le|g_{\mathbf{P}}(x_1,x_2)|^2\le B<\infty \tag{7.20}$$

almost everywhere in the square $0\le x_1,x_2<1$ iff $\{g^{[m,n]}\}$ is a frame for $L^2(R)$ with frame bounds A,B. (By Theorem 7.2 this frame is actually a basis for $L_2(R)$.).

Proof. If (7.20) holds then $g_{\mathbf{P}}$ is a bounded function on the square. Hence for any $f\in L_2(R)$, $f_{\mathbf{P}}\bar{g}_{\mathbf{P}}$ is a periodic function, in x_1,x_2 on the square. Thus

$$\begin{aligned}\sum_{m,n=-\infty}^{\infty}|\langle f,g^{[m,n]}\rangle|^2 &= \sum_{m,n=-\infty}^{\infty}|(f_{\mathbf{P}},E_{n,m}g_{\mathbf{P}})|^2\\ &=\sum_{m,n=-\infty}^{\infty}|(f_{\mathbf{P}}\bar{g}_{\mathbf{P}},E_{n,m})|^2=||f_{\mathbf{P}}\bar{g}_{\mathbf{P}}||^2\\ &=\int_0^1\int_0^1|f_{\mathbf{P}}|^2|g_{\mathbf{P}}|^2dx_1dx_2.\end{aligned} \tag{7.21}$$

(Here we have used the Plancherel theorem for the exponentials $E_{n,m}$) It follows from (7.20) that

$$A||f||^2\le\sum_{m,n=-\infty}^{\infty}|\langle f,g^{[m,n]}\rangle|^2\le B||f||^2, \tag{7.22}$$

so $\{g^{[m,n]}\}$ is a frame.

Conversely, if $\{g^{[m,n]}\}$ is a frame with frame bounds A,B, it follows from (7.22) and the computation (7.21) that

$$A||f_{\mathbf{P}}||^2\le\int_0^1\int_0^1|f_{\mathbf{P}}|^2|g_{\mathbf{P}}|^2dx_1dx_2\le B||f_{\mathbf{P}}||^2$$

for an **arbitrary** $f_{\mathbf{P}}$ in the lattice Hilbert space. (Here we have used the fact that $||f||=||f_{\mathbf{P}}||$, since $\mathbf{P}$ is a unitary transformation.) Thus the inequalities (7.20) hold almost everywhere. □

Frames of the form $\{g^{[ma,nb]}\}$ are called **Weyl-Heisenberg** (or **W-H**) **frames**. The Weyl-Brezin-Zak transform is not so useful for the study of W-H frames with general frame parameters (a,b). (Note from (7.1) that it is only the product ab that is of significance for the W-H frame parameters. Indeed, the change of variable $t'=t/a$ in (7.1) converts the frame parameters (a,b) to $(a',b')=(1,ab)$.) An easy consequence of the general definition of frames is the following:

Theorem 7.5. *Let $g\in L_2(R)$ and $a,b,A,B>0$ such that*

1) $0<A\le\sum_m|g(x+ma)|^2\le B<\infty$, a.e.,
2) *g has support contained in an interval I where I has length b^{-1}.*

Then the $\{g^{[ma,nb]}\}$ are a W-H frame for $L_2(R)$ with frame bounds $b^{-1}A, b^{-1}B$.

Proof. For fixed m and arbitrary $f \in L_2(R)$ the function $F_m(t) = f(t)\overline{g(t+ma)}$ has support in the interval $I_m = \{t+ma : x \in I\}$ of length b^{-1}. Thus $F_m(t)$ can be expanded in a Fourier series with respect to the basis exponentials $E_{nb}(t) = e^{2\pi ibnt}$ on I_m. Using the Plancherel formula for this expansion we have

$$\begin{aligned}\sum_{m,n} |\langle f, g^{[ma,nb]}\rangle|^2 &= \sum_{m,n} |\langle F_m, E_{nb}\rangle|^2 \\ &= \frac{1}{b}\sum_m |\langle F_m, F_m\rangle| = \frac{1}{b}\sum_m \int_{I_m} |f(t)|^2 |g(t+ma)|^2 dt \\ &= \frac{1}{b}\int_{-\infty}^{\infty} |f(t)|^2 \sum_m |g(t+ma)|^2 dt.\end{aligned}$$

From property 1) we have then

$$\frac{A}{b}||f||^2 \leq \sum_{m,n} |\langle f, g^{[ma,nb]}\rangle|^2 \leq \frac{B}{b}||f||^2,$$

so $\{g^{[ma,nb]}\}$ is a W-H frame. □

There are no W-H frames with frame parameters (a,b) such that $ab > 1$, [BBGK], [R3]. For some insight into this case we consider the example $(a,b) = (N,1), N > 1$, N an integer. Let $g \in L_2(R)$. There are two distinct possibilities:

1) There is a constant $A > 0$ such that $A \leq |g_{\mathbf{P}}(x_1, x_2)|$ almost everywhere.
2) There is no such $A > 0$.

Let M be the closed subspace of $L_2(R)$ spanned by the functions $\{g^{[mN,n]}, m,n = 0 \pm 1, \pm 2, \cdots\}$ and suppose $f \in L_2(R)$. Then

$$\langle f, g^{[mN,n]}\rangle = (f_{\mathbf{P}}, E_{n,mN} g_{\mathbf{P}}) = (f_{\mathbf{P}} \bar{g}_{\mathbf{P}}, E_{n,mN}).$$

If possibility 1) holds, we set $f_{\mathbf{P}} = \bar{g}_{\mathbf{P}}^{-1} E_{n_0,1}$. Then $f_{\mathbf{P}}$ belongs to the lattice Hilbert space and $0 = (E_{n_0,1}, E_{n,mN}) = (f_{\mathbf{P}} \bar{g}_{\mathbf{P}} E_{n,mN}) = \langle f, g^{[mN,n]}\rangle$ so $f \in M^{\perp}$ and $\{g^{[mN,n]}\}$ is not a frame. Now suppose possibility 2) holds. Then according to the proof of Theorem 7.4, g cannot generate a frame $\{g^{[m,n]}\}$ with frame parameters $(1,1)$ because there is no $A > 0$ such that $A||f||^2 < \sum_{m,n} |\langle f, g^{[m,n]}\rangle|^2$. Since the $\{g^{[mN,n]}\}$ corresponding to frame parameters $(1,N)$ is a proper subset of $\{g^{[m,n]}\}$, it follows that $\{g^{[mN,n]}\}$ cannot be a frame either.

For frame parameters (a,b) with $0 < ab < 1$ it is not difficult to construct W-H frames $\{g^{[ma,nb]}\}$ such that $g \in L_2(R)$ is a smooth function [DGM], [H3], [HW]. Taking the case $a = 1, b = \frac{1}{2}$, for example, let v be an infinitely differentiable function on R such that

$$v(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x \geq 1 \end{cases}$$

and $0 < v(x) < 1$ if $0 < x < 1$. Set

$$g(x) = \begin{cases} 0, & x \leq 0 \\ v(x), & 0 < x < 1 \\ [1 - v^2(x-1)]^{\frac{1}{2}}, & 1 \leq x \leq 2 \\ 0, & 2 < x \end{cases} \tag{7.23}$$

Then $g \in L_2(R)$ is infinitely differentiable and with support contained in the interval $[0, 2]$. Moreover, $||g||^2 = 1$ and $\sum_n |g(x+m)|^2 \equiv 1$. It follows immediately from Theorem 7.5 that $\{g^{[m,n/2]}\}$ is a W-H frame with frame bounds $A = B = 2$.

We conclude this section by deriving some identities related to cross-ambiguity functions evaluated on lattices of the Heisenberg group.

Theorem 7.6 [S1], [S2]. *Let $f, g \in L_2(R)$ such that $|f_{\mathbf{P}}(x_1, x_2)|$ $|g_{\mathbf{P}}(x_1, x_2)|$ are bounded almost everywhere. Then*

$$\sum_{m,n} |\langle f, g^{[m,n]}\rangle|^2 = \sum_{m,n} \langle f, f^{[m,n]}\rangle \langle g^{[m,n]}, g\rangle.$$

Proof. Since $\langle f, g^{[m,n]}\rangle = (f_{\mathbf{P}}, E_{n,m} g_{\mathbf{P}}) = (f_{\mathbf{P}}\bar{g}_{\mathbf{P}}, E_{n,m})$ we have the Fourier series expansion

$$f_P(x_1, x_2)\overline{g_P(x_1, x_2)} = \sum_{m,n} \langle f, g^{[m,n]}\rangle E_{n,m}(x_1, x_2). \tag{7.24}$$

Since $|f_P|, |g_P|$ are bounded, $f_P \bar{g}_P$ is square integrable with respect to the measure $dx_1 dx_2$ on the square $0 \leq x_1, x_2 < 1$. From the Plancherel formula for double Fourier series, we obtain the identity

$$\int_0^1 \int_0^1 |f_{\mathbf{P}}|^2 |g_{\mathbf{P}}|^2 dx_1 dx_2 = \sum_{m,n} |\langle f, g^{[m,n]}\rangle|^2. \tag{7.24}$$

Similarly, we can obtain expansions of the form (7.24) for $f_{\mathbf{P}}\bar{f}_{\mathbf{P}}$ and $g_{\mathbf{P}}\bar{g}_{\mathbf{P}}$. Applying the Plancherel formula to these two functions we find

$$\int_0^1 \int_0^1 |f_{\mathbf{P}}|^2 g_{\mathbf{P}}|^2 dx_1 dx_2 = \sum_{m,n} \langle f, f^{[m,n]}\rangle \langle g^{[m,n]}, g\rangle. \tag{7.25}$$

□

7.6 Exercises.

7.1 Verify that if $g \in L_2(R)$, $||g|| = 1$ and g is centered about (t_0, ω_0) in phase space, then $g^{[x_1,x_2]}$ is centered about $(t_0 - x_1, \omega_0 + x_2)$.

7.2 Given the function

$$g(t) = \begin{cases} 1, & |t| \leq \frac{1}{2} \\ 0, & |t| \geq \frac{1}{2}, \end{cases}$$

show that the set $\{g^{[m,n]}\}$ is an ON basis for $L_2(R)$.

§8. AFFINE FRAMES AND WAVELETS

8.1 Wavelets. Here we work out the analog for the affine group of the Weyl-Heisenberg frame for the Heisenberg group. Let $g \in L_2(R)$ with $||g|| = 1$ and define the affine translation of g by

$$g^{(a,b)}(t) = a^{-1/2} g\left(\frac{t+b}{a}\right) = \mathbf{L}_0[a,b]g(t) \tag{8.1}$$

where $a > 0$ and $\mathbf{L}_0$ is the unitary rep (6.10) of the affine group. Recall that $\mathbf{L}_0 \approx \mathbf{R}_+ + \mathbf{R}_-$ is reducible. Indeed $L_2(R) = H^+ \oplus H^-$ where H^+ consists of the functions f_+ such that the Fourier transform $\mathbf{F}f_+(y)$ has support on the positive y-axis and the functions f_- in H^- have Fourier transform with support on the negative y-axis. Thus the functions $\{g^{(a,b)}\}$ will not necessarily span $L_2(R)$. However, if we choose two functions $g_\pm \in H^\pm$ with $||g_\pm|| = 1$ then the functions $\{g_+^{(a,b)}, g_-^{(a,b)} : a > 0\}$ **will** span $L_2(R)$. By translation in t if necessary, we can assume that $\int_{-\infty}^{\infty} t|g_\pm(t)|^2 dt = 0$. Let $k_+ = \int_0^\infty y|\mathbf{F}g_+(y)|^2 dy$, $k_- = \int_{-\infty}^0 y|\mathbf{F}g_-(y)|^2 dy$. Then $g_\pm$ are centered about the origin in position space and about $k_\pm$ in momentum space. It follows that

$$\int_{-\infty}^{\infty} t|g_\pm^{(a,b)}(t)|^2 dt = -b, \quad \pm\int_0^\infty y|Fg_\pm^{(a,b)}(\pm y)|^2 dy = a^{-1}k_\pm.$$

To define a lattice in the affine group space we choose two nonzero real numbers $a_0, b_0 > 0$ with $a_0 \neq 1$. Then the lattice points are $a = a_0^m, b = nb_0a_0^m$, $m, n = 0, \pm 1, \cdots$, so

$$g^{mn}(t) = g^{(a_0^m, nb_0a_0^m)}(t) = a_0^{-m/2} g(a_0^{-m}t + nb_0). \tag{8.2}$$

Thus $g_\pm^{mn}$ is centered about $-nb_0a_0^m$ in position space and about $a_0^{-m}k_\pm$ in momentum space. Note that if g has support contained in an interval of length ℓ then the support of g^{mn} is contained in an interval of length $a_0^{-m}\ell$. Similarly, if $\mathbf{F}g$ has support contained in an interval of length L then the support of $\mathbf{F}g^{mn}$ is contained in an interval of length $a_0^m L$. (Note that this behavior is very different from the behavior of the Heisenberg translates $g^{[ma,nb]}$. In the Heisenberg case the support of g in either position or momentum space is the same as the support of $g^{[ma,nb]}$. In the affine case the sampling of position-momentum space is on a logarithmic scale. There is the possibility, through the choice of m and n, of sampling in smaller and smaller neighborhoods of a fixed point in position space, [C], [D4].)

The affine translates $g_\pm^{(a,b)}$ are called **wavelets** and each of the functions $g_\pm$ is a **mother wavelet**. The map $\mathbf{T} : f \to \langle f, g_\pm^{mn}\rangle$ is the **wavelet transform**

8.2 Affine frames. The general definitions and analysis of frames presented in Chapter 7 clearly apply to wavelets. However, there is no affine analog of the Weil-Brezin-Zak transform which was so useful for Weyl-Heisenberg frames. Nonetheless we can prove the following result directly.

Lemma 8.1 [DGM]. *Let $g \in L_2(R)$ such that the support of $\mathbf{F}g$ is contained in the interval $[\ell, L]$ where $0 < \ell < L < \infty$, and let $a_0 > 1, b_0 > 0$ with $(L - \ell)b_0 \leq 1$. Suppose also that*

$$0 < A \leq \sum_m |\mathbf{F}g(a_0^m y)|^2 \leq B < \infty$$

for almost all $y \geq 0$. Then $\{g^{mn}\}$ is a frame for H^+ with frame bounds $A/b_0, B/b_0$.

Proof. The demonstration is analogous to that of Theorem 7.5. Let $f \in H^+$ and note that $g \in H^+$. For fixed m the support of $\mathbf{F}f(a_0^m y)\overline{Fg}(y)$ is contained in the interval $\ell \leq y \leq \ell + 1/b_0$ (of length $1/b_0$). Then

$$\begin{aligned}\sum_{m,n} |\langle f, g^{mn}\rangle|^2 &= \sum_{m,n} |\langle \mathbf{F}f, Fg^{mn}\rangle|^2 \\ &= \sum_{m,n} a_0^{-m} \left| \int_{-0}^{\infty} \mathbf{F}f(a_0^{-m}y)\overline{\mathbf{F}g}(y)e^{-inb_0 y}dy\right|^2 \\ &= (\text{ Plancherel theorem }) \sum_m \frac{a_0^{-m}}{b_0} \int_{\ell}^{\ell+1/b_0} |\mathbf{F}f(a_0^{-m}y)\mathbf{F}g(y)|^2 dy \\ &= \frac{1}{b}\sum_m \int_0^\infty |\mathbf{F}f(y)\mathbf{F}g(a_0^m y)|^2 dy \\ &= \frac{1}{b}\int_0^\infty |\mathbf{F}f(y)|^2 \left(\sum_m |\mathbf{F}g(a_0^m y)|^2\right) dy.\end{aligned}$$

Since $||f||^2 = \int_0^\infty |\mathbf{F}f(y)|^2 dy$ for $f \in H^+$, the result

$$A||f||^2 \leq \sum_{m,n} |\langle f, g^{mn}\rangle|^2 \leq B||f||^2$$

follows. □

A very similar result characterizes a frame for H^-. (Just let y run from $-\infty$ to 0.) Furthermore, if $\{g_+^{mn}\}, \{g_-^{mn}\}$ are frames for H^+, H^-, respectively, corresponding to lattice parameters a_0, b_0, then $\{g_+^{mn}, g_-^{mn}\}$ is a frame for $L_2(R)$

Example 1. For lattice parameters $a_0 = 2, b_0 = 1$, choose $g_+ = \chi_{[1,2)}$ and $g_- = \chi_{(-2,-1]}$. Then g_+ generates a tight frame for H^+ with $A = B = 1$ and g_- generates a tight frame for H^- with $A = B = 1$. Thus $\{g_+^{mn}, g_-^{mn}\}$ is a tight frame for $L_2(R)$. (Indeed, one can verify directly that $\{g_\pm^{mn}\}$ is an ON basis for $L_2(R)$.

Example 2. Let g be the function such that

$$\mathbf{F}g(y) = \frac{1}{\sqrt{\ln a}} \begin{cases} 0 & \text{if } y \leq \ell \\ \sin \frac{\pi}{2} v\left(\frac{y-\ell}{\ell(a-1)}\right) & \text{if } \ell < y \leq a\ell \\ \cos \frac{\pi}{2} v\left(\frac{y-a\ell}{a\ell(a-1)}\right) & \text{if } a\ell < y \leq a^2\ell \\ 0 & \text{if } a^2\ell < y \end{cases}$$

where $v(x)$ is defined as in (7.23). Then $\{g^{mn}\}$ is a tight frame for H^+ with $A = B = \frac{1}{b \ln a}$. Furthermore, if $g_+ = g$ and $g_- = \bar{g}$ then $\{g_\pm^{mn}\}$ is a tight frame for $L_2(R)$.

Suppose $g \in L_2(R)$ such that $\mathbf{F}g(y)$ is bounded almost everywhere and has support in the interval $\left[-\frac{1}{2b}, \frac{1}{2b}\right]$. Then for any $f \in L_2(R)$ the function

$$a_0^{-m/2}\mathbf{F}f(a_0^{-m}y)\overline{\mathbf{F}g(y)}$$

has support in this same interval and is square integrable. Thus

$$\sum_{m,n} |\langle f, g_{mn}\rangle|^2 = \sum_{m,n} |a_0^{-m/2} \int_{-\infty}^{\infty} \mathbf{F}f(a_0^{-m}y)\overline{\mathbf{F}g(y)}e^{-2\pi i y b_0}dy|^2$$

$$= b_0^{-1}\sum_m \int_{-\infty}^{\infty} a_0^{-m}|\mathbf{F}f(a_0^{-m}y)\mathbf{F}g(y)|^2 dy$$

$$= \frac{1}{b_0}\int_{-\infty}^{0} |\mathbf{F}f(y)|^2 \sum_m |\mathbf{F}g(a_0^m y)|^2 dy$$

$$+ \frac{1}{b_0}\int_{0}^{\infty} |\mathbf{F}f(y)|^2 \sum_m |\mathbf{F}g(a_0^m y)|^2 dy.$$

It follows from the computation that if there exist constants $A, B > 0$ such that

$$A \le \sum_m |\mathbf{F}g(a_0^m y)|^2 \le B$$

for almost all y, then the single mother wavelet g generates an affine frame.

We conclude this section with two examples of wavelets whose properties do not follow directly from the preceding theory. The first is the Haar basis generated by the mother wavelet

$$g(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } 0 \le t < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \le t \le 1 \\ 0 & \text{if } 1 < t, \end{cases}$$

where $a = 2, b = 1$. One can check directly that $\{g^{mn}\}$ is not only a frame, it is an ON basis for $L_2(R)$.

The Haar wavelets have discontinuities. However, Y. Meyer discovered an ON basis for $L_2(R)$ whose mother wavelet g is an infinitely differential function such that $\mathbf{F}g$ has compact support. The lattice is $a = 2, b = 1$. The Meyer wavelet is defined by $\mathbf{F}g(y) = e^{iy/2}\omega(|y|)$ where

$$\omega(|y|) = \begin{cases} 0 & \text{if } y \le \frac{1}{3} \\ \sin\frac{\pi}{2}v(3y-1) & \text{if } \frac{1}{3} \le y \le \frac{2}{3} \\ \cos\frac{\pi}{2}v\left(\frac{3y}{2}-1\right) & \text{if } \frac{2}{3} \le y \le \frac{4}{3} \\ 0 & \text{if } \frac{4}{3} \le y \end{cases}$$

and v is defined as in (7.23), except that in addition we require $v(y)+v(1-y) = 1$ for $0 \le y \le 1$. One can check that $||g||^2 = 1$ and $\sum_m |\mathbf{F}g(2^m y)|^2 = 1$. Moreover, it can be shown that g generates a tight frame for $L_2(R)$ with frame bounds $A = B = 1$, and, indeed, that $\{g^{mn}\}$ is an ON basis for $L_2(R)$.

A theory which "explains" the orthogonality found in these last two examples is multiresolution analysis [HW], [LM], [D2]; it is beyond the scope of these notes.

8.3 Exercises.

8.1 Suppose $g \in L_2(R)$ with $||g|| = 1$ and g is centered about $(0, k)$ in the position-momentum space. Show that $g^{(a,b)}$ is centered about $(-b, a^{-1}k)$.

8.2 Prove directly that the Haar basis is an ON basis for $L_2(R)$.

8.3 For $g(t) = e^{-t^2}$, show that the functions $g^{(a,b)}$ are dense in $L_2(R)$.

§9. THE SCHRÖDINGER GROUP

9.1 Automorphisms of H_R. We have already seen that the infinite-dimensional irred unitary reps $\mathbf{T}^\lambda$ of the Heisenberg group H_R extend naturally to irred reps of the four-parameter oscillator group and that a study of the oscillator group reps provides insight into the behavior of the H_R reps, §5.7. In fact, the H_R reps extend to reps of the six-parameter Schrödinger group. An understanding of the action of the Schrödinger group provides an explanation for a number of the "deep" transformation properties of objects such as radar ambiguity functions and Jacobi Theta functions.

We start by searching for **automorphisms** of H_R, i.e. one-to-one maps ρ of H_R onto itself such that $\rho(AB) = \rho(A)\rho(B)$ for $A, B \in H_R$. Using the usual coordinate representation

$$A(x,y,z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \tag{9.1}$$

for H_R, so that the group product is

$$A(x,y,z)A(x',y',z') = A(x+x', y+y', z+z'+xy'),$$

we can write

$$\rho(A)(x,y,z) = A(\rho_1(x,y,z), \rho_2(x,y,z), \rho_3(x,y,z)) \tag{9.2}$$

where

$$\begin{aligned} a)\ & \rho_1(x+x', y+y', z+z'+xy') = \rho_1(x,y,z) + \rho_1(x',y',z') \\ b)\ & \rho_2(x+x', y+y', z+x'+xy') = \rho_2(x,y,z) + \rho_2(x',y',z') \\ c)\ & \rho_3(x+x', y+y', z+z'+xy') = \rho_3(x,y,z) + \rho_3(x',y',z') \\ & \qquad + \rho_1(x,y,z)\rho_2(z',y',z'). \end{aligned} \tag{9.3}$$

Under the assumption that the ρ_j are continuously differentiable functions, we shall determine all such automorphisms.

Before proceeding with this task, let us see why it could be relevant to radar and sonar. The radar cross-ambiguity function takes the form $F(x,y,z) = \langle \mathbf{T}^1[x,y,z]f, g\rangle$ (up to a harmless exponential factor arising from the z coordinate) where $f, g \in L_2(R)$ and $\mathbf{T}^1[x,y,z] = \mathbf{T}^1(A(x,y,z))$ is the irred rep (5.5) of H_R. If ρ is an automorphism of H_R then $\mathbf{T}^1(\rho(A)(x,y,z)) = \mathbf{T}^1_\rho[x,y,z]$ also defines an irred unitary rep of H_R on $L_2(R)$. (A rep since ρ preserves the group multiplication property and irreducible since the operators $\mathbf{T}^1_\rho$ are just a reordering of the operators $\mathbf{T}^1$.) Thus $\mathbf{T}^1_\rho$ is equivalent, hence unitary equivalent, to one of the standard unitary irred reps $\mathbf{T}^\lambda$ of H_R that we have already studied. Suppose this rep is $\mathbf{T}^1$ itself, i.e., suppose there is a unitary operator $\mathbf{U}$ such that $\mathbf{T}^1_\rho[x,y,z] = \mathbf{U}^{-1}\mathbf{T}^1[x,y,z]\mathbf{U}$. Then $\langle \mathbf{T}^1(\rho(A))f, g\rangle = \langle \mathbf{T}^1_\rho[x,y,z]f, g\rangle = \langle \mathbf{U}^{-1}\mathbf{T}^1[x,y,z]\mathbf{U}f, g\rangle = \langle \mathbf{T}^1[x,y,z]\mathbf{U}f, \mathbf{U}g\rangle = F'(x,y,z)$, which is again an ambiguity function. Thus if $F(x,y,z)$ is an ambiguity function, then so is $F(\rho_1(\mathbf{x}), \rho_2(\mathbf{x}), \rho_3(\mathbf{x}))$.

Now we compute the possible automorphisms ρ. Differentiating (9.3a) with respect to x' we find

$$\partial_1\rho_1(x+x', y+y', z+z'+xy') = \partial_1\rho_1(x', y', z').$$

Since the right-hand side of this equation is independent of x, y, z, we have $\partial_1\rho_1(x,y,z) = \alpha$, a constant. Similarly, differentiating (9.3a) with respect to y we have $\partial_2\rho_1(x,y,z) = \beta$. Differentiating (9.3a) with respect to x yields

$$\alpha + y'\partial_3\rho_1(x+x', y+y', z+z'+xy') = \alpha.$$

Since this equation holds for general x', y', z', it follows that $\partial_3\rho_1(x,y,z) = 0$. Thus $\rho_1(x,y,z) = \alpha x + \beta y + k$ where k is a constant. Substituting this expression back into (9.3a) we see that $k = 0$. Similarly, equation (9.3b) has only the solution $\rho_2(x,y,z) = \gamma x + \delta y$ where γ, δ are constants. The computation for equation (9.3c) is just as straightforward, although the details are a bit more complicated. The final result is

$$\begin{aligned} &\rho_1(x,y,z) = \alpha x + \beta y, \;\; \rho_2(x,y,z) = \gamma x + \delta y, \\ &\rho_3(x,y,z) = ax + by + \frac{1}{2}(\alpha x + \beta y)(\gamma x + \delta y) + (\alpha\delta - \beta\gamma)(z - \frac{1}{2}xy). \end{aligned} \tag{9.4}$$

Here, $\alpha, \beta, \gamma, \delta, a, b$ are real constants such that $\alpha\delta - \beta\gamma \neq 0$, so that ρ is 1-1.

Some of the automorphisms of H_R are **inner automorphisms**. These are the automorphisms of the form $\rho_B(A) = B^{-1}AB$ for $A \in H_R$, where B is a fixed member of H_R. (Clearly, ρ_B maps H_R onto itself and is one-to-one. Furthermore $\rho_B(A_1A_2) = B^{-1}A_1A_2B = (B^{-1}A_1B)(B^{-1}A_2B) = \rho_B(A_1)\rho_B(A_2)$, so ρ_B is a group homomorphism.) If $B = B(a', b', c')$ then

$$\begin{aligned} \rho_B(A) = B^{-1}A(x,y,z)B &= \begin{pmatrix} 1, & x, & z - a'y + b'x \\ 0, & 1, & y \\ 0, & 0, & 1 \end{pmatrix} \\ &= A(x, y, z - a'y + b'x) \end{aligned}$$

so the transformations

$$\begin{aligned} &\rho_1(x,y,z) = z, \quad \rho_2(x,y,z) = y \\ &\rho_3(x,y,z) = b'x - a'y + z \end{aligned} \tag{9.5}$$

correspond to inner automorphisms. We are not very interested in inner automorphisms because they can easily be understood in terms of the Heisenberg group itself. Thus we set $a = b = 0$ in (9.4) and concentrate on the **outer automorphisms**

$$\begin{aligned} &\rho_1(x,y,z) = \alpha x + \beta y, \quad \rho_2(x,y,z) = \gamma x + \delta y, \\ &\rho_3(x,y,z) = \frac{1}{2}(\alpha x + \beta y)(\gamma x + \delta y) - (\alpha\delta + \beta\gamma)(z - \frac{1}{2}xy). \end{aligned} \tag{9.6}$$

Recall that the infinite-dimensional irred unitary reps $\mathbf{T}^\lambda$ of H_R take the form, (5.5),

$$\mathbf{T}^\lambda[x,y,z]\mathbf{f}(t) = e^{2\pi i\lambda(z+ty)}\mathbf{f}(t+x)$$

for $\mathbf{f} \in L_2(R)$, where λ is a nonzero real constant. Note that the operators corresponding to the center C of H_R are just multiples of the identity operator: $\mathbf{T}^\lambda[0,0,z] = e^{2\pi i\lambda z}\mathbf{E}$. Since $\mathbf{T}^\lambda_\rho[0,0,z] = e^{2\pi i\lambda(\alpha\delta-\beta\gamma)z}\mathbf{E}$ the irred rep $\mathbf{T}^\lambda_\rho$ can possibly be equivalent to $\mathbf{T}^\lambda$ only if $\alpha\delta - \beta\gamma = 1$, so we now restrict our attention to this case.

With this restriction $\mathbf{T}^\lambda_\rho$ must be equivalent to $\mathbf{T}^\lambda$. Indeed, the reps $\mathbf{T}^\lambda, \mathbf{T}^\lambda_\rho$ coincide on the center C. Furthermore, the matrix elements $T_{\rho,jk}[x,y,0]$ are square integrable with respect to the measure $dxdy$ in the plane. (In fact, these matrix elements differ from $T^\lambda_{jk}[x,y,0]$ only by a factor of absolute value 1 and a change of variables with Jacobian $\alpha\delta-\beta\gamma = 1$.) Thus, if $\mathbf{T}^\lambda_\rho$ is not equivalent to $\mathbf{T}^\lambda$ we can use Corollary 3.1 and repeat the arguments leading to (5.8) for $\mathbf{T}^{(\mu)} \equiv \mathbf{T}^\lambda, \mathbf{T}^{(\nu)} \equiv \mathbf{T}^\lambda_\rho, \mu \neq \nu$, and measure $dxdy$ to obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T^\lambda_{\rho,j\ell}[x,y,0]\overline{T^\lambda_{sk}}[x,y,0]dxdy = 0$$

for all j, ℓ, s, k. Thus the matrix elements of $\mathbf{T}^\lambda_\rho$ are orthogonal to those of $\mathbf{T}^\lambda$ in $L_2(R^2)$. However, as we have shown in §5.6, the matrix elements $T^\lambda_{jk}(x,y,0)$ form a basis for $L_2(R^2)$. This contradiction proves that $\mathbf{T}^\lambda_\rho \cong \mathbf{T}^\lambda$, hence that there exist unitary operators $\mathbf{U}(x,y)$ such that

$$\begin{aligned}
\mathbf{T}^\lambda_\rho[x,y,z] &= \mathbf{U}_D^{-1}\mathbf{T}^\lambda[x,y,z]\mathbf{U}_D \\
&= \mathbf{T}^\lambda\left[\alpha x+\beta y, \gamma x+\delta y, \frac{1}{2}(\alpha x+\beta y)(\gamma x+\delta y)+z-\frac{1}{2}xy\right] \\
&= \exp\left[\pi i\lambda\left((\alpha x+\beta y)(\gamma x+\delta y)-xy\right)\right]\mathbf{T}^\lambda[\alpha x+\beta y,\gamma x+\delta y, z]
\end{aligned} \tag{9.7}$$

where $D = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ and $\det D = \alpha\delta-\beta\gamma = 1$.

Theorem 9.1. *Suppose $F^\lambda(x,y,z)$ is a matrix element of the irred unitary rep $\mathbf{T}^\lambda$ of H_R. Then*

$$\begin{aligned}
G^\lambda_D(x,y,z) = \exp\left[\pi i\lambda\left((\alpha x+\beta y)(\gamma x+\delta y)-xy\right)\right] \times \\
F^\lambda(\alpha x+\beta y, \gamma x+\delta y, z)
\end{aligned} \tag{9.8}$$

is also a matrix element of $\mathbf{T}^\lambda$ for any $D = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ with $\det D = \alpha\delta-\beta\gamma = 1$.

Proof. Suppose $F^\lambda(x,y,z) = \langle\mathbf{T}^\lambda[x,y,z]\mathbf{f}_1,\mathbf{f}_2\rangle$ for $\mathbf{f}_1,\mathbf{f}_2 \in L_2(R)$. Setting $\mathbf{g}_j = \mathbf{U}\mathbf{f}_j, j = 1,2$, we obtain (9.8) from (9.7) with $G^\lambda(x,y,z) = \langle\mathbf{T}^\lambda[x,y,z]\mathbf{g}_1,\mathbf{g}_2\rangle$. □

Note that (9.8) gives us information about the structure of the set of ambiguity and of cross-ambiguity functions.

9.2 The metaplectic representation. Next we turn to the problem of actually computing the operators $\mathbf{U}_D$. First of all, note from (9.7) that for any phase factor $e^{i\varphi(D)}$ (with $|e^{i\varphi}| = 1$) the unitary operators $\mathbf{U}'_D = e^{i\varphi}\mathbf{U}_D$ also satisfy (9.7). Indeed, a simple argument using Theorem 5.1 shows that the operators $\mathbf{U}_D$ are uniquely determined up to a phase factor. We shall find that it is possible to choose the operators $\mathbf{U}_D$ such that the mapping $\mathbf{f} \to \mathbf{U}_D\mathbf{f}$ is continuous in the norm as a function of the local parameters $\alpha, \beta, \gamma, \delta$ for every $\mathbf{f} \in L_2(R)$.

It is no accident that we have arranged the parameters $\alpha, \beta, \gamma, \delta$ in the form of the matrix $D \in SL(2, R)$, since $\det D = 1$. Indeed, it is straightforward to check that if

$$\mathbf{T}^\lambda_\rho[\mathbf{x}] = \mathbf{U}_D^{-1}\mathbf{T}^\lambda[\mathbf{x}]\mathbf{U}_D, \quad \mathbf{T}^\lambda_{\rho'}[\mathbf{x}] = \mathbf{U}_{D'}^{-1}T^\lambda[\mathbf{x}]\mathbf{U}_{D'}$$

for automorphisms ρ and ρ' of H_R, then the automorphism $\rho\rho' : \mathbf{x} \to \rho(\rho'(\mathbf{x}))$ corresponds to the matrix $DD' \in SL(2, R)$, (matrix product). However,

$$\begin{aligned}\mathbf{T}^\lambda_{\rho\rho'}[\mathbf{x}] = \mathbf{T}^\lambda[\rho(\rho'(\mathbf{x})] = \mathbf{T}^\lambda_\rho[\rho'(\mathbf{x})] &= \mathbf{U}_D^{-1}\mathbf{T}^\lambda[\rho'(\mathbf{x})]\mathbf{U}_D \\ &= \mathbf{U}_D^{-1}\mathbf{T}^\lambda_{\rho'}[\mathbf{x}]\mathbf{U}_D = \mathbf{U}_D^{-1}U_{D'}^{-1}\mathbf{T}^\lambda[\mathbf{x}]\mathbf{U}_{D'}\mathbf{U}_D \\ &= [\mathbf{U}_{D'}\mathbf{U}_D]^{-1}\mathbf{T}^\lambda[\mathbf{x}](\mathbf{U}_{D'}\mathbf{U}_D),\end{aligned}$$

so

$$\mathbf{U}_{DD'} = e^{e\psi(D',D)}\mathbf{U}_{D'}\mathbf{U}_D \tag{9.9}$$

for some phase factor $e^{i\psi(D',D)}$. (Note the reversal of order in (9.9).) It follows from (9.9) that the operators $\mathbf{U}_D$ determine a **projective representation** of $SL(2, R)$, i.e., a rep up to a phase factor.

It is easy to verify the operator identity

$$\mathbf{R}(a)^{-1}\mathbf{T}^\lambda[x, y, z]\mathbf{R}(a) = \mathbf{T}^\lambda_{D_1}(a)[x, y, z] \tag{9.10}$$

where

$$D_1(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

and $\mathbf{R}(a)\mathbf{f}(t) = e^{i\pi\lambda a t^2}\mathbf{f}(t)$.

Furthermore, defining the unitary operator $\mathbf{V}(b)$ by

$$\mathbf{V}(b)\mathbf{f}(t) = b^{1/2}\mathbf{f}(t), \quad \mathbf{f} \in L_2(R), \quad b > 0,$$

we find

$$\mathbf{V}^{-1}(b)\mathbf{T}^\lambda[x, y, z]\mathbf{V}(b) = \mathbf{T}^\lambda[by, b^{-1}y, z] = \mathbf{T}^\lambda_{D_2(b)}[\mathbf{x}] \tag{9.11}$$

where

$$D_2(b) = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}.$$

Matrices of the form $D_1(a)$, $D_2(b)$ generate a two-dimensional subgroup of $SL(2, R)$. To generate the full group we need a third one-parameter subgroup.

We have already derived such operators, the $\mathbf{U}(\alpha)$ in (5.43). However, from the form (5.43) it is not easy to verify relations (9.7). It is much easier to use the unitary transformation $\mathbf{A} : L_2(R) \to F$ to realize the rep $\mathbf{T}^\lambda$ on the Bargmann-Segal Hilbert space F. Recall that $\mathbf{U}'(\alpha) = \mathbf{A}\mathbf{U}(\alpha)\mathbf{A}^{-1}$ takes the form (5.41):

$$\mathbf{U}'(\alpha)\mathbf{f}(w) = \mathbf{f}(e^{i\alpha}w), \quad \mathbf{f} \in F.$$

The action of $\mathbf{T}^\lambda$ on F is given by (5.35):

$$\mathbf{T}'^{\lambda}(\mathbf{x})\mathbf{f}(w) = \exp\Big[-\frac{1}{4}(x^2 + 4\pi^2\lambda^2 y^2) + 2^{-1/2}(x - 2\pi\lambda i y)w \\ -\pi\lambda i x y + 2\pi\lambda i z\Big]\mathbf{f}(w - 2^{-1/2}[x + 2\pi\lambda i y]).$$

Now it is easy to verify the identity

(9.12)
$$\begin{aligned}\mathbf{U}'(-\alpha)\mathbf{T}'^{\lambda}[x,y,z]\mathbf{U}'(\alpha) \\ =\mathbf{T}'^{\lambda}\Big[& x\cos\alpha + 2\pi\lambda y\sin\alpha, \frac{-x}{2\pi\lambda}\sin\alpha + y\cos\alpha, \\ & z + \frac{1}{2}(x\cos\alpha + 2\pi\lambda y\sin\alpha)\left(\frac{-x}{2\pi\lambda}\sin\alpha + y\cos\alpha\right) - \frac{1}{2}xy\Big] \\ =\mathbf{T}'^{\lambda}_{D_3(\lambda,\alpha)}[\mathbf{x}]\end{aligned}$$

where

$$D_3(\lambda,\alpha) = \begin{pmatrix} \cos\alpha & 2\pi\lambda\sin\alpha \\ -\sin\alpha/2\pi\lambda & \cos\alpha \end{pmatrix}.$$

Transforming back to $L_2(R)$ we see that the operators $\mathbf{U}(\alpha)$ in (5.43) must satisfy

$$\mathbf{U}(\alpha)^{-1}\mathbf{T}^{\lambda}[\mathbf{x}]\mathbf{U}(\alpha) = \mathbf{T}^{\lambda}_{D_3(\lambda,\alpha)}[\mathbf{x}].$$

Since

$$\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} b^{-1} & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} \cos\alpha & b^2\sin\alpha \\ -b^{-2}\sin\alpha & \cos\alpha \end{pmatrix},$$

setting $b = \sqrt{2\pi\lambda}$ in the case where $\lambda > 0$ we find the operator

$$\mathbf{W}(\alpha) = \mathbf{V}\left(\sqrt{2\pi\lambda}\right)\mathbf{U}(\alpha)\mathbf{V}\left(\frac{1}{\sqrt{2\pi\lambda}}\right)$$

or

(9.13)
$$\mathbf{W}(\alpha)\mathbf{f}(t) = \sqrt{\frac{\lambda}{2}}\lim_{n\to\infty}\int_{-n}^{n}\frac{e^{-i\epsilon\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}}{(|\sin\alpha|)^{1/2}}\exp\left[\pi\lambda i(\cot\alpha)\left(\frac{t^2+\tau^2}{2}\right) - \frac{\pi\lambda i t\tau}{\sin\alpha}\right]\mathbf{f}(\tau)d\tau,$$

where $\alpha = 2k\pi + \epsilon\beta$, k an integer, $\epsilon = \pm 1$, $0 < \beta < \pi$ and $\mathbf{f} \in L_2(R)$. Here $\mathbf{W}(\alpha)$ satisfies

$$\mathbf{W}(-\alpha)\mathbf{T}^{\lambda}[\mathbf{x}]\mathbf{W}(\alpha) = \mathbf{T}^{\lambda}_{D_3(\alpha)}[\mathbf{x}]$$

where

$$D_3(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}.$$

NOTE: The operators $e^{2\pi\lambda iat^2}$, $\mathbf{V}(b)$ and $\mathbf{W}(\alpha)$ do not generate a rep of $SL(2,R)$ but the first two types of operators and the operators $\mathbf{W}'(\alpha) = \mathbf{W}(\alpha)e^{-i\alpha/2}$ do generate a rep of a two-fold covering group $\widetilde{SL(2,R)}$ of $SL(2,R)$. This rep is called the **metaplectic representation**. See [M6] and [S2] for more details. For $2\pi\lambda = 1$ the rep $\mathbf{T}^\lambda$ of H_R together with the metaplectic rep of $\widetilde{SL(2,R)}$ extends uniquely to an irred unitary rep of the 6-parameter **Schrödinger group**, the semi-direct product of H_R and $\widetilde{SL(2,R)}$. The Schrödinger group is the symmetry group of the time-dependent Schrödinger equations for each of the free-particle, the harmonic oscillator and the linear potential in two-dimensional space time. See [M6] for a detailed analysis.

The formula

$$\begin{pmatrix} 1 & 0 \\ \tan\theta & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1/\cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & \cos\theta\sin\theta \\ 0 & 1 \end{pmatrix}$$

shows that the unitary operator $\mathbf{Z}(\tau)$ corresponding to the matrix $D = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$ can be defined (unique to within a phase factor) by

$$\begin{aligned} \mathbf{Z}(\tau)\mathbf{f}(x) &= \mathbf{V}\left(\frac{1}{\cos\theta}\right)\mathbf{W}'(\theta)\mathbf{R}(\tan\theta)\mathbf{f}(x) \\ &= \sqrt{\frac{\lambda}{i\tau}}\lim_{n\to\infty}\int_{-n}^{n}\exp\left[-\frac{\pi\lambda(t-y)^2}{2i\tau}\right]\mathbf{f}(y)dy \end{aligned}$$

where $\tau = \sin\theta\cos\theta$. Clearly this operator is well defined and unitary for $|\tau| \le 1$, since it is a product of unitary operators. Indeed $\mathbf{Z}(\tau)$ is well defined and unitary for all real τ. To show this we use the fact that the family of all functions of the form $\mathbf{f}(x) = e^{-b(x-a)^2}$, for $b > 0$ and a real, spans $L_2(R)$, i.e., the set of all dilations and translations of e^{-x^2} spans $L_2(R)$. (See [K1, page 494] and Exercise 8.3.) Now the integral

$$\text{(9.14)} \qquad \mathbf{Z}(\tau)\mathbf{f}(x) = \sqrt{\frac{\lambda}{2i\tau}}\int_{-\infty}^{\infty} e^{-\pi\lambda(x-y)^2/2i\tau}\mathbf{f}(y)dy$$

is well-defined for all τ and agrees with the preceding integral for $|\tau| \le 1$. An explicit evaluation of the integral yields

$$\text{(9.14}'\text{)} \qquad \mathbf{Z}(\tau)\mathbf{f}(x) = \frac{1}{\sqrt{1+\frac{2ib\tau}{\pi\lambda}}}e^{-b(x-a)^2/(1+\frac{2ib\tau}{\pi\lambda})},$$

so

$$\langle \mathbf{Z}(\tau)\mathbf{f}_1, \mathbf{Z}(\tau)\mathbf{f}_2\rangle = \langle \mathbf{f}_1, \mathbf{f}_2\rangle = \sqrt{\frac{\pi}{b_1+b_2}}e^{-b_1b_2(a_1-a_2)^2/(b_1+b_2)}.$$

(Here the parameters a_i, b_i correspond to the functions f_i.) Furthermore,

$$\mathbf{Z}(\tau_1)[\mathbf{Z}(\tau_2)f](x) = \left[1+\frac{2ib(\tau_1+\tau_2)}{\pi\lambda}\right]^{-1/2} \exp\left[-b(x-a)^2/\left(1+\frac{2ib(\tau_1+\tau_2)}{\pi\lambda}\right)\right]$$
$$= \mathbf{Z}(\tau_1+\tau_2)f(x),$$

so

$$\mathbf{Z}(\tau_1)\mathbf{Z}(\tau_2) = \mathbf{Z}(\tau_1+\tau_2)$$

for all τ_1, τ_2. Since $\mathbf{Z}(\tau)$ is unitary for $|\tau| \le 1$ it follows easily that $\mathbf{Z}(\tau)$ is unitary for all τ. Note also that $\mathbf{Z}(0) = \mathbf{E}$.

By explicit differentiation in (9.14′) we see that for $\mathbf{g}(x,\tau) = \mathbf{Z}(\tau)\mathbf{f}(x)$, $\partial_\tau \mathbf{g} = \frac{i}{2\pi\lambda}\partial_{xx}\mathbf{g}$, $\mathbf{g}(x,0) = \mathbf{f}(x)$. Thus, $\mathbf{Z}(\tau)\mathbf{f}(x)$ gives the unique solution of the Cauchy problem for the time dependent free particle Schrödinger equation. In particular

$$\mathbf{Z}(\tau)\mathbf{f}(x) = e^{\frac{i\tau}{2\pi\lambda}\partial_{xx}}\mathbf{f}(x)$$

where $e^{i\tau\mathbf{H}}$ is the unitary operator generated by the self-adjoint operator $\mathbf{H}$ via the spectral theorem, [K1]. In [M6] it is shown that the Schrödinger group acts as the symmetry group of the time dependent Schrödinger equation, i.e., it maps solutions into solutions of this equation, and that the possible solutions which are obtainable by separation of variables can be characterized by the group action. Similarly, the operator $\mathbf{W}'(\tau)$ satisfies the equation

$$\partial_\tau \mathbf{g} = \frac{i}{2\pi\lambda}\left(\partial_{xx} + \frac{x^2}{4}\right)\mathbf{g}, \quad \mathbf{g}(x,0) = \mathbf{f}(x)$$

where $\mathbf{g}(x,\tau) = \mathbf{W}'(\tau)\mathbf{f}(x)$. Thus, $\mathbf{g}$ is the unique solution of the Cauchy problem for the time dependent Schrödinger equation equation with a harmonic oscillator potential, [M6]. Such considerations are beyond the scope of these notes.

9.3 Theta functions and the lattice Hilbert space. For another application of the use of the metaplectic formula (9.6) let us reconsider our construction of the lattice representation of H_R. According to (5.45) this rep is defined on functions $f[\mathbf{x}] = f(A(\mathbf{x}))$ on H_R such that

$$\mathbf{f}(a_1+x_1, a_2+x_2, y_3+x_3+a_1x_2) = e^{2\pi i y_3}\mathbf{f}(x_1,x_2,x_3) \tag{9.15}$$

where a_1, a_2 are integers. For ρ an automorphism (9.6) of H_R, with $\alpha\delta-\beta\gamma=1$, it is natural to look for the conditions such that $\mathbf{f}_\rho(\mathbf{x}) \equiv \mathbf{f}(\rho(\mathbf{x}))$ belongs to the lattice Hilbert space for every $\mathbf{f}$ belonging to this Hilbert space. If ρ corresponds to the matrix

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \quad \alpha\delta-\beta\gamma=1,$$

the conditions are
(9.16)

$$\mathbf{f}\Big(\alpha x_1+\beta x_2+\alpha a_1+\beta a_2,\ \ \gamma x_1+\delta x_2+\gamma a_1+\delta a_2,$$
$$y_3+x_3+a_1x_2+\frac{1}{2}(\alpha x_1+\beta x_2+\alpha a_1+\beta a_2)(\gamma x_1+\delta x_2+\gamma a_1+\delta a_2)$$
$$-\frac{1}{2}(a_1+x_1)(a_2+x_2)\Big)$$
$$=e^{2\pi i y_3}\mathbf{f}\left(\alpha x_1+\beta x_2,\gamma x_1+\delta x_2,x_3+\frac{1}{2}(\alpha x_1+\beta x_2)(\gamma x_1+\delta x_2)-\frac{1}{2}x_1x_2\right).$$

Since $\mathbf{f}$ satisfies only (9.15) we see that $\alpha,\beta,\gamma,\delta$ must be integers. Then (9.15) implies

$$\begin{aligned}\mathbf{f}([\alpha x_1+\beta x_2]+[\alpha a_1+\beta a_2],\ [\gamma x_1+\delta x_2]+[\gamma a_1+\delta a_2],\ \tilde{y}_3+\tilde{x}_3\\ +[\alpha a_1+\beta a_2][\gamma x_1+\delta x_2])=e^{2\pi i\tilde{y}_3}\mathbf{f}(\alpha x_1+\beta x_2,\gamma x_1+\delta x_2,\tilde{x}_3).\end{aligned}\tag{9.17}$$

Setting $\tilde{x}_3=x_3+\frac{1}{2}[\alpha x_1+\beta x_2][\gamma x_1+\delta x_2]-\frac{1}{2}x_1x_2, \tilde{y}_3=x_3+\frac{1}{2}(\alpha a_1+\beta a_2)(\gamma a_1+\delta a_2)-\frac{1}{2}a_1a_2$ in (9.17), we recover (9.16) provided $(\alpha a_1+\beta a_2)(\gamma a_1+\delta a_2)-a_1a_2=\alpha\gamma a_1^2+2\beta\gamma a_1a_2+\beta\delta a_2^2$ is an even integer for all integers a_1,a_2. This will be the case if and only if

$$\alpha\gamma\equiv\beta\delta\equiv 0\mod 2,\tag{9.18}$$

i.e., $\alpha\gamma$ and $\beta\delta$ must be even integers.

Thus we see that if $D=\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\in SL(2,Z)$, i.e., if $D\in SL(2,R)$ and the matrix elements of D are integers, and if conditions (9.18) are satisfied, then $\mathbf{f}_{\rho_D}$ belongs to the lattice Hilbert space whenever $\mathbf{f}$ so belongs. Reduced to the space of functions $\varphi(x_1,x_2)$ where $\mathbf{f}(x_1,x_2,x_3)=\varphi(x_1,x_2)e^{2\pi i x_3}$, so that

$$\varphi(x_1+a_1,x_2+a_2)=e^{-2\pi i a_1x_2}\varphi(x_1,x_2)\tag{9.19}$$

for $a_1,a_2\in Z$, the action is

$$\varphi_{\rho_D}(x_1,x_2)=\varphi(\alpha x_1+\beta x_2,\gamma x_1+\delta x_2)e^{\pi i[(\alpha x_1+\beta x_2)(\gamma x_1+\delta x_2)-x_1x_2]}.\tag{9.20}$$

The action of $SL(2,Z)$ on the lattice Hilbert space will lead us to a number of interesting transformation formulas for Theta functions.

As we showed earlier, (7.10), the ground state wave function $\boldsymbol{\psi}_0(t)=\pi^{-1/4}e^{-x^2/2}\in L_2(R)$ is mapped by the Weil-Brezin-Zak transform to

$$\mathbf{P}\boldsymbol{\psi}(x_1,x_2)=\pi^{-1/4}e^{-x_1^2/2}\theta_3(x_2+\frac{ix_1}{2\pi}\mid\frac{i}{2\pi})\tag{9.21}$$

where θ_3 is the Jacobi theta function, [EMOT1], [WW],

$$\theta_3(z\mid\tau)=\sum_{n=-\infty}^{\infty}\exp[\pi i\tau n^2+2\pi i n z].\tag{9.22}$$

Here for τ such that $\mathrm{Im}\tau > 0$, θ_3 is an entire function of z. Moreover, the function $\Theta^\tau(t) = e^{\pi i \tau t^2} \in L_2(R)$, with $\mathrm{Im}\tau > 0$ is mapped to

$$\hat{\Theta}^\tau(x_1, x_2) = \mathbf{P}\Theta^\tau(x_1, x_2) = e^{\pi i \tau x_1^2}\theta_3(x_1\tau + x_2 \mid \tau) \tag{9.23}$$

in the lattice Hilbert space. An elementary complex variable argument [WW] shows that $\hat{\Theta}^\tau(x_1, x_2)$ vanishes precisely once in the square $0 \le x_1 < 1$, $0 \le x_2 < 1$, with a simple zero at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$. Thus by Theorem 7.1 the functions $\hat{\Theta}^\tau(x_1, x_2)e^{2\pi i(m_1x_1+m_2x_2)}$, $m_1, m_2 \in Z$, span the lattice Hilbert space. Another way to state this is to say that every element in the lattice Hilbert space can be written in the form $\hat{\Theta}^\tau(x_1, x_2)h(x_1, x_2)$ where h is a periodic function in x_1 and $x_2 : h(x_1 + a_1, x_2 + a_2) = h(x_1, x_2)$ for integers a_1, a_2. Since $\hat{\Theta}^\tau$ belongs to the lattice Hilbert space, so does $\hat{\Theta}^\tau_{\rho_D}$ where $D \in SL(2, Z)$, and satisfies (9.18). Thus

$$\hat{\Theta}^\tau_{\rho_D}(\mathbf{x}) = \hat{\Theta}^{\tau'}(\mathbf{x})h^{\tau'}(\mathbf{x}) \tag{9.24}$$

for some periodic function $h^{\tau'}$. Expression (9.24) describes the framework for a family of transformation formulas obeyed by the Theta functions. Note that τ' need not be the same as τ. In the derivations to follow we will choose τ' so that the expressions for $h^{\tau'}$ are as simple as possible.

As a nontrivial example we take the case $D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, see Exercise 9.3. The result is

$$e^{-2\pi i x_1 x_2} e^{\pi i \tau x_2^2}\theta_3(x_2\tau - x_1 \mid \tau) = e^{-i\frac{\pi x_1^2}{\tau}}\sqrt{\frac{i}{\tau}}\,\theta_3\left(-\frac{x_1}{\tau} + x_2 \mid -\frac{1}{\tau}\right).$$

(We have chosen $\tau' = -1/\tau$.) This is equivalent to the transformation formula

$$\theta_3(z \mid \tau) = \sqrt{\frac{i}{\tau}}e^{-\pi i z^2/\tau}\theta_3(\frac{z}{\tau} \mid \frac{-1}{\tau}). \tag{9.25}$$

As a second example we take $D = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Then

$$\begin{aligned}
\hat{\Theta}^\tau_{\rho_D}(\mathbf{x}) &= e^{2\pi i x_1^2}e^{\pi i \tau x_1^2}\theta_3(x_1\tau + 2x_1 + x_2 \mid \tau) \\
&= e^{2\pi i x_1^2}e^{\pi i \tau x_1^2}\sum_n e^{i\pi\tau n^2 + 2\pi i n(x_1\tau + 2x_1 + x_2)} \\
&= e^{2\pi i x_1^2}\sum_n e^{i\pi\tau(n+x_1)^2 + 2\pi i n(2x_1 + 1/2)} = \sum_n e^{i\pi(\tau+2)(n+x_1)^2}e^{2\pi i n x_2} \\
&= e^{i\pi\tau' x_1^2}\theta_3(\tau' x_1 + x_2 \mid \tau'), \quad \tau' = \tau + 2.
\end{aligned}$$

Thus,

$$\theta_3(z \mid \tau) = \theta_3(z \mid \tau + 2). \tag{9.26}$$

Even in the cases where the parity conditions (9.18) don't hold, we get useful information. For example, consider the case $D = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. With this automorphism we are replacing the function $\varphi(x_1, x_2)$

$$\varphi(x_1, x_2+1) = \varphi(x_1, x_2), \quad \varphi(x_1+1, x_2) = e^{-2\pi i x_2}\varphi(x_1, x_2) \tag{9.27}$$

in the lattice Hilbert space by the function

$$\eta(x_1, x_2) = \varphi(x_1, x_1 + x_2)e^{\pi i x_1^2}.$$

Now it is easy to check that

$$\eta(x_1, x_2+1) = \eta(x_1, x_2), \quad \eta(x_1+1, x_2) = -e^{-2\pi i x_2}\eta(x_1, x_2), \tag{9.28}$$

so η doesn't belong to the lattice Hilbert space. However it is straightforward to show that $\tilde{\Theta}^\tau(x_1, x_2) = \hat{\Theta}^\tau\left(x_1, x_2 + \frac{1}{2}\right)$ transforms according to (9.28). It follows from this remark that any square integrable (on the unit square) function η satisfying (9.28) can be written in the form $\hat{\Theta}^{\tau'}\left(x_1, x_2 + \frac{1}{2}\right) h^{\tau'}(x_1, x_2)$ where $h^{\tau'}$ is periodic in x_1, x_2. Indeed we find

$$\begin{aligned}
\hat{\Theta}^\tau_{\rho_D}(\mathbf{x}) &= e^{\pi i x_1^2} e^{\pi i \tau x_1^2}\theta_3(x_1\tau + x_1 + x_2 |\ \tau) = e^{\pi i x_1^2} e^{\pi i \tau x_1^2} \sum_2 e^{i\pi\tau n^2 + 2\pi i n(x_1\tau + x_1 + x_2)} \\
&= e^{\pi i x_1^2} \sum_n e^{i\pi\tau(n+x_1)^2 + 2\pi i n(x_1+x_2)} = \sum_n e^{i\pi(\tau+1)(n+x_1)^2} e^{2\pi i n\left(x_2 + \frac{1}{2}\right)} \\
&= e^{i\pi\tau' x_1^2}\theta_3\left(x_1\tau' + x_2 + \frac{1}{2} \mid \tau'\right), \quad \tau' = \tau + 1.
\end{aligned}$$

(Here we have used the fact that $e^{-i\pi n^2} = e^{i\pi n}$ for any integer n.) Thus we have the transformation formula

$$\theta_3(z|\tau) = \theta_3\left(z + \frac{1}{2} \mid \tau + 1\right). \tag{9.29}$$

Note that by using (9.29) twice we get (9.26), in accordance with the fact that $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$. Note: The other three basic Jacobi Theta functions θ_1, θ_2, and θ_4 (or θ_0) can easily be expressed in terms of θ_3, [EMOT1], [WW].

Since the modular group elements $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ generate $SL(2, Z)$, see for example [H3, pages 168-171], it follows that all the $SL(2, Z)$ transformation formulas can be derived by repeated use of (9.25) and (9.29). See [AT1] and [EMOT1] for details. It is worth remarking that the appropriate τ' corresponding to each $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, Z)$ is

$$\tau' = \frac{\delta\tau + \gamma}{\beta\tau + \alpha}.$$

In the preceding discussion we have been concerned with nonorthogonal bases for the lattice Hilbert space. For completeness we also compute the ON basis

$$\varphi_n(x_1,x_2) = \mathbf{P}\boldsymbol{\psi}_n(x_1,x_2), \quad n = 0,1,2,\cdots$$

corresponding, via the Weyl-Brezin-Zak transform, to the ON basis (5.26) for $L_2(R)$:

$$\boldsymbol{\psi}_n(t) = \pi^{-1/4}(n!)^{-1/2}(-1)^n 2^{-n/2} e^{-t^2/2} H_n(t),$$

where $H_n(t)$ is a Hermite polynomial. We have already seen that the ground state wave function $\boldsymbol{\psi}_0(t) = \pi^{-1/4}e^{-t^2/2}$ maps to, (9.21),

$$\varphi_0(x_1,x_2) = \pi^{-1/4}e^{-x_1^2/2}\theta_3(x_2 + \frac{ix_1}{2\pi} \mid \frac{i}{2\pi}).$$

Applying the transform $\mathbf{P}$ to both sides of the generating function (5.25) for the $\boldsymbol{\psi}_n(t)$ and using the fact that

$$\mathbf{Pf}(x_1,x_2) = \pi^{-1/4}e^{-\beta^2-2\beta x_1-\frac{1}{2}x_1^2}\theta_3\left(x_2 + \frac{i}{2\pi}[x_1+2\beta] \mid \frac{i}{2\pi}\right)$$

for $\mathbf{f}(t) = \pi^{-1/4}\exp(-\beta^2 - 2\beta t - \frac{1}{2}t^2)$, we obtain

$$\pi^{-1/4}\exp(-\beta^2 - 2\beta x_1 - \tfrac{1}{2}x_1^2)\theta_3(x_2 + \tfrac{i}{2\pi}[x_1+2\beta] \mid \tfrac{i}{2\pi})$$
$$= \sum_{n=0}^{\infty} \frac{2^{n/2}\beta^n}{(n!)^{1/2}}\varphi_n(x_1,x_2).$$

The left-hand side of this expression is an entire function of β.

With this brief look at the Schrödinger group, an interesting group for future study which contains both H_R and the affine group as subgroups, we conclude these notes.

9.4 Exercises.

9.1 Compute the automorphism group of G_A. Does G_A have any outer automorphisms?

9.2 For $\gamma > 0$ verify the identity

$$D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = D_1(\frac{\alpha}{\gamma})D_3(\frac{1}{2\pi}, -\frac{\pi}{2})D_2(\gamma)D_1(\frac{\delta}{\gamma}).$$

Find a similar factorization for $\gamma < 0$ and $\gamma = 0$. Show that the automorphisms of H_R are generated by $D_1(\alpha)$, $D_3(\frac{1}{2\pi}, -\frac{\pi}{2})$, and $D_2(\gamma)$.

9.3 Apply the automorphism $D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ to the Theta function (9.23) in the lattice Hilbert space to derive the formula

$$e^{-2\pi i x_1 x_2}e^{\pi i \tau x_2^2}\theta_3(x_2\tau - x_1 \mid \tau) = e^{-i\frac{\pi x_1^2}{\tau}}\sqrt{\frac{i}{\tau}}\,\theta_3\left(-\frac{x_1}{\tau} + x_2 \mid -\frac{1}{\tau}\right).$$

9.4 Show that the functions $e^{\pi i \tau x_1^2}\theta_3(x_1\tau + x_2 \mid \tau)e^{2\pi i(m_1x_1+m_2x_2)}$ form an ON basis for the lattice Hilbert space. What is the corresponding ON basis for $L_2(R)$ under the inverse Weil-Brezin-Zak transform?

9.5 Express the relation, Exercise 4.7,

$$\frac{1}{\sqrt{2\pi t}}\sum_{n=-\infty}^{\infty} e^{-\frac{n^2}{2t}} = \sum_{n=-\infty}^{\infty} e^{-2\pi^2 n^2 t}$$

as a Theta function identity. Compare with equation (9.25).

References

[AG1] I. Akhiezer and I. M. Glazman, *The Theory of Linear Operators in Hilbert Space Vol I*, Frederick Ungar, New York, 1961.

[AG2] I. Akhiezer and I. M. Glazman, *The Theory of Linear Operators in Hilbert Space Vol II*, Frederick Ungar, New York, 1963.

[AK] W. Aslaksen and J. R. Klauder, *Unitary representation of the affine group*, J. Math. Physics **9** (1968), 208–211.

[AB] Auslander and T. Brezin, *Fiber bundle structures and harmonic analysis of compact Heisenberg manifolds*, Conference on Harmonic Analysis, Lecture Notes in Mathematics 266, Springer Verlag, New York (1971).

[AG3] Auslander and I. Gertner, *Wide-band ambiguity functions and $a \cdot x + b$ group*, Signal Processing, Part I: Signal Processing Theory, Vol 22. IMA Volumes in Mathematics and its Applications, L. Auslander, T. Kailath and S. Mitter, eds., Springer Verlag, New York (1990), 1–12.

[AT1] Auslander and R. Tolimieri, *Abelian harmonic analysis, theta functions and function algebras on a nilmanifold*, Lecture Notes in Mathematics 436, Springer-Verlag, New York (1975).

[AT2] Auslander and R. Tolimieri, *Characterizing the radar ambiguity functions*, IEEE Transactions on Information Theory **IT-30 (6), November** (1984), 832–836.

[AT3] Auslander and R. Tolimieri, *Radar ambiguity function and group theory*, SIAM J. Math. Anal. **16, No. 3** (1985), 577–601.

[AT4] Auslander and R. Tolimieri, *Computing decimated finite cross-ambiguity functions*, IEEE Transactions on Acoustics, Speech, and Signal Processing **vol 36, No** (March 1988), 359–363.

[AT5] Auslander and R. Tolimieri, *On finite Gabor expansions of signals*, Signal Processing, Part I: Signal Processing Theory, Vol 22. IMA Volumes in Mathematics and its Applications, L. Auslander, T. Kailath and S. Mitter, eds., Springer Verlag, New York (1990), 13–23.

[BGZ] Bacry, A. Grossman, and J. Zak, *Proof of completeness of lattice states in the kq representation*, Phys. Rev. B. **12** (1975), 1118–1120.

[B1] R. Balian, *Un principe d'incertitude fort en théorie du signal on en mécanique quantique*, C.R. Acad. Sci. Paris **292** (1981), 1357–1362.

[B2] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform, I*, Comm. Pure Appl. Math. **14** (1961), 187–214.

[BBGK] Bargmann, P. Butera, L. Girardello and J.r. Klauder, *On the completeness of coherent states*, Rep. Math. Phys. **2** (1971), 221–228.

[B3] M.J. Bastians, *The expansion of an optical signal into a discrete set of Gaussian beams*, Optik **57, No. 1** (1980), 95–102.

[B4] M.J. Bastians, *Gabor's expansion of a signal onto Gaussian elementary signals*, IEEE Proc. **68** (1980).

[B5] J. Benedetto, *Gabor representations and wavelets*, Commutative Harmonic Analysis, D. Colella, Ed., Contemp. Math. 19, American Mathematical Society, Providence (1989), 9–27.

[BJ] W.M. Bergmans and A.J.E.M. Janssen, *Robust data equalization, fractional tap spacing and the Zak transform*, Phillips J. Res. **42** (1987), 351–398.

[B6] M. Bernfeld, *Chirp Doppler radar*, Proceedings of the IEEE **72(4), April** (1984), 540–541.

[B7] H. Boerner, *Representations of Groups*, North-Holland, Amsterdam, 1969.

[BZ] M. Boon and J. Zak, *Amplitudes on von Neumann lattices*, J. Math. Phys. **22** (1981), 1090–1099.

[BZZ] Boon, J. Zak, and I.J. Zucker, *Rational von Neumann lattices*, J. Math. Phys. **24** (1983), 316–323.

[B8] J. Brezin, *Function theory on metabelian solvmanifolds*, J. Func. Anal. **10** (1972), 33–51.

[B9] N.G. de Bruijn, *Uncertainty principles in Fourier analysis*, in Inequalities (ed. O. Shisha) Academic Press, New York (1967), 55–71.

[CM] A.C.M. Claassen and W.F.G. Mecklenbräuker, *The Wigner distributions — A tool for time-frequency signal analysis*, Phillips J. Res. **35** (1980), 217–250, 276–300, 372–389.

[C] R.R. Coifman, *Wavelet analysis and signal processing*, Signal Processing, Part I: Signal Processing Theory, Vol 22. IMA Volumes in Mathematics and its Applications, L. Auslander, T. Kailath and S. Mitter, eds., Springer Verlag, New York (1990), 59–68.

[CR] R. Coifman and R. Rochberg, *Representation theorems for holomorphic and harmonic functions in* L^p, Astérique **77** (1980), 11–66.

[CB] E. Cook and M. Bernfeld, *Radar Signals*, Academic Press, New York (1967).

[D1] I. Daubechies, *Discrete sets of coherent states and their use in signal analysis*, In International Conferences on Differential Equations and Mathematical Physics, Birmingham, Alabama (1986).

[D2] I. Daubechies, *Orthonormal bases of compactly supported wavelets*, Comm. Pure Appl. Math. **41** (1988), 909–996.

[D3] I. Daubechies, *Time-frequency localization operators: a geometric phase space approach*, IEEE Trans. Inform. Theory **34** (1988), 605–612.

[D4] I. Daubechies, *The wavelet transform, time-frequency localization and signal analysis*, IEEE Trans. Inform. Theory **36** (1990), 961–1005.

[DGM] Daubechies, A. Grossman, and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys. **27** (1986), 1271–1283.

[DP] Daubechies and T. Paul, *Time-frequency localization operators-a geometric phase space approach: II The use of dilations*, Inverse Problems **4** (1988), 661–680.

[DG] E. Davison and F.A. Grünbaum, *Tomographic reconstruction with arbitrary directions*, Communications on Pure and Applied Math **34** (1981), 77–120.

[DS1] J. Duffin and A.C. Schaeffer, *A class of nonharmonic Fourier series, Trans. Amer. Math. Soc.* **72** (1952), 341–366.

[DM1] Duflo and C.C. Moore, *On the regular representation of a nonunimodular locally compact group*, Journal of Functional Analysis **21** (1976), 209–243.

[DS2] Dunford and J. T. Schwartz, *Linear Operators*, Interscience Publishers, New York (1958).

[DM2] Dym and H.P. McKean, *Fourier Series and Integrals*, Academic Press, New York (1972).

[EMOT1] Erdélyi, W. Magnus, F. Oberhettinger and F. Tricomi, *Higher Transcendental Functions, Vol. II*, McGraw-Hill, New York, 1953.

[EMOT2] Erdélyi, W. Magnus, F. Oberhettinger and F. Tricomi, *Tables of Integral Transforms, Vol. I*, McGraw-Hill, New York, 1954.

[FG] Feig and F. A. Grünbaum, *tomographic methods in range-Doppler radar*, Inverse Problems **2(2)** (1986), 185–195.

[G1] S. Gaal, *Linear Analysis and Representation Theory*, Springer-Verlag, New York, 1973.

[G2] D. Gabor, *Theory of communication*, J. Inst. Electr. Engin. (London) **93 (III)** (1946), 429–457.

[GMS] Gel'fand, R.A. Minlos and Z.Ya. Shapiro, *Representations of the Rotation and Lorentz Groups and their Applications*, Pergamon, New York, 1963.

[G3] R.J. Gleiser, *Doppler shift for a radar echo*, American Journal of Physics **47(8), August** (1979), 735.

[G4] A. Grossman, *Wavelet transforms and edge detection*, Stochastic Processing in Physics and Engineering, S. Albeverio et al., eds., D. Reidel, Dordrecht, the Netherlands (1988), 149–157.

[GM1] Grossman and J. Morlet, *Decomposition of Hardy functions into square integrable wavelets of constant shape*, SIAM Journal of Mathematical Analysis **15** (1984), 723–726.

[GM2] Grossman and J. Morlet, *Decomposition of functions into wavelets of constant shape, and related transforms*, in Mathematics and Physics, Lectures on recent results, World Scientific (Singapore) (1985).

[GMP1] Grossman and J. Morlet, and T. Paul, *Transforms associated to square integrable group representations. Part 1: General results*, Journal of Mathematical Physics **26 (10), October** (1985), 2473–2479.

[GMP2] Grossman and J. Morlet, and T. Paul, *Transforms associated to square integrable group representations. Part 2: examples*, Ann. Inst. Henri Poincaré **45(3)** (1986), 293–309.

[HS] M. Hausner and J. T. Schwartz, *Lie Groups, Lie Algebras*, 1968.

[H1] C. E. Heil, *Generalized harmonic analysis in higher dimensions; Weyl-Heisenberg frames and the Zak transform*, Ph.D. thesis, University of Maryland, College Park, MD (1990).

[H2] C. E. Heil, *Wavelets and frames*, Signal Processing, Part I: Signal Processing Theory, Vol 22. IMA Volumes in Mathematics and its Applications, L. Auslander, T. Kailath and S. Mitter, eds., Springer Verlag, New York (1990), 147–160.

[[HW]] C.E. Heil and D. F. Walnut, *Continuous and discrete wavelet transforms*, SIAM Review **Vol 31, No 4** (December 1989), 628–666.
[[H3]] E. Hille, *Analytic Function Theory, Volume II*, Ginn and Company, Boston, 1962.
[[J1]] J.D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, New York (1975).
[[J2]] A.J.E.M. Janssen, *Weighted Wigner distributions vanishing on lattices*, J. of Mathematical Analysis and Applications **80, No. 1** (1981), 156–167.
[[J3]] A.J.E.M. Janssen, *Gabor representation of generalized functions*, J. of Mathematical Analysis and Applications **83** (1981), 377–394.
[[J4]] A.J.E.M. Janssen, *Bargmann transform, Zak transform, and coherent states*, J. Math. Phys. **23** (1982), 720–731.
[[J5]] A.J.E.M. Janssen, *The Zak transform: a signal transform for sampled time-continuous signals*, Philips J. Res. **Vol. 43** (1988), 23–69.
[[K1]] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1966.
[[K2]] Y. Katznelson, *An Introduction to Harmonic Analysis*, John Wiley, New York, 1968.
[[KK]] J.B. Keller and H.B. Keller, *Determination of reflected and transmitted fields by geometrical optics*, Journal of the Optical Society of America **40** (1949), 48–52.
[[K3]] E.J. Kelly, *The radar measurement of range, velocity and acceleration*, IRE Transactions on Military Electronics **MIL-5, April** (1961), 51–57.
[[K4]] I. Khalil, *Sur l'analyse harmonique du groupe affine de la droite*, Studia Mathematica **LI(2)** (1974), 139–167.
[[K5]] A.A. Kirillov, *Elements of the Theory of Representations*, Springer-Verlag, New York, 1976.
[[K6]] J.R. Klauder, *The design of radar signals having both high range resolution and high velocity resolution*, The Bell System Technical Journal **July** (1960), 808–819.
[[KS]] J.R. Klauder and B.S. Skagerstam, *Coherent states*, World Scientific (Singapore) (1985).
[[K7]] A. Kohari, *Harmonic analysis on the group of linear transformations of the straight line*, Proceedings of the Japanese Academy **37** (1961), 250–254.
[[K8]] J. Korevaar, *Mathematical Methods, Vol. 1*, Academic Press, New York, 1968.
[[KMG]] R. Kronland-Martinet, J. Morlet, and A. Grossmann, *Analysis of sound patterns through wavelet transforms*, Internet. J. Pattern Recog. Artif. Int. **1** (1987), 273–302.
[[LM]] P.G. Lemarié and Y. Meyer, *Ondelettes et bases hilbertiennes*, Rev. Mat. Iberoamericana **2** (1986), 1–18.
[[L]] F. Low, *Complete sets of wave-packets*, in A passion for physics - Essays in honor of Geoffrey Chew, World Scientific (Singapore) (1985), 17–22.
[[M1]] Y. Meyer, *Principe d'incertitude, bases hilbertiennes et algèbres d'opérateurs, Séminaire Bourbaki* (1985–1986), 662.
[[M2]] Y. Meyer, *Ondelettes, function splines, et analyses graduées*, Univ. of Torino (1986).
[[M3]] W. Miller, Jr., *On the special function theory of occupation number space*, Comm. Pure Appl. Math. (1965), 679–696.
[[M4]] W. Miller, Jr., *Lie Theory and Special Functions*, Academic Press, New York, 1968.
[[M5]] W. Miller, Jr., *Symmetry Groups and their Applications*, Academic Press, New York, 1972.
[[M6]] W. Miller, Jr., *Symmetry and Separation of Variables*, Addison-Wesley, Reading, Massachusetts, 1977.
[[MAFG]] J. Morlet, G. Arehs, I. Forugeau and D. Giard, *Wave propagation and sampling theory*, Geophysics **47** (1982), 203–236.
[[N1]] L. Nachbin, *Haar Integral*, Van Nostrand, Princeton (1965).
[[N2]] M. Naimark, *Normed Rings*, English Transl. Noordhoff, Groningen, The Netherlands, 1959.
[[N3]] H. Naparst, *Radar signal choice and processing for a dense target environment*, Signal Processing, Part II: Control Theory and Applications, Vol 23, IMA Volumes in Mathematics and its Applications, F.A. Grunbaum, J.W. Helton and P. Khargonekar, eds., Springer Verlag, New York (1990), 293–319.
[[N4]] F. Natterer, *On the inversion of the attenuated Radon transform*, Numerische Mathematik **32** (1979), 431–438.
[[NS]] A. Naylor and G. Sell, *Linear Operator Theory in Engineering and Science*, Holt, New York, 1971.
[[P1]] A. Papoulis, *Ambiguity function in Fourier optics*, Journal of the Optical Society of America, June (1974), 779–788.

[P2] A. Papoulis, *Signal Analysis*, McGraw-Hill, New York (1977).

[P3] T. Paul, *Functions analytic on the half-plane at quantum mechanical states*, J. Math. Phys. **25** (1984), 3252–3263.

[P4] A.M. Perelomov, *Note on the completeness of systems of coherent states*, Teor. I. Matem. Fis. **6** (1971), 213–224.

[RR] T. Rado and P. Reichelderfer, *Continuous Transformations in Analysis*, Springer-Verlag, Berlin, New York (1955).

[R1] H. Reiter, *Classical harmonic Analysis and Locally Compact Groups*, Oxford University Press, Oxford, 1968.

[R2] A. W. Ribaczek, *Radar resolution of moving targets*, IEEE Transactions on Information Theory **IT-13(1)** (1967), 51–56.

[R3] M. Rieffel, *Von Neumann algebras associated with pairs of lattices in Lie groups*, Math. Ann. **257** (1981), 403–418.

[RN] F. Riesz and B. Sz.-Nagy, *Functional Analysis*, English Transl. Ungar, New York, 1955.

[R4] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, New York, 1964.

[S1] W. Schempp, *Radar ambiguity functions, the Heisenberg group, and holomorphic theta series*, Proc. Amer. Math. Soc. **92** (1984), 103–110.

[S2] W. Schempp, *Harmonic analysis on the Heisenberg nilpotent Lie group with applications to signal theory*, Longman Scientific and technical, Pitman Research Notes in Mathematical Sciences **147, Harlow, Essex, UK** (1986).

[S2'] W. Schempp, *Neurocomputer architectures*, Resultate der Mathematik – Results in Mathematics ((To appear)).

[S3] J. M. Speiser, *Wideband ambiguity functions*, IEEE Transactions on information Theory **IT-13, January** (1967), 122–123.

[S4] M. Spivak, *Calculus on Manifolds*, Benjamin, New York, 1965.

[S5] S. Sussman, *Least square synthesis of radar ambiguity functions*, IRE Trans. Info. Theory (April 1962), 246–254.

[S6] D. A. Swick, *An ambiguity function independent of assumption about bandwidth and carrier frequency*, NRL Report 6471, Naval Research Laboratory, Washington, D.C., December (1966).

[S7] D. A. Swick, *A Review of Wideband Ambiguity Functions*, NRL Report 6994, Naval Research Laboratory, Washington, D.C., December (1969).

[T1] N. Tatsuuma, *Plancherel formula for nonunimodular locally compact groups*, J. Math. Kyoto University **12** (1972), 179–261.

[T2] P. Tchamitchian, *Calcul symbolique sur les opérateurs de Caldéron-Zygmund et bases inconditionnelles de $L^2(\mathbb{R}^n)$*, C.R. Acad. Sc. Paris **303, série 1** (1986), 215–218.

[V] N. Ja. Vilenkin, *Special Functions and the Theory of Group Representations*, American Mathematical Society, Providence, Rhode Island (1966).

[W1] D. Walnut, *Weyl-Heisenberg wavelet expansions: existence and stability in weighed spaces*, Ph.D. thesis, University of Maryland, College Park, MD (1989).

[W2] A. Weil, *Sur certains groupes d'operatours unitaires*, Acta Math. **111** (1964), 143–211.

[WW] E.T. Whittaker and G.N. Watson, *A Course in Modern Analysis* (1958), Cambridge University Press, Cambridge.

[W3] N. Wiener, *The Fourier integral and certain of its applications*, MIT Press, Cambridge (1933).

[W4] E.P. Wigner, *On the Quantum correction for thermodynamic equilibrium*, Phys. Rev. (1932), 749–759.

[W5] C. H. Wilcox, *The Synthesis Problem for Radar Ambiguity Functions*, MRC Technical Summary Report 157, Mathematics Research Center, United States Army, University of Wisconsin, Madison, Wisconsin, April (1960).

[W6] P.M. Woodward, *Probability and Information theory, with Applications to Radar*, Pergamon Press, New York (1953).

[Y] R. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York (1980).

[Z1] J. Zak, *Finite translations in solid state physics*, Phys. Rev. Lett. **19** (1967), 1385–1397.

[Z2] J. Zak, *Dynamics of electrons in solids in external fields*, Phys. rev. **168** (1968), 686–695.

[Z3] J. Zak, *The kq-representation in the dynamics of electrons in solids*, Solid State Physics **27** (1972), 1–62.

[Z4] J. Zak, *Lattice operators in crystals for Bravais and reciprocal vectors*, Phys. Rev. B **12** (1975), 3023–3026.