Lie theory and separation of variables. 5. The equations $iU_t + U_{xx} = 0$ and $iU_t + U_{xx} - c/x^2 U = 0$

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A detailed study of the group of symmetries of the time-dependent free particle Schrödinger equation in one space dimension is presented. An orbit analysis of all first order symmetries is seen to correspond in a well-defined manner to the separation of variables of this equation. The study gives a unified treatment of the harmonic oscillator (both attractive and repulsive), Stark effect, and free particle Hamiltonians in the time dependent formalism. The case of a potential c/x^2 is also discussed in the time dependent formalism. Use of representation theory for the symmetry groups permits simple derivation of expansions relating various solutions of the Schrödinger equation, several of which are new.

INTRODUCTION

The present paper is one of a series investigating the connection between separation of variables and Lie symmetry groups. In this work we make a detailed study of the free particle Schrödinger equation in the time-dependent formalism, i.e., the equation

$$(*)\quad u_{xx} + iu_t = 0,$$

and of the radial equation for a free particle,

$$(**)\quad u_{xx} - \frac{c}{x^2}u + iu_t = 0.$$

Anderson *et al.*[1] (with some errors) and Boyer[2] have classified all equations of the form

$$(***)\quad u_{xx} - V(x)u + iu_t = 0$$

which admit a nontrivial symmetry algebra of first order differential operators. It is known, e.g., Neiderer,[3] that among these equations, those corresponding to the harmonic oscillator and the linear potential are actually equivalent to (*). Here we show in a very explicit manner that every equation (***) admitting symmetries is equivalent to either (*) or (**). The equations (***) are exactly those obtained from (*) and (**) by taking all possible separations of variables.

In Sec. 1 we rederive the known six-parameter symmetry group G of equation (*).[1,2,4,5] Here G is a semidirect product of the three-parameter Weyl group W and $SL(2,R)$. We determine the global action of G and compute the orbit structure of its Lie algebra under the adjoint representation.

In Sec. 2 we classify all coordinate systems such that variables separate in equation (*) and relate them one-to-one with the G orbits. It is found necessary to include R separation as well as ordinary separation in this analysis. The orbits are essentially labelled by the attractive and repulsive harmonic oscillator, linear potential, and free particle Hamiltonians. Although all our coordinates systems are already known,[4] the proof that they are exhaustive and their explicit relation to orbits appears to be new.

In Secs. 3 and 4 we give the basis in a one-parameter model for a representative of each G orbit. The calculation of the basis functions in the Hilbert space of functions depending on x and t, and the overlap functions between the various bases are also given. We show that our knowledge of the G structure of (*) greatly simplifies the derivation of the spectral representations of various associated Hamiltonians as well as expansion theorems relating different solutions of (*). Several of the overlap functions are new and our proofs of the L_2-expansion theorems for parabolic cylinder and Airy functions are much simpler than the standard derivations. This work can be considered as the Hilbert space analogy of Weisner's work[6] on analytic expansions in Hermite functions. The papers of Whittaker[7] and Erdélyi[8] are also related to our procedure.

Finally, in Sec. 5 we give a corresponding analysis of the equation (**). The methods of Barut[9] for computing the spectra of Hamiltonians through the use of representation theory are closely related to our approach.

The analysis presented in this paper is preliminary to the treatment of the time-dependent Schrödinger equations in two and three space variables, which admit symmetries. There the theory is much richer. In particular, degenerate eigenvalues appear and it is necessary to associate separable coordinates with both first and second order symmetry operators. Nevertheless, as we shall show in forthcoming papers, the same general approach can be utilized.

All special functions appearing in this work are normalized as in the Bateman project.[10]

1. SYMMETRIES OF THE EQUATION $iu_t + u_{xx} = 0$

Let X be the differential operator

$$X = i\partial_t + \partial_{xx} \tag{1.1}$$

acting on the space $\mathcal{F}$ of locally C^∞ functions of the real variables x, t. We wish to find the maximal symmetry algebra of the equation

$$iu_t = -u_{xx}, \tag{1.2}$$

i.e., we wish to compute all linear differential operators

$$L = a(x,t)\partial_x + b(x,t)\partial_t + c(x,t), \quad a,b,c \in \mathcal{F} \tag{1.3}$$

such that $Lu(x,t)$ satisfies (1.2) whenever u does. As is well known[1,2,11] a necessary and sufficient condition for L to be a symmetry is

$$[L, X] = r(x,t)X \tag{1.4}$$

for some $r \in \mathcal{F}$. By equating coefficients of ∂_{xx}, ∂_t, ∂_x, and 1 on both sides of (1.4), one obtains a system of differential equations for a, b, c, and r. We omit the details which can be found in several references.[1,2,4] The final result is that the allowable L form a six-dimensional complex Lie algebra $\mathcal{G}^c$ with basis

$$K_2=-t^2\partial_t-tx\partial_x-t/2+ix^2/4,\quad K_1=-t\partial_x+ix/2,$$
$$K_0=i,\quad K_{-1}=\partial_x,\quad K_{-2}=\partial_t,\quad K^3=x\partial_x+2t\partial_t+\tfrac{1}{2} \tag{1.5}$$

and commutation relations

$$[K^3,K_j]=jK_j,\quad j=\pm 2,\ \pm 1, 0,\quad [K_{-1},K_1]=\tfrac{1}{2}K_0,$$
$$[K_{-1},K_2]=K_1,\quad [K_{-2},K_1]=-K_{-1},\quad [K_{-2},K_2]=-K^3. \tag{1.6}$$

In this paper we will be concerned only with the *real* Lie algebra $\mathcal{G}$ whose basis is (1.5). A second convenient basis for $\mathcal{G}$ is S_j, L_k, E, where

$$S_1=K_{-1},\quad S_2=K_1,\quad L_3=K_{-2}-K_2,$$
$$L_1=K^3,\quad L_2=K_{-2}+K_2,\quad E=K_0. \tag{1.7}$$

The commutation relations become

$$[L_1,L_2]=-2L_3,\quad [L_3,L_1]=2L_2,\quad [L_2,L_3]=2L_1,$$
$$[S_1,S_2]=\tfrac{1}{2}E,\quad [L_3,S_1]=S_2,\quad [L_3,S_2]=-S_1,$$
$$[L_2,S_1]=[S_2,L_1]=-S_2,\quad [L_1,S_1]=[L_2,S_2]=-S_1$$

where E generates the center of $\mathcal{G}$. Clearly, the operators L_1, L_2, L_3 form a basis for a subalgebra of $\mathcal{G}$ isomorphic to $sl(2,R)$ and the operators S_1, S_2, E form a basis for the Weyl algebra $\mathcal{W}$. Furthermore, $\mathcal{G}$ is the semidirect product of $sl(2,R)$ and $\mathcal{W}$.

Using standard results from Lie theory,[12] one can exponentiate the differential operators of $\mathcal{G}$ to obtain a local Lie group G of operators acting on $\mathcal{F}$. The action of the Weyl group W is given by operators

$$T(u,v,\rho)=\exp[\rho+(uv/4)]E\exp(uS_2)\exp(vS_1) \tag{1.8}$$

with multiplication

$$T(u,v,\rho)T(u',v',\rho')=T(u+u',v+v',\rho+\rho'+(vu'-uv')/4) \tag{1.9}$$

where

$$[T(u,v,\rho)f](x,t)=\exp\{i[\rho+(uv+2ux-u^2t)/4]\}\times f(x+v-ut,t),\quad f\in\mathcal{F}.$$

The action of $SL(2,R)$ is given by operators

$$[T(A)f](x,t)=\exp\left[i\left(\frac{x^2\beta/4}{\delta+t\beta}\right)\right](\delta+t\beta)^{-1/2}\times f\left[\frac{x}{\delta+t\beta},\frac{\gamma+t\alpha}{\delta+t\beta}\right] \tag{1.10}$$

where

$$A=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL(2,R),$$

i.e., A is a real matrix with determinant $+1$. Furthermore,

$$T\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}=\exp(\beta K_2),\quad T\begin{pmatrix}1 & 0\\ \gamma & 1\end{pmatrix}=\exp(\beta K_{-2}),$$
$$T\begin{pmatrix}e^\alpha & 0\\ 0 & e^{-\alpha}\end{pmatrix}=\exp(\alpha K^3),\quad T\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}=\exp(\theta L_3) \tag{1.11}$$
$$T\begin{pmatrix}\cosh\phi & \sinh\phi\\ \sinh\phi & \cosh\phi\end{pmatrix}=\exp(\phi L_2).$$

Finally, the action of $SL(2,R)$ on W via the adjoint representation is

$$T^{-1}(A)T(u,v,\rho)T(A)=T(u\delta+v\beta,u\gamma+v\alpha,\rho). \tag{1.12}$$

This defines G as a semidirect product of $SL(2,R)$ and W:

$$g=(A,w)\in G,\quad A\in SL(2,R),\quad w=(u,v,\rho)\in W,$$
$$T(g)=T(A)T(w), \tag{1.13}$$
$$T(g)T(g')=T(AA')[T(A')^{-1}T(w)T(A')]T(w')=T(gg').$$

It follows from general Lie theory that $T(g)$ maps solutions of (1.2) into solutions.[11]

The group G acts on the Lie algebra $\mathcal{G}$ of differential operators K via the adjoint representation:

$$K\to K^g=T(g)KT^{-1}(g).$$

This action splits $\mathcal{G}$ into G orbits. For our purposes the operator $K_0=i$ is trivial so we will merely study the orbit structure of the factor algebra $\mathcal{G}'=\mathcal{G}/\{K_0\}$ where $\{K_0\}$ is the center of $\mathcal{G}$.

This computation was carried out by Weisner[6] for the complexification of $\mathcal{G}$ and needs only minor modification to adopt it to $\mathcal{G}$. Let

$$K=A_2K_2+A_1K_1+A_{-1}K_{-1}+A_{-2}K_{-2}+A_3K^3$$

be a nonzero element of $\mathcal{G}'$ and set $\alpha=A_2A_{-2}+A_3^2$. It is straightforward to show that α is invariant under the adjoint representation. In the table below we give a complete set of orbit representatives. That is, K lies on the same G orbit as a real multiple of exactly one of the five operators in the list.

$$\text{Case 1}(\alpha<0):\ K_{-2}-K_2=L_3,$$
$$\text{Case 2}(\alpha>0):\ K_3, \tag{1.14}$$
$$\text{Case 3}(\alpha=0):\ K_2+K_{-1},\ K_{-2},\ K_{-1}.$$

Note that there are essentially five orbits.

It is well-known that knowledge of the symmetry algebra of a differential equation permits one to obtain solutions of the equation via separation of variables.[13,14] Indeed, in our case for given $K\in\mathcal{G}$ and $\lambda\in R$ the system of equations

$$Ku=i\lambda u,\quad Xu=0 \tag{1.15}$$

leads to a separation of variables in the Schrödinger equation. It is clear that two operators K, K' on the same G orbit lead to equivalent separation of variables via (1.15). Furthermore, since $K_{-2}u=iK_{-1}^2u$ whenever $Xu=0$, the orbits containing K_{-1} and K_{-2} lead to essentially equivalent separations. Thus Eqs. (1.15) lead to separation of variables in four distinct coordinate systems associated with the orbit representatives K_3, L_3, K_2+K_{-1}, and K_{-1}. In Sec. 2 we shall classify all coordinate systems in which variables separate for $Xu=0$ and show that there exist only the four obtainable from (1.15). Thus separation of variables for $Xu=0$ is explainable in terms of the symmetry algebra alone. (Note that for equations such as $u_{xx}+u_{yy}+k^2u=0$ and $-iu_t=u_{xx}+u_{yy}$ it is necessary to use quadratic elements in the enveloping algebra of the symmetry algebra to describe separation of variables.[15,16]

The real six-dimensional symmetry algebra $\mathcal{G}''$ of the heat equation

$$u_t = u_{xx} \tag{1.16}$$

can be obtained by a computation analogous to that for the free-particle Schrödinger equation.[4] One finds that the operators

$$\begin{aligned} &K_2' = t^2\partial_t + tx\partial_x + t/2 + x^2/4, \quad K_1' = t\partial x + x/2, \\ &K_0' = 1, \quad K_{-1}' = \partial_x, \quad K_{-2}' = \partial_t, \quad K'^3 = x\partial_x + 2t\partial_t + \tfrac{1}{2} \end{aligned} \tag{1.17}$$

form a basis for $\mathcal{G}''$ where K_0' spans the center of $\mathcal{G}''$ and

$$[K'^3, K_j'] = jK_j', \quad j = \pm 2, \ \pm 1, \ 0, \quad [K_1', K_2'] = [K_{-1}', K_{-2}'] = 0$$

$$[K_{-1}', K_2'] = K_1', \quad [K_{-1}', K_1'] = \tfrac{1}{2}K_0, \quad [K_{-2}', K_1'] = K_{-1}',$$

$$[K_{-2}', K_2'] = K'^3.$$

There are five orbits in $\mathcal{G}''/\{K_0'\}$ under the adjoint representation with corresponding orbit representatives $K_3', K_2' + K_{-2}', K_{-2}' + K_1', K_{-2}', K_{-1}'$. Since $K_{-2}' = (K_{-1}')^2$ for solutions of the heat equation, only four coordinate systems in which variables separate are associated with the five orbits.

2. SEPARATION OF VARIABLES FOR THE EQUATION $XU = 0$ AND THE HEAT EQUATION $u_t = u_{xx}$

In this section we examine the problem of the separation of variables for Eq. (1.2). As opposed to the corresponding problem for the Helmholtz equation there is no established method of approach here (i.e., no associated differential form and corresponding obvious, group of motions as in the case of, say, the Euclidean plane.[17]) We therefore proceed directly and examine the possibilities.

Choosing a new set of real variables v_1 and v_2 where

$$x = G(v_1, v_2), \quad t = H(v_1, v_2) \tag{2.1}$$

and G, H are real invertable functions, Eq. (1.2) can be written in the form

$$(a_{11}\partial_{11} + a_{12}\partial_{12} + a_{22}\partial_{22} + a_1\partial_1 + a_2\partial_2)u = 0, \tag{2.2}$$

where

$$a_{11} = \left(\frac{H_2}{D}\right)^2, \quad a_{12} = -\frac{2H_1H_2}{D^2}, \quad a_{22} = \left(\frac{H_1}{D}\right)^2,$$

and $D = G_1H_2 - H_1G_2$ (subscripts denote differentiation with respect to v_i), a_1 and a_2 are complicated functions whose explicit form we do not need for general G and H. From the form of (2.2) we see that a necessary condition for a separable solution (see definition below) of the form $u = A(v_1)B(v_2)$ is that at least one of the coefficients a_{11}, a_{12}, a_{22} be zero, i.e., either H_1 or H_2 is zero. Without loss of generality we can take $H_1 = 0$ and write $t = v_2$ (as H cannot then be a constant function). With these assumptions (1.2) assumes the form (2.2) where

$$a_{11} = \frac{1}{G_1^2}, \quad a_1 = \frac{iG_2}{G_1} - \frac{G_{11}}{G_1^3}, \quad a_2 = i \tag{2.3}$$

and all other coefficients zero. In order that this equation separate we have the additional constraints

$$\frac{1}{G_1} = f(v_2)g(v_1), \quad \frac{G_2}{G_1} = f^2(v_2)h(v_1). \tag{2.4}$$

From these equations we have

$$G_{12} = \frac{1}{g}\partial_2\left(\frac{1}{f}\right) = f\partial_1\left(\frac{h}{g}\right) \tag{2.5}$$

and hence

$$\frac{1}{f}\partial_2\left(\frac{1}{f}\right) = \tfrac{1}{2}b \tag{2.6}$$

with b a constant real number. There are two cases to consider:

(i) $b \neq 0$. Then $1/f = \sqrt{bv_2 + c}$. Without loss of generality we can take $c = 0$ as our defining equation is translation invariant. The function G then has the form $G = \bar{g}(v_1)v_2^{1/2}$ where $\bar{g}$ is a nonconstant real function. Accordingly we can define $\bar{g}(v_1) = v_1$. The system of coordinates is then

$$t = v_2, \quad x = v_1v_2^{1/2}. \tag{2.7}$$

(ii) $b = 0$. From the equation $G_2 = f(h/g)$ we see that $G = cv_2 + \bar{g}(v_1)$ and hence the coordinate system in this case is

$$t = v_2, \quad x = cv_2 + v_1. \tag{2.8}$$

One point that should be mentioned here is that the full equation does admit a separable solution when the functions A and B are exponentials and the new variables are given by

$$t = av_1 + bv_2, \quad x = cv_1 + dv_2 \tag{2.9}$$

with $ad - bc \neq 0$. In our definition of separation, however, we require that in the associated coordinate system the Eq. (1.2) can be replaced by two ordinary (nontrivial) differential equations in each of the separable variables. Then only the subclass of coordinates given by (2.8) is admissible as strictly separable. We accordingly make no further comment on the choice of variables (2.9).

In addition to considering separable coordinates for (1.2) it is also of interest to consider R-separable solutions of this equation. These are coordinates which admit solutions of the form $\exp[Q(v_1, v_2)]A(v_1)B(v_2)$ where Q is not expressible in the form $g(v_1) + h(v_2)$ and is not a constant. With the inclusion of such a multiplier term e^Q, Eq. (1.2) for the product $A(v_1)B(v_2)$ assumes the form (2.2) with an extra term a_0u added to the left-hand side. The conditions for R-separability are the same as for strict separability so that $a_{22} = a_{12} = 0$.

The nonzero coefficients are given by

$$\begin{aligned} &a_{11} = \frac{1}{G_1^2}, \quad a_1 = \frac{2Q_1}{G_1^2} - i\frac{G_2}{G_1} - \frac{G_{11}}{G_1^3}, \quad a_2 = i, \\ &a_0 = \frac{(Q_{11} + Q_1^2)}{G_1^2} - Q_1\left(i\frac{G_2}{G_1} + \frac{G_{11}}{G_1^3}\right) + iQ_2. \end{aligned} \tag{2.10}$$

The conditions for separability then become upon writing $Q = R + iS$ (R and S real)

$$1/G_1 = f(v_2)/g_1(v_1), \tag{2.11a}$$

$$2R_1/G_1^2 = f^2(v_2)w(v_1), \tag{2.11b}$$

$$(2S_1/G_1^2) - (G_2/G_1) = f^2(v_2)K(v_1). \tag{2.11c}$$

Equation (2.11b) allows us to take $R=0$, since its solution is of the form $r_1(v_1)+r_2(v_2)$. The remaining conditions simplify to

$$\frac{S_1^2}{G_1^2}-S_1\frac{G_2}{G_1}+S_2=f^2(v_2)q(v_1)+p(v_2), \tag{2.12a}$$

$$\frac{S_{11}}{G_1^2}-S_1\frac{G_{11}}{G_1^3}=f^2(v_2)\,r(v_1)+s(v_2). \tag{2.12b}$$

[Note: $g_1(v_1)=\partial_1 g(v_1)$ for some g.] From (2.11a) the form of G is $G=g/f+h(v_2)$ and $g\neq$ const. We are then free to take $g=v_1$. From (2.11c) we see that

$$2S_1=-\frac{f_2}{f^3}v_1+\frac{h_2}{f}+K. \tag{2.13}$$

We can therefore write the form of S as

$$S=-\frac{f_2}{4f^3}v_1^2+\frac{h_2}{2f}v_1. \tag{2.14}$$

[Remember that terms of the form $\bar{g}(v_1)+\bar{h}(v_2)$ in the expression for S can be dropped as they do not contribute to strict R-separation.] We now evaluate the possibilities.

(i) $f=$ const. Then we can put $f=1$. Equation (2.12a) implies $h_{22}=2a\neq 0$. Without loss of generality we can then take $h=av_2^2$. The corresponding coordinate system is

$$t=v_2,\quad x=v_1+av_2^2,\quad a>0, \tag{2.15}$$

and $S=av_1v_2$.

(ii) $f_2/f^3=-\frac{1}{2}a\neq 0$. In this case we can take $f=v_2^{-1/2}$, the constant a being absorbed in the definition of the variable v_1. Substitution into (2.12a) then requires $h_{22}=-\frac{1}{4}bv_2^{-3/2}$ for some constant b, so that

$$h=bv_2^{1/2}+cv_2. \tag{2.16}$$

We may take $b=0$ by redefining v_1. The resulting coordinate system is then

$$t=v_2,\quad x=v_1v_2^{1/2}+cv_2 \tag{2.17}$$

with

$$S=\tfrac{1}{2}cv_1v_2^{1/2}.$$

This is seen to be a generalization of the coordinate system (2.7).

(iii) $f_2/f^3\neq$ const. In this case, substituting into (2.12a) we obtain the equations given below as requirements for the functions f and h:

$$ff_{22}-2f_2^2=\alpha f^6, \tag{2.18a}$$

$$h_{22}=\beta f^3 \tag{2.18b}$$

with α,β real constants. We consider two possibilities.

(1) $\alpha=0$. In this case $f=av_2^{-1}$ and $h=b/v_2+cv_2$. In particular, we can take $a=1$ and $c=0$ effectively absorbing c into the definition of v_1. The resulting coordinate system is

$$t=v_2,\quad x=v_1v_2+\frac{b}{v_2},\quad b\geq 0, \tag{2.19}$$

with

$$S=\tfrac{1}{4}v_2v_1^2-bv_1/2v_2.$$

(2) $\alpha\neq 0$. In this case (2.18a) has the solution

$$f=(av_2^2+b)^{-1/2} \tag{2.20}$$

and h has a solution of the form $h=c(av_2^2+b)^{1/2}+dv_2$. We can put $c=0$, effectively absorbing this term in the definition of v_1. This results in two distinct types of coordinates depending on the relative sign of a and b.

(a) $t=v_2,\quad x=v_1\sqrt{1+v_2^2}+dv_2$ (2.21)

where

$$S=\tfrac{1}{4}v_1^2v_2+\tfrac{1}{2}dv_1\sqrt{1+v_2^2}$$

and

(b) $t=v_2,\quad x=v_1\sqrt{1-v_2^2}+dv_2$ (2.22a)

with

$$S=-\tfrac{1}{4}v_1^2v_2+\tfrac{1}{2}dv_1\sqrt{1-v_2^2},$$

$$t=v_2,\quad x=v_1\sqrt{v_2^2-1}+dv_2 \tag{2.22b}$$

with

$$S=\tfrac{1}{4}v_1^2v_2+\tfrac{1}{2}dv_1\sqrt{v_2^2-1}.$$

The coordinate system (b) is the only system which requires two distinct parametrizations to cover the entire range of variation of v_2. This then exhausts the classification of all coordinate systems which are R-separable and separable for (1.2). In particular, it is to be noticed that in each case the operator $X=\partial_{xx}+i\partial_t$ can be written $X=f(v_1,v_2)(L+K)$ where L and K are operators in v_1 and v_2, respectively. In particular, K is a first order operator such that $XKu=0$ and so can always be expressed as a linear combination of the generators K_i. In Table I we give all the coordinate systems we have found together with the associated operators K. It is clear that in this classification we have not made use of the full invariance group of (1.2) apart from translational invariance. If we do include this group in our definition of equivalence all the coordinate systems we have found are equivalent to ones whose representative basis defining operators are one of the forms (1.14). In particular, we see that under this equivalence more than one coordinate system may be on the same orbit. This is a consequence of the fact that the group action has not been accounted for in the classification of sep-

TABLE I. Separable coordinate systems for the Schrödinger equation $Xu=0$ and their associated basis defining operators. (Note only the x coordinate is given as we always have $t=v_2$.)

Coordinate system	Multiplier e^{iS}	Basis operator K
1. $x=cv_2+v_1,\ c\geq 0$	$S=0$	$K=K_{-2}+cK_{-1}$
2. $x=v_1+av_2^2,\ a>0$	$S=av_1v_2$	$K=K_{-2}-2aK_1$
3. $x=v_1v_2^{1/2}+cv_2,\ c\in R$	$S=\frac{1}{2}cv_1v_2^{1/2}$	$K=K^3-cK_1$
4. $x=v_1v_2+b/v_2,\ b\geq 0$	$S=\frac{1}{4}v_2v_1^2-bv_1/2v_2$	$K=K_2+2bK_{-1}$
5. $x=v_1\sqrt{1+v_2^2}+dv_2,\ d\geq 0$	$S=\frac{1}{4}v_1^2v_2+\frac{1}{2}dv_1\sqrt{1+v_2^2}$	$K=K_2-K_{-2}-dK_{-1}$
6. $x=v_1\sqrt{1-v_2^2}+dv_2$	$S=-\frac{1}{4}v_1^2v_2+\frac{1}{2}dv_1\sqrt{1-v_2^2}$	$K=K_2+K_{-2}+dK_{-1}$
$x=v_1\sqrt{v_2^2-1}+dv_2,\ d\geq 0$		$S=\frac{1}{4}v_1^2v_2+\frac{1}{2}dv_1\sqrt{v_2^2-1}$

TABLE II. Separable coordinate systems for the heat equation $U_t = U_{xx}$ (for all multipliers $S=0$).

Coordinate system	Multiplier	Operator
1. $x=v_1$	0	K'_{-2}
2. $x=v_1 v_2^{1/2}$	0	K'^3
3. $x=v_1\sqrt{1+v_2^2}$	$R=-\frac{1}{4}v_2 v_1^2$	$K'_2+K'_{-2}$
4. $x=v_1+\frac{1}{2}v_2^2$	$R=-\frac{1}{2}v_1 v_2$	$K'_{-2}+K'_1$

arable systems. In the next section we deal with those bases corresponding to inequivalent orbits. In that section we give the solutions of (1.2) in the corresponding coordinates.

Finally, in this section we list in Table II the separable coordinate systems for the heat equation (1.16) corresponding to representatives of the inequivalent orbits of basis defining symmetry operators.

3. ONE AND TWO-VARIABLE MODELS

We now show that the operators (1.5) can be interpreted as a Lie algebra of skew-Hermitian operators on the Hilbert space $L_2(R)$ of complex-valued Lebesgue square-integrable functions on the real line. To do this we consider t as a fixed parameter and, in view of (1.2), replace ∂_t by $i\partial_{xx}$ in expressions (1.5). It is easy to show that the resulting operators restricted to the domain of C^∞-functions with compact support and multiplied by i are symmetric and essentially self-adjoint. Indeed the operators (1.5) are real linear combinations of the operators

$$\mathcal{K}_2=ix^2/4,\quad \mathcal{K}_1=ix/2,\quad \mathcal{K}_0=i,\quad \mathcal{K}_{-1}=\partial_x,\quad \mathcal{K}_{-2}=i\partial_{xx}$$
$$\mathcal{K}^3=x\partial_x+\tfrac{1}{2} \tag{3.1}$$

and $i\mathcal{K}_j$, $i\mathcal{K}^3$ are essentially self-adjoint. Moreover, when the parameter t is set equal to zero, K_j becomes $\mathcal{K}_j$ and K^3 becomes $\mathcal{K}^3$. It follows that the operators $\mathcal{K}_j$, $\mathcal{K}_3$ satisfy the commutation relations (1.6).

From Stone's theorem[18] we know that to each skew-Hermitian $\mathcal{H}\in\mathcal{G}$ there corresponds a one-parameter group $U(\alpha)=\exp(\alpha\mathcal{H})$ of unitary operators on $L_2(R)$. This group in turn acts on $\mathcal{G}$ via $\mathcal{K}\to U(\alpha)\mathcal{K}U(-\alpha)$. In particular, one can easily verify that

$$\begin{aligned}&[\exp(t\mathcal{K}_{-2})]\mathcal{K}_j[\exp(-t\mathcal{K}_{-2})]=K_j,\\ &[\exp(t\mathcal{K}_{-2})]\mathcal{K}^3[\exp(-t\mathcal{K}_{-2})]=K^3.\end{aligned} \tag{3.2}$$

Thus if $f\in L_2(R)$ then $u=\exp(t\mathcal{K}_{-2})f$ satisfies $u_t=\mathcal{K}_{-2}u$ or $iu_t=-u_{xx}$ (for almost every t) whenever f is in the domain of $\mathcal{K}_{-2}$, and $u(0)=f$. Also it is easy to show that the unitary operators $\exp(\alpha K)$ $=\exp(t\mathcal{K}_{-2})\exp(\alpha\mathcal{K})\exp(-t\mathcal{K}_{-2})$ map such a u into v $=\exp(\alpha\mathcal{K})u$ which also satisfies $v_t=\mathcal{K}_{-2}v$. Thus the unitary operators $\exp(\alpha K)$ are symmetries of (1.2).

Later we will show that the operators $\mathcal{K}_j$, $\mathcal{K}^3$ generate a global unitary representation of the group G on $L_2(R)$. Assuming this for the moment, let $U(g)$, $g\in G$, be the corresponding unitary operators and set $T(g)$ $=\exp(t\mathcal{K}_{-2})U(g)\exp(-t\mathcal{K}_{-2})$. Again it is easy to demonstrate that the $T(g)$ are unitary symmetries of (1.2) and that the associated infinitesimal operators are $K=\exp(t\mathcal{K}_{-2})\mathcal{K}\exp(-t\mathcal{K}_{-2})$.

Now consider the operator $\mathcal{L}_3=\mathcal{K}_{-2}-\mathcal{K}_2=i\partial_{xx}-ix^2/4$ $\in\mathcal{G}$. If $f\in L_2(R)$ then $u(t)=\exp(t\mathcal{L}_3)f$ satisfies $u_t=\mathcal{L}_3u$ or $iu_t=-u_{xx}+x^2u/4$ and $u(0)=f$. Similarly, the unitary operators $V(g)=\exp(t\mathcal{L}_3)U(g)\exp(-t\mathcal{L}_3)$ are symmetries of this equation, the Schrödinger equation for the harmonic oscillator, and one can verify that the associated infinitesimal operators $\exp(t\mathcal{L}_3)\mathcal{K}\exp(-t\mathcal{L}_3)$ can be expressed as first order differential operators in t and x. Continuing in this manner we consider the operator $\mathcal{L}_2=\mathcal{K}_{-2}+\mathcal{K}_2=i\partial_{xx}-ix^2/4\in\mathcal{G}$. If $f\in L_2(R)$ then $u(t)$ $=\exp(t\mathcal{L}_2)f$ satisfies $u_t=\mathcal{L}_2u$ or $iu_t=-u_{xx}-x^2u/4$ and $u(0)=f$. The operators $W(g)=\exp(t\mathcal{L}_2)U(g)\exp(-t\mathcal{L}_2)$ form the unitary symmetry group of this equation, repulsive harmonic oscillator potential, and the associated infinitesimal operators $\exp(t\mathcal{L}_2)\mathcal{K}\exp(-t\mathcal{L}_2)$ are first order in x and t. Finally, we consider the operator $\mathcal{H}$ $=\mathcal{K}_{-2}-\mathcal{K}_1=i\partial_{xx}-ix/2\in\mathcal{G}$. If $f\in L_2(R)$ then $u(t)=\exp(t\mathcal{H})$ satisfies $u_t=\mathcal{H}_u$ or $iu_t\pm-u_{xx}+xu/2$ and $u(0)=f$. The unitary operators $X(g)=\exp(t\mathcal{H})\mathcal{K}\exp(-t\mathcal{H})$ are symmetries of this Schrödinger equation for the linear potential and the infinitesimal operators $\exp(t\mathcal{H})\mathcal{K}\exp(-t\mathcal{H})$ are first order in x and t.

Note further from (1.14) the operators $\mathcal{K}_{-2}$, $\mathcal{L}_3$, $\mathcal{L}_2$, and $\mathcal{K}_{-2}-\mathcal{K}_1$ corresponding to the free particle, attractive and repulsive harmonic oscillator, and linear potential Hamiltonians, lie on the same G orbits as the four representatives $\mathcal{K}_{-2}$, $\mathcal{L}_3$, $\mathcal{K}_3$ and $\mathcal{K}_2+\mathcal{K}_{-1}$, respectively. Thus these four Hamiltonians correspond exactly to the four systems of coordinates in which Eq. (1.2) separates. We see that these Hamiltonians form a complete set of orbit representatives in $\mathcal{G}$ in the sense explained following Eq. (1.15).

Note that if two operators lie on the same G orbit then the first operator is unitary equivalent to a real constant times the second operator. Thus two suitably normalized operators on the same orbit necessarily have the same spectrum. In particular, if $\mathcal{K},\mathcal{K}'\in\mathcal{G}$ with $\mathcal{K}'=U(g)\mathcal{K}U(g^{-1})$ and the self-adjoint operator $i\mathcal{K}$ has a complete set of (possibly generalized) eigenvectors $f_\lambda(x)$ with

$$i\mathcal{K}f_\lambda=\lambda f_\lambda,\quad (f_\lambda,f_\mu)=\delta_{\lambda\mu} \tag{3.3}$$

where

$$(h_1,h_2)=\int_{-\infty}^{\infty}h_1(x)\overline{h_2(x)}\,dx,\quad h_j\in L_2(R), \tag{3.4}$$

then for $f'_\lambda=U(g)f_\lambda$ we have

$$i\mathcal{K}'f'_\lambda=\lambda f'_\lambda,\ (f'_\lambda,f'_\mu)=\delta_{\lambda\mu} \tag{3.5}$$

and the f'_λ form a complete set of eigenvectors for $i\mathcal{K}'$.[19] These remarks imply that, if we wish to compute the spectrum corresponding to each operator $\mathcal{K}\in\mathcal{G}$, it is enough to determine the spectra of the four Hamiltonians listed above. Moreover, we may be able to choose another operator $\mathcal{K}$ on the same G orbit as a given Hamiltonian such that the spectral decomposition of $\mathcal{K}$ is especially easy. The spectral decomposition of the Hamiltonian and the corresponding eigenfunction expansions then follow from those of $\mathcal{K}$ by application of a group operator $U(g)$.

As a special case of these remarks consider the operator $\mathcal{K}_{-2}=i\partial_{xx}$. If $\{f_\lambda\}$ is the basis of generalized eigenvectors for some operator $\mathcal{K}\in\mathcal{G}$, then $\{f'_\lambda(t)$

$=\exp(t\mathcal{K}_{-2})f_\lambda\}$ is the basis of generalized eigenvectors for $K=\exp(t\mathcal{K}_{-2})\mathcal{K}\exp(-t\mathcal{K}_{-2})$ and the $f'_\lambda(t)$ satisfy the equation $iu_t=-u_{xx}$. Similar remarks hold for the other Hamiltonians.

We begin our explicit computations by determining the spectral resolution of the operator $\mathcal{L}_3=\mathcal{K}_{-2}-\mathcal{K}_2$. The results are well-known.[18] The eigenfunction equation is

$$i\mathcal{L}_3 f=\lambda f,\quad (-\partial_{xx}+x^2/4)f=\lambda f,$$

and the normalized eigenfunctions are

$$f^{(1)}_{\lambda_n}(x)=[n!\,\sqrt{2\pi}\,2^n]^{-1/2}\exp(-x^2/4)H_n(x2^{-1/2}),$$
$$\lambda_n=n+\tfrac12,\quad n=0,1,2,\ldots,\quad (f^{(1)}_{\lambda_n},f^{(1)}_{\lambda_m})=\delta_{nm} \tag{3.6}$$

where $H_n(x)$ is a Hermite polynomial.

It is now easy to show that the $\mathcal{K}$ operators exponentiate to a global unitary irreducible representation of G. Indeed, from the known recurrence formulas for the Hermite polynomials one can check that the operators $\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3$ acting on the $f^{(1)}$-basis define a reducible representation of $sl(2,R)$ belonging to the discrete series. The value of the Casimir operator is $\frac14(\mathcal{L}_1^2+\mathcal{L}_2^2-\mathcal{L}_3^2)=-3/16$. As first shown by Bargmann,[20] this Lie algebra representation extends to a global unitary reducible representation of $SL(2,R)$. Similarly, the operators $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{L}_3$, acting on the $f^{(1)}$-basis define the irreducible representation $(\lambda,l)=(-\frac12,1)$ of the Lie algebra of the harmonic oscillator group S.[21] Again this Lie algebra representation is known to generate a global unitary irreducible representation of S.[21,22] Finally, since every unitary operator from $SL(2,R)$ can be written in the form $\exp(\alpha\mathcal{L}_3)\exp(\beta\mathcal{L}_1)\exp(\gamma\mathcal{L}_3)$,[20] where $\exp(\alpha\mathcal{L}_3)$ also belongs to S, and since $\mathcal{L}_1$ is a first order operator whose exponential is easily determined, we can check that the identity (1.12) holds in general. Thus our representation of G extends to a global unitary representation U of G which is irreducible since $U|S$ is already irreducible. The matrix elements of the operators $U(g)$ in the $f^{(1)}$-basis can be found in numerous references, e.g., Refs. 20, 22, 23.

The unitary operators $U(g)$ on $L_2(R)$ are easily computed. The operators

$$U(u,v,\rho)=\exp[\rho+(uv/4)]\,\mathcal{E}\exp(u\mathcal{S}_2)\exp(v\mathcal{S}_1)$$

defining an irreducible representation of W take the form

$$[U(u,v,\rho)f](x)=\exp\left[i\left(\rho+\frac{uv}{4}+\frac{ux}{2}\right)\right]f(x+v),\quad f\in L_2(R). \tag{3.7}$$

The operators $U(A)$, $A\in SL(2,R)$, are more complicated. From Ref. 24 (p. 493) we have

$$\exp(a\mathcal{K}_{-2})f(x)=\text{l.i.m.}\,\frac{1}{\sqrt{4\pi i a}}\int_{-\infty}^{\infty}\exp[-(x-y)^2/4ia]f(y)\,dy, \tag{3.8}$$

and it is elementary to show

$$\exp(b\mathcal{K}^3)f(x)=\exp(b/2)f(e^b x),$$
$$\exp(c\mathcal{K}_2)f(x)=\exp(icx^2/4)f(x). \tag{3.9}$$

Relations (1.11) imply

$$\exp(\phi\mathcal{L}_2)=\exp(\tanh\phi\,\mathcal{K}_2)\exp(\sinh\phi\cosh\phi\,\mathcal{K}_{-2})\times\exp(-\ln\cosh\phi\,\mathcal{K}_3),$$

so (3.8) and (3.9) yield

$$\exp(\phi\mathcal{L}_2)f(x)=\frac{\exp[(ix^2/4)\tanh\phi]}{(4\pi i\sinh\phi)^{1/2}}\times\text{l.i.m.}\int_{-\infty}^{\infty}\exp[-(x-y\cosh\phi)^2/4i\sinh\phi\cosh\phi]f(y)\,dy. \tag{3.10}$$

A similar computation for $\exp(\theta\mathcal{L}_3)$ gives

$$\exp(\theta\mathcal{L}_3)f(x)=\frac{\exp[(ix^2/4)\cot\theta]}{(4\pi i\sin\theta)^{1/2}}\times\text{l.i.m.}\int_{-\infty}^{\infty}\exp[-(y^2\cos\theta-2xy)/4i\sin\theta]f(y)\,dy. \tag{3.11}$$

Using (3.8) we see that the basis functions $f^{(1)}_{\lambda_n}(x)$ map to the ON basis functions $F^{(1)}_{\lambda_n}(x,t)=\exp(t\mathcal{K}_{-2})f^{(1)}_{\lambda_n}(x)$ or

$$F^{(1)}_{\lambda_n}(x,t)=[n!\,2^n\sqrt{2\pi(1+t^2)}]^{-1/2}\exp\left(\frac{i}{4}\frac{x^2t}{1+t^2}-\frac{x^2}{4(1+t^2)}-i\lambda_n\arctan t\right)H_n[x/\sqrt{2(1+t^2)}] \tag{3.12}$$

which are solutions of (1.2).

Next we study the spectral theory for the orbit containing the operators $\mathcal{K}_{-2}+\mathcal{K}_2$ (repulsive oscillator) and $\mathcal{K}_3$. Since the spectral analysis for $\mathcal{K}_3$ is elementary we study it first. [The corresponding results for $\mathcal{K}_{-2}+\mathcal{K}_2$ then follow by application of an appropriate group operators $U(g)$.] The eigenfunction equation is

$$i\mathcal{K}^3f=\lambda f,\quad \mathcal{K}^3=x\partial_x+\tfrac12.$$

The spectral resolution for this operator is well-known.[25] It is obtained by considering $L_2(R)$ as the direct sum $L_2(R+)\oplus L_2(R-)$ of square-integrable functions on the positive and negative reals, respectively, and taking the Mellin transform of each component. Then $i\mathcal{K}_3$ transforms into multiplication by the transform variable. The spectrum is continuous and covers the real axis with multiplicity two. The generalized eigenfunctions are

$$f^{(2)\pm}_\lambda(x)=\frac{1}{\sqrt{2\pi}}x_\pm^{-i\lambda-1/2},\quad \lambda\in R, \tag{3.13}$$
$$(f^{(2)\pm}_\lambda,f^{(2)\pm}_\mu)=\delta(\mu-\lambda),\quad (f^{(2)\pm}_\lambda,f^{(2)\mp}_\mu)=0,$$

where

$$x_+^\alpha=\begin{cases}x^\alpha & \text{if } x>0\\ 0 & \text{if } x<0,\end{cases}\qquad x_-^\alpha=\begin{cases}0 & \text{if } x>0\\ (-x)^\alpha & \text{if } x<0\end{cases}.$$

From (3.8) we find $\exp(t\mathcal{K}_{-2})f^{(2)\pm}_\lambda=F^{(2)\pm}_\lambda(x,t)$ where

$$F^{(2)\pm}_\lambda(x,t)=\exp\left(\frac{x^2}{4it}+\frac{\pi\lambda}{4}+\frac{i\pi}{8}\right)\times\frac{(2t)^{-i\lambda/2+1/4}}{\sqrt{8\pi^2 it}}\,\Gamma(\tfrac12-i\lambda)D_{i\lambda-\frac12}\left(\frac{\mp xe^{-i\pi/4}}{\sqrt{2t}}\right),\quad t>0, \tag{3.14}$$

$\Gamma(z)$ is a gamma function, and $D_\nu(z)$ is a parabolic cylinder function.[10] [These results follow from (3.8) by moving the integration contour from the positive real axis to a ray making an angle of $\pi/4$ with the real axis. We can also use the fact that we know the differential equations characterizing the function (3.14).] Also, we have

$$\text{(a) } F_\lambda^{(2)+}(x,t) = F_{-\lambda}^{(2)+}(x,-t), \qquad \text{(b) } F_\lambda^{(2)-}(x,t) = F_\lambda^{(2)+}(-x,t). \tag{3.15}$$

It follows immediately from (3.13) that

$$(F_\lambda^{(2)\pm}, F_\mu^{(2)\pm}) = \delta(\mu-\lambda), \quad (F_\lambda^{(2)\pm}, F_\mu^{(2)\mp}) = 0. \tag{3.16}$$

Application of these orthogonality and completeness relations to expand an arbitrary $f \in L_2(R)$ yields the Hilbert space version of Cherry's theorem,[10,26] which is an expansion in terms of parabolic cylinder functions. Note that our expansion is simply related to the spectral resolution of the operator $K^3 = 2t\partial_t + x\partial_x + \frac{1}{2} = 2it\partial_{xx} + x\partial_x + \frac{1}{2}$.

The next orbit we consider contains the operators $\mathcal{K}_{-2} + \mathcal{K}_1$ (linear potential) and $\mathcal{K}_2 + \mathcal{K}_{-1}$. Since the spectral analysis for the second operators is simpler, we study it. The eigenfunction equation is

$$i(\mathcal{K}_2 + \mathcal{K}_{-1})f = \lambda f, \quad \mathcal{K}_2 + \mathcal{K}_{-1} = ix^2/4 + \partial_x.$$

The spectral resolution is easily obtained from the Fourier integral theorem. The spectrum is continuous and covers the real axis, and the generalized eigenfunctions are

$$f_\lambda^{(3)}(x) = \frac{1}{\sqrt{2\pi}} \exp[-i(\lambda x + x^3/12)], \quad \lambda \in R, \tag{3.17}$$

$$(f_\lambda^{(3)}, f_\mu^{(3)}) = \delta(\mu - \lambda).$$

We find that

$$F_\lambda^{(3)}(x,t) = \exp(-i\pi/4)\, 2^{1/6} \exp\left[\frac{i}{4}\left(-\frac{1}{8v_2^2} + v_2 v_1^2 - \frac{v_1}{v_2}\right) - \frac{i\lambda}{v_2}\right] \times \mathrm{Ai}[2^{2/3}(\tfrac{1}{2}v_1 + \lambda)] \tag{3.18}$$

with v_1 and v_2 as in Table I, system 4 with $b = \frac{1}{2}$.

$\mathrm{Ai}(z)$ is a Airy function. These are the basis functions for the operator $K_2 + K_{-1} = -it^2\partial_{xx} + (1 - tx)\partial_x - t/2 + ix^2/4$. For the orbit containing K_{-1} the complete set of eigenfunctions is

$$f^{(4)} = \frac{1}{\sqrt{2\pi}} \exp(-i\lambda x), \quad \lambda \in R, \tag{3.19}$$

with the usual orthogonality properties. It is not hard to show that

$$F_\lambda^{(4)}(x,t) = \frac{1}{\sqrt{2\pi}} \exp[i(\lambda^2 t - \lambda x)]. \tag{3.20}$$

The case of the remaining orbit K_{-2} differs so little from this last case that we do not treat it here.

If $\{f_\lambda(x)\}$ is a basis of (generalized) eigenfunctions of some $\mathcal{K} \in \mathcal{G}$ and $F_\lambda(x,t) = \exp(t\mathcal{K}_{-2}) f_\lambda(x)$ then $F_\lambda(x,\tau) = \exp([\tau - t]\mathcal{K}_{-2})\, F_\lambda(x,t)$ and we have the Hilbert space expansions

$$k(x-y,t) = \int F_\lambda(x,t)\overline{f_\lambda(y)}\, d\lambda, \tag{3.21}$$

$$k(x-y,\tau-t) = \int F_\lambda(x,\tau)\overline{F_\lambda(y,t)}\, d\lambda$$

where the integration domain is the spectrum of $i\mathcal{K}$ and

$$k(x,t) = \frac{1}{\sqrt{4\pi i t}} \exp(-x^2/4it)$$

is the kernel of the integral operator $\exp(t\mathcal{K}_{-2})$. These expansions are known as continuous generating functions.[7,8]

4. OVERLAP FUNCTIONS

In this section we compute the overlap functions $(f_\lambda^{(i)}, f_\mu^{(j)})$ which allow us to expand eigenfunctions $f_\lambda^{(i)}$ in terms of eigenfunctions $f_\mu^{(j)}$. Since $(U(g)f_\lambda^{(i)}, U(g)f_\mu^{(j)}) = (f_\lambda^{(i)}, f_\mu^{(j)})$, the same expressions allow us to expand eigenfunctions $U(g)f_\lambda^{(i)}$ in terms of eigenfunctions $U(g)f_\mu^{(j)}$. We give here then those overlap functions corresponding to bases $f_\lambda^{(i)}$ that we have taken as standard:

$$(f_{\lambda_n}^{(1)}, f_\lambda^{(2)\pm}) = \frac{(\pm 2)^{n+i\lambda-1/2}\,\Gamma(i\lambda/2 + \frac{1}{4} + \frac{1}{2}n)}{2\pi\sqrt{2^n n!}} \times {}_2F_1(-\tfrac{1}{2}n, \tfrac{1}{2} - \tfrac{1}{2}n, \tfrac{3}{4} - i\lambda/2 - \tfrac{1}{2}n; \tfrac{1}{2}). \tag{4.1}$$

For the calculation of the overlap functions $(f_{\lambda_n}^{(1)}, f_\lambda^{(3)})$ it is convenient to give a generating function rather than an explicit expression. The result is

$$2^{2/3}\exp[-i(\tfrac{1}{6} + \lambda + \sqrt{2y})]\,\mathrm{Ai}[2^{2/3}(\tfrac{1}{4} - i\lambda - i\sqrt{2y})] = \sum_{n=0}^\infty \frac{(\sqrt{2}iy)^n}{\sqrt{n!}} (f_{\lambda_n}^{(1)}, f_\lambda^{(3)}). \tag{4.2}$$

This expression follows from the form of the generating function of Hermite polynomials given by Ref. 10.

$$(f_{\lambda_n}^{(1)}, f_\lambda^{(4)}) = [n!(-2)^n\pi]^{-1/2}\exp(-\lambda^2)H_n(\sqrt{2}\lambda), \tag{4.3}$$

$$(f_{\lambda'}^{(3)}, f_\lambda^{(2)+}) = \frac{1}{2\pi}(12i)^{(1/6-i\lambda/3)}\sum_{n=0}^\infty \frac{\Gamma\{[(n-i\lambda)/3] + \frac{1}{6}\}}{n!} \times [\exp(5i\pi/6)\lambda']^n (12)^{n/3}, \tag{4.4}$$

where

$$(f_{\lambda'}^{(3)}, f_\lambda^{(2)-}) = (-1)^{i\lambda-1/2}\,\overline{(f_{\lambda'}^{(3)}, f_{-\lambda}^{(2)+})}. \tag{4.5}$$

$$(f_{\lambda'}^{(3)}, f_\lambda^{(4)}) = 2^{2/3}\mathrm{Ai}(2^{2/3}[\lambda - \lambda']),$$

$$(f_{\lambda'}^{(4)}, f_{\lambda'}^{(2)\pm}) = \frac{\pm 1}{2\pi}\exp(\mp i\lambda'\pi/2)\,\Gamma(-\lambda'+1)(\lambda \pm i0)^{-\lambda'-1}. \tag{4.6}$$

The general overlap function relating an eigenbasis on one orbit to an eigenbasis on another orbit is of the form $(U(g)f_\lambda^{(i)}, f_\mu^{(j)})$. Indeed, a general eigenbasis $\{h_\lambda^{(i)}\}$ on orbit i can be expressed as $h_\lambda^{(i)} = U(g_h)f_\lambda^{(i)}$. Thus, $(h_\lambda^{(i)}, k_\mu^{(j)}) = (U(g_h)f_\lambda^{(i)}, U(g_k)f_\mu^{(j)}) = (U(g_k^{-1}g_h)f_\lambda^{(i)}, f_\mu^{(j)})$. These expressions are known as "mixed basis matrix elements."[27] Their knowledge allows us to expand any eigenfunction of an operator in $\mathcal{G}$ in terms of eigenfunctions of any other operator in $\mathcal{G}$. Since the inner product is invariant under the unitary operators $U(g)$, the knowledge of the matrix elements for fixed i, j, and g can lead to a variety of different expansions. We shall not tabulate these elements here but merely note that they are of some interest. Indeed, they yield Hilbert space analogies of the analytic function expansions derived by Weisner in Ref. 6. However, the Hilbert space theory is richer and more complicated since one can derive expansions in all bases, not just Hermite function bases as used by Weisner.

As an example we give the mixed basis elements:

$$(\exp(t\mathcal{K}_{-2})f^{(1)}_{\lambda_n},\ f^{(2)\pm}_{\mu}) = (f^{(1)}_{\lambda_n},\ \exp(-t\mathcal{K}_{-2})f^{(2)\pm}_{\mu})$$

$$=\frac{(\pm 2)^{n+i\mu-1/2}(1+it)^{i\mu/2}\exp(-i\lambda_n \arctan t)}{2\pi\sqrt{2^n n!}\,(1-it)^{n/2+i\mu/2+1/4}}$$

$$\times\Gamma\left(\frac{i\mu}{2}+\frac{1}{4}+\frac{n}{2}\right)$$

$$\times{}_2F_1\left(-\frac{n}{2},\ \frac{1}{2}-\frac{n}{2},\ \frac{3}{4}-\frac{i\mu}{2}-\frac{n}{2};\ \frac{1-it}{2}\right).$$

These elements allow us to expand Hermite polynomials as an integral over parabolic cylinder functions and parabolic cylinder functions in series of Hermite polynomials.

5. THE EQUATION $iu_t + u_x - cu/x^2 = 0$

Here we apply the methods discussed in the previous sections to the differential operator

$$Y = i\partial_t + \partial_{xx} - c/x^2, \quad c \neq 0. \tag{5.1}$$

We first compute the maximal symmetry algebra of the equation $Yu=0$. Thus, we find all operators L, Eq. (1.3), such that $Y(Lu)=0$ whenever $Yu=0$. A straightforward calculation shows that the symmetry algebra $\mathcal{H}^c$ is three-dimensional with basis

$$K_{-2}=\partial_t, \quad K_2 = -t^2\partial_t - tx\partial_x - t/2 + ix^2/4, \tag{5.2}$$
$$K^3 = 2t\partial_t + x\partial_x + 1/2$$

and commutation relations

$$[K^3, K_{\pm 2}] = \pm 2K_{\pm 2}, \quad [K_2, K_{-2}] = K^3.$$

For the basis L_j where

$$L_1 = K^3, \quad L_2 = K_{-2} + K_2, \quad L_3 = K_{-2} - K_2,$$

we have the relations

$$[L_1, L_2] = -2L_3, \quad [L_3, L_1] = 2L_2, \quad [L_3, L_2] = -2L_1. \tag{5.3}$$

It is clear that the real Lie algebra generated by these basis elements is $sl(2,R)$. The corresponding group action of $SL(2,R)$ on functions $f(x,t)$ is given by the operators (1.10), and the explicit relation between the group and Lie algebra operators by (1.11).

The group $SL(2,R)$ acts on $sl(2,R)$ via the adjoint representation and splits the Lie algebra into orbits. Let

$$K = A_2K_2 + A_{-2}K_{-2} + A_3K^3 \in sl(2,R)$$

and set $\alpha = A_2A_{-2} + A_3^2$. It is straightforward to check that α is invariant under the adjoint representation and that K lies on the same $SL(2,R)$ orbit as a real multiple of exactly one of the three operators in the following list:

$$\text{Case } 1(\alpha<0):\ K_{-2}-K_2 = L_3,$$
$$\text{Case } 2(\alpha>0):\ K^3, \tag{5.4}$$
$$\text{Case } 3(\alpha=0):\ K_2.$$

We see that there are essentially three orbits.

The evaluation of all separable coordinate systems proceeds as for the free particle case except that now we have the added restriction that $G_1/G = h(u_1)$. The resulting coordinate systems, multipliers, and basis defining operator are then listed in Table III.

In analogy with our argument in Sec. 3 we can interpret the operators (5.2) as a Lie algebra of skew-Hermitian operators on the Hilbert space $L_2(R+)$ of complex-valued Lebesgue square-integrable functions $f(x)$ on the positive real line, $0 < x < \infty$. This is accomplished by considering t as a fixed parameter and replacing ∂_t by $i\partial_{xx} - ic/x^2$ in expressions (5.2). The resulting operators when multiplied by i and restricted to the domain of C^∞ functions with compact support in $R+$ are via Weyl's lemma,[28] easily seen to be essentially self-adjoint provided $c \geq 2$. In the remainder of this paper we assume that the constant c satisfies this inequality. The operators $K_{\pm 2}, K^3$ are real linear combinations of the skew-Hermitian operators

$$\mathcal{K}_{-2} = i\partial_{xx} - ic/x^2, \quad \mathcal{K}_2 = ix^2/4, \quad \mathcal{K}^3 = x\partial_x + 1/2 \tag{5.5}$$

to which they reduce when $t=0$. Similarly, the skew-Hermitian operators

$$\mathcal{L}_1 = \mathcal{K}^3 = x\partial_x + \tfrac{1}{2}, \quad \mathcal{L}_2 = \mathcal{K}_{-2} + \mathcal{K}_2 = i\partial_{xx} - ic/x^2 + ix^2/4, \tag{5.6}$$
$$\mathcal{L}_3 = \mathcal{K}_{-2} - \mathcal{K}_2 = i\partial_{xx} - ic/x^2 - ix^2/4$$

satisfy relations (5.3) and the L_j reduce to $\mathcal{L}_j$ when $t=0$.

In analogy with Sec. 3, one finds

$$\exp(t\mathcal{K}_{-2})\mathcal{K}_j\exp(-t\mathcal{K}_{-2}) = K_j,$$
$$\exp(t\mathcal{K}_{-2})\mathcal{L}_j\exp(-t\mathcal{K}_{-2}) = L_j. \tag{5.7}$$

Thus for any $f \in L_2(R+)$ the vector $u(t) = \exp(t\mathcal{K}_{-2})f$ satisfies $u_t = \mathcal{K}_{-2}u$ or $iu_t = -u_{xx} + cu/x^2$ and $u(0)=f$. Also the unitary operators $\exp(\alpha K) = \exp(t\mathcal{K}_{-2})\exp(\alpha\mathcal{K})\exp(-t\mathcal{K}_{-2})$, $\mathcal{K} \in sl(2,R)$, map solutions of the equation $u_t = \mathcal{K}_{-2}u$ into other solutions.

We will soon demonstrate that the operators $\mathcal{K}_{\pm 2}$, $\mathcal{K}^3$ generate a global unitary irreducible representation of the universal covering group J of $SL(2,R)$ by operators $U(g)$, $g \in J$, on $L_2(R+)$. Assuming this we see that the operators $T(g) = \exp(t\mathcal{K}_{-2})U(g)\exp(-t\mathcal{K}_{-2})$ define a group of unitary symmetries of the equation $Yu=0$, with associated infinitesimal operators $K = \exp(t\mathcal{K}_{-2})\mathcal{K}\exp(-t\mathcal{K}_{-2})$. This discussion shows the relationship between our Lie algebra of $\mathcal{K}$-operators and the Schrödinger equation for the radial free particle.

Next consider the operator $\mathcal{L}_3 \in sl(2,R)$. If $f \in L_2(R+)$ then $u(t) = \exp(t\mathcal{L}_3)f$ satisfies $u_t = \mathcal{L}_3u$ or $iu_t = -u_{xx} + cu/x^2 + x^2u/4$, the Schrödinger equation for the radial harmonic oscillator. The unitary operators $V(g) = \exp(t\mathcal{L}_3)U(g)\exp(-t\mathcal{L}_3)$ are symmetries of this equation and the associated infinitesimal operators

TABLE III. Separable coordinate systems for the equation $Yu=0$.

Coordinate	Multiplier e^{iS}	Basis operator
1. $x = v_1$	$S=0$	K_{-2}
2. $x = v_1v_2^{1/2}$	$S=0$	K^3
3. $x = v_1v_2$	$S=\frac{1}{4}v_2v_1^2$	K_2
4. $x = v_1\sqrt{1+v_2^2}$	$S=\frac{1}{4}v_2v_1^2$	$K_2 - K_{-2}$
5. $x = v_1\sqrt{\pm(1-v_2^2)}$	$S=\pm\frac{1}{4}v_2v_1^2$	$K_2 + K_{-2}$

$\exp(t\mathcal{L}_3)\mathcal{K}\exp(-t\mathcal{L}_3)$ are first order linear differential operators in x and t. Similarly, if $f\in L_2(R+)$ then $u(t)=\exp(t\mathcal{L}_2)f$ satisfies $u_t=\mathcal{L}_2u$ or $iu_t=-u_{xx}+cu/x^2-x^2u/4$, the Schrödinger equation for the repulsive radial oscillator. The operators $W(g)=\exp(t\mathcal{L}_2)U(g)\exp(-t\mathcal{L}_2)$ determine the symmetry group of this equation and the associated infinitesimal operators $\exp(t\mathcal{L}_2)\mathcal{K}\exp(-t\mathcal{L}_2)$ are first order in x and t.

From (5.4) it follows that the operators $\mathcal{K}_{-2},\mathcal{L}_3,\mathcal{L}_2$ corresponding to the radial free particle, attractive and repulsive harmonic oscillator Hamiltonians lie on the same J orbits, as the three orbit representatives $\mathcal{K}_2$, $\mathcal{L}_3$ and $\mathcal{K}^3$, respectively. Our three Hamiltonians correspond to the three J orbits of $sl(2,R)$. The remarks concerning expressions (3.3)—(3.5) and the invariance of spectra for operators on an orbit carry over without change to this case except that the inner product is now

$$(h_1,h_2)=\int_0^\infty h_1(x)\overline{h_2(x)}\,dx,\quad h_j\in L_2(R+). \tag{5.8}$$

Note that if $\{f_\lambda\}$ is the basis of generalized eigenvectors for some $\mathcal{K}\in sl(2,R)$ then $\{f_\lambda'(t)=(\exp t\mathcal{K}_{-2})f_\lambda\}$ is the basis of eigenvectors for $K=\exp(t\mathcal{K}_{-2})\mathcal{K}\exp(-t\mathcal{K}_{-2})$ and the $f_\lambda'(t)$ satisfy the Schrödinger equation for the radial free particle. Similar remarks hold for the other Hamiltonians.

We first present the well-known results for the spectrum of $\mathcal{L}_3$. The eigenfunction equation is

$$i\mathcal{L}_3f=\lambda f,\quad(-\partial_{xx}+c/x^2+x^2/4)f=\lambda f$$

and the normalized eigenfunctions are

$$f_{\lambda_n}^{(1)}(x)=\left(\frac{n!\,2^{-\mu/2}}{\Gamma(n+1+\mu/2)}\right)^{1/2}e^{-x^2/4}x^{(\mu+1)/2}L_n^{(\mu/2)}(x^2/2), \tag{5.9}$$

$$\lambda_n=-2n-\mu/2-1,\quad c=(\mu^2-1)/4,\quad \mu\geqslant 3,$$
$$n=0,1,2,\cdots,$$

where $L_n^{(\alpha)}(z)$ is a generalized Laguerre polynomial. The $\{f_{\lambda_n}^{(1)}\}$ form an ON basis for $L_2(R+)$.

Using the recurrence relations for the Laguerre polynomials one can check that the operators $\mathcal{L}_j$ acting on the $f^{(1)}$ basis define an irreducible representation of $sl(2,R)$ belonging to the discrete series. The Casimir operator is $\frac{1}{4}(\mathcal{L}_1^2+\mathcal{L}_2^2-\mathcal{L}_3^2)=-3/16+c/4$. As is well-known,[20,23] this Lie algebra representation extends to a global unitary irreducible representation of J. The matrix elements of the operators $U(g)$ in a $f^{(1)}$ basis can be found in Refs. 23 or 29.

We now compute the operators $U(g)$ directly. Clearly,

$$\exp(a\mathcal{K}^3)f(x)=\exp(a/2)f(e^ax),$$
$$\exp(\alpha\mathcal{K}_2)f(x)=\exp(i\alpha x^2/4)f(x).$$

Furthermore,

$$\exp(\beta\mathcal{L}_3)f(x)=\frac{\exp(\mp i\pi(\mu+2)/4}{2|\sin\beta|}\,\text{l.i.m.}\int_0^\infty(xy)^{1/2}\times\exp\left(\pm\frac{i}{4}(x^2+y^2)\left|\cot\beta\right|\right)\times J_{\mu/2}\left(\frac{xy}{2|\sin\beta|}\right)f(y)\,dy,\quad 0<|\beta|<\pi, \tag{5.10}$$

where we take the upper sign for $\beta>0$ and the lower for $\beta<0$. [Here $J_\mu(z)$ is a Bessel function.] The additional relation $\exp(\pi\mathcal{L}_3)=\exp[-i\pi(1+\mu/2)]$ allows us to determine $\exp(\beta\mathcal{L}_3)$ for any β. To prove these results we apply the integral operator (5.10) to an $f^{(1)}$ basis element, and use the Hille—Hardy formula[22] and the fact that $\exp(\beta\mathcal{L}_3)f_{\lambda_n}^{(1)}=\exp[-i(2n+\mu/2+1)\beta]f_{\lambda_n}^{(1)}$ to check its validity. Since (5.10) is valid on an ON basis and $\exp(\beta\mathcal{L}_3)$ is unitary, the expression must be true for all $f\in L_2(R+)$.

The group multiplication formula

$$\exp\gamma\mathcal{K}_{-2}=\exp(-\sin\theta\cos\theta\mathcal{K}_2)\exp(\ln\cos\theta\mathcal{K}^3)\exp(\theta\mathcal{L}_3)$$

with $\gamma=\tan\theta$ and expressions (5.9), (5.10) easily yield

$$\exp(\gamma\mathcal{K}_{-2})f(x)=\frac{\exp[\mp(i/4)\pi(\mu+2)]}{2|\gamma|}\,\text{l.i.m.}\int_0^\infty(xy)^{1/2}\times\exp\left(\frac{i(x^2+y^2)}{4\gamma}\right)J_{\mu/2}\left(\frac{xy}{2|\gamma|}\right)f(y)\,dy, \tag{5.11}$$

where we take the upper sign for $\gamma>0$ and the lower for $\gamma<0$. A similar group theoretic calculation gives

$$\exp(\phi\mathcal{L}_2)f(x)=\frac{\exp[\mp(i/4)\pi(\mu+2)]}{2|\sinh\phi|}\,\text{l.i.m.}\int_0^\infty(xy)^{1/2}\times\exp\left(\frac{i}{4}(x^2+y^2)\coth\phi\right)\times J_{\mu/2}\left(\frac{xy}{2|\sinh\phi|}\right)f(y)\,dy. \tag{5.12}$$

From (5.11) we find that the basis functions $f_{\lambda_n}^{(1)}(x)$ map to the ON basis functions $F_{\lambda_n}^{(1)}(x,t)=\exp(t\mathcal{K}_{-2})f_{\lambda_n}^{(1)}(x)$

$$F_{\lambda_n}^{(1)}(x,t)=2(-1)^n\exp[\pm(i/4)\pi(\mu+2)]\left(\frac{x^2}{1+t^2}\right)^{(\mu+1)/4}\times(t-i)^{-\mu/4-3/4-n}(t+i)^{\mu/4+1/4+n}\times\exp\left(\frac{1}{4}\frac{x^2}{1+t^2}(-1+i\gamma)\right)L_n^{\mu/2}\left(\frac{1}{2}\frac{x^2}{1+t^2}\right)\quad\text{for } t\lessgtr 0 \tag{5.13}$$

which are solutions F of $YF=0$.

The J orbit containing the operator $\mathcal{L}_2$ (repulsive radial oscillator) also contains $\mathcal{K}^3$ so we merely study the spectral theory for $\mathcal{K}^3$. The results are well-known.[35] The eigenfunction equation is

$$i\mathcal{K}^3f=\lambda f,\quad \mathcal{K}^3=x\partial_x+\tfrac{1}{2}.$$

The spectrum is continuous and covers the real axis with multiplicity one. The generalized eigenfunctions are

$$f_\lambda^{(2)}(x)=\frac{1}{\sqrt{2\pi}}x^{-i\lambda-1/2},\quad\lambda\in R, \tag{5.14}$$
$$(f_\lambda^{(2)},f_\mu^{(2)})=\delta(\mu-\lambda).$$

Again using (5.11) we find $F_\lambda^{(2)}(x,t)=\exp(t\mathcal{K}_2)f_\lambda^{(2)}(x)$ where

$$F_\lambda^{(2)}(x,t)=\frac{1}{\sqrt{2\pi}}\frac{\Gamma(i\lambda/2+\mu/4+\frac{1}{2})}{\Gamma(1+\mu/2)}\exp[\mp(\pi/4)(i\mu+i+\lambda)]$$
$$\times t^{i\lambda/2-1/4}(x^2/t)^{-1/4}\exp\left(\frac{ix^2}{8t}\right)M_{(i\lambda/2),(\mu/4)}\left(\frac{ix^2}{t}\right) \tag{5.15}$$

for $t \gtrless 0$. Here $M_{\chi,\eta}(z)$ is a solution of Whittaker's equation.[10] If follows from our procedure that the basis functions satisfy

$$(F_\lambda^{(2)},\ F_\mu^{(2)})=\delta(\mu-\lambda)$$

and can be used to expand any $f\in L_2(R+)$.

Finally, the orbit containing $\mathcal{K}_{-2}$, corresponding to the radial free particle, also contains $\mathcal{K}_2$. The spectral theory for $\mathcal{K}_2$ is elementary because $\mathcal{K}_2$ is already diagonalized in our realization. The generalized eigenfunctions are (symbolically)

$$f_\lambda^{(3)}=\delta(x-\lambda),\quad i\mathcal{K}_2 f_\lambda^{(3)}=(\lambda^2/4)f_\lambda^{(3)},\quad \lambda\geqslant 0.$$

The spectrum is continuous and covers the positive real axis with multiplicity one. We have

$$F_\lambda^{(3)}(x,t)=\exp(t\mathcal{K}_{-2})f_\lambda^{(3)}(x)$$

or

$$F_\lambda^{(3)}(x,t)=\frac{\exp(\mp i(\pi/4)(\mu+2)}{2|t|}(x\lambda)^{1/2}$$
$$\times\exp\left(\frac{i(x^2+\lambda^2)}{4t}\right)J_{\mu/2}\left(\frac{x\lambda}{2|t|}\right) \tag{5.17}$$

with $(F_\lambda^{(3)},\ F_\mu^{(3)})=\delta(\mu-\lambda)$. Expansions in the basis $\{F_\lambda^{(3)}\}$ are equivalent to the inversion theorem for the Hankel transform. The $F_\lambda^{(3)}$ are basis functions for the operator K_2.

Each of our bases has continuous generating functions of the form (3.19) where now

$$k(x,y,t)=\frac{\exp(\pm i(\pi/4)(\mu+2)}{2|t|}(xy)^{1/2}$$
$$\times\exp\left(\frac{i(x^2+y^2)}{4t}\right)J_{\mu/2}\left(\frac{xy}{2|t|}\right) \tag{5.18}$$

(see Ref. 8).

The overlap functions $(f_\lambda^{(i)},\ f_\mu^{(j)})$ have the same significance as in Sec. 4. Because of the simplicity of the basis $f_\lambda^{(3)}$ the only overlap of interest is

$$(f_{\lambda_n}^{(1)},\ f_\lambda^{(2)})=\tfrac{1}{2}\left(\frac{\Gamma(n+1+\frac{1}{2}\mu)2^{i\lambda}}{\pi n!}\right)^{1/2}\frac{\Gamma(i\lambda/2+\mu/4+\frac{1}{2})}{\Gamma(1+\frac{1}{2}\mu)}$$
$$\times{}_2F_1\left(-n,\frac{i\lambda}{2}+\frac{\mu}{4}+\frac{1}{2};\ 1+\frac{1}{2}\mu;2\right). \tag{5.19}$$

In particular, we notice that the overlap functions are dependent on the representatives f_λ^i, $f_\mu^{(j)}$ that have been chosen on each orbit. From this we see that the most general way to define an overlap function is as the mixed basis matrix element $(f_\lambda^{(i)}, U(g)f_\mu^{(j)})$ where g is a general group element. This problem has been treated for the group $SL(2,R)$, Ref. 27, where a corresponding group parametrization has been given for each choice of $i\neq j$ in the above expression. In particular, the resulting expressions for the mixed basis matrix elements proved quite tractable to calculate and amounted to the calculation of the mixed basis matrix element of a one parameter subgroup in each case. We refer to the original article[27] for further details.

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[1]R. L. Anderson, S. Kumei, and C. E. Wulfman, Rev. Mex. Fis. 21, 1 (1972).
[2]C. P. Boyer, "The maximal kinematical invariance group for an arbitrary potential," preprint CIMAS 1973/53, Mexico.
[3]U. Niederer, Helv, Phys. Acta 46, 191 (1973); U. Niederer, "The group theoretical equivalence of the free particle, the Harmonic oscillator and the free fall," *Proceedings of the 2nd International Colloquium on Group Theoretical Methods in Physics*, 1973, University of Nijmegen, The Netherlands.
[4]G.W. Blumen and J.D. Cole, J. Math. Mech. 18, 1025 (1969).
[5]L. V. Ovsjannikov, *Gruppovye svoystva differentsialny uravneni*, (Novosibirsk, Moscow, 1962).
[6]L. Weisner, Can. J. Math. 11, 141 (1959).
[7]E. Whittaker, Proc. Roy. Soc. Edin. 61, A, 1 (1941).
[8]A. Erdélyi, Proc. Roy. Soc. Edin. 61, 61 (1941).
[9]A. O. Barut, SIAM J. Appl. Math. 25, 247 (1973); A. O. Barut and G. Bornzin, J. Math. Phys. 12, 841 (1971).
[10]A. Erdélyi *et al.*, *Higher Trancendental Functions* (McGraw-Hill, New York, 1953), Vols. 1, 2.
[11]W. Miller, Jr., SIAM J. Math. Anal. 4, 314 (1973).
[12]W. Miller, Jr., *Symmetry Groups and their Applications* (Academic, New York, 1973).
[13]G. Birkhoff, *Hydrodynamics* (Princeton U. P. Princeton, 1960).
[14]A. J. A. Morgan, Quart. J. Math. Oxford Ser. (2), 3, 250 (1952).
[15]P. Winternitz, I. Lukăc, and Y. Smorodinskiĭ, Sov. J. Nucl. Phys. 7, 139 (1968).
[16]W. Miller, Jr., SIAM J. Math. Anal. (to appear).
[17]E. G. Kalnins, "On the separation of variables for the Laplace equation $\Delta\psi+k^2\psi=0$ in two and three dimensional Minkowski space," Université de Montréal preprint CRM-319.
[18]N. Dunford and J. Schwartz, *Linear Operators*. Part II (Wiley-Interscience, New York, 1963).
[19]K. Maurin, *General Eigenfunction Expansions and Unitary Representations of Topological Groups* (Polish Scientific, Warsaw, 1968).
[20]V. Bargmann, Ann. Math. 48, 569 (1947).
[21]W. Miller, Jr., J. Math. Phys. 13, 648 (1972).
[22]W. Miller, Jr., *Lie Theory and Special Functions* (Academic, New York, 1968).
[23]P. Sally, "Analytic Continuation of the Irreducible Unitary Representations of the Universal Covering Group of SL(2,*R*), AMS Mem., No. 69, Providence, R.I., 1967.
[24]T. Kato, *Perturbation Theory for Linear Operators* (Springer, New York, 1966).
[25]N. Y. Vilenkin *Special Functions and the Theory of Group Representations* (AMS Transl., Providence, R.I., 1968).
[26]T. M. Cherry, Proc. Edin. Math. Soc. (2), 8, 50 (1949).
[27]E. G. Kalnins, J. Math. Phys. 14, 654 (1973).
[28] I. Stakgold, *Boundary Value Problems of Mathematical Physics* (Macmillan, New York, 1967), Vol. 1.
[29]W. Miller, Jr., J. Math. Phys. 13, 827 (1972).