Special Functions and the Complex Euclidean Group in 3-Space. I

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It is shown that the general addition theorems of Gegenbauer, relating Bessel functions and Gegenbauer polynomials, are special cases of identities for special functions obtained from a study of certain local irreducible representations of the complex Euclidean group in 3-space. Among the physically interesting results generalized by this analysis are the expansion for a plane wave as a sum of spherical waves and the addition theorem for spherical waves. This paper is one of a series attempting to derive the special functions of mathematical physics and their basic properties from the representation theory of Lie symmetry groups.

INTRODUCTION

The cylindrical (Bessel) functions obey two distinct types of addition theorems: those of Graf and Gegenbauer.[1] Graf's addition theorems are closely related to the representation theory of the Euclidean group in the plane and are obtained from a study of the solutions of the wave equation in 2-space.[2–4] On the other hand, the addition theorems of Gegenbauer are usually considered as by-products of the representation theory of the Euclidean group in n-space and are ordinarily derived from a study of the wave equation in n-space. It will be shown, however, that the Gegenbauer theorems can be derived (and even extended) from a study of certain representations of the Euclidean group in 3-space alone.

The results presented here are part of a continuing program by the author to uncover the relationship between Lie symmetry groups and the special functions of mathematical physics.[5,6] In this program, symmetry groups are considered as fundamental objects, while special functions and their properties are derived in a systematic fashion from the representation theory of the symmetry groups. The special functions associated with a given group arise in two ways: as matrix elements corresponding to a representation of the group, and as basis vectors in a model of such a representation. To the extent that matrix elements and models can be derived systematically for a given group, a large part of special function theory can be derived systematically from the theory of Lie groups.

In this paper, we examine a restricted class of irreducible representations of the complex Euclidean group in 3-space and obtain identities relating Bessel functions and Gegenbauer polynomials. In future papers, we shall examine other representations of this group and derive identities relating Whittaker functions and Jacobi polynomials.

1. REPRESENTATIONS OF THE EUCLIDEAN GROUP

We denote by $\mathcal{T}_6$ the 6-dimensional complex Lie algebra with generators p^+, p^-, p^3, j^+, j^-, and j^3 commutation relations as follows:

$$\begin{aligned}
&[j^3, j^\pm] = \pm j^\pm,\ [j^+, j^-] = 2j^3,\\
&[j^3, p^\pm] = [p^3, j^\pm] = \pm p^\pm,\\
&[j^+, p^+] = [j^-, p^-] = [j^3, p^3] = \mathcal{O},\\
&[j^+, p^-] = [p^+, j^-] = 2p^3,\\
&[p^3, p^\pm] = [p^+, p^-] = \mathcal{O}.
\end{aligned} \tag{1.1}$$

The elements j^+, j^-, j^3 generate a subalgebra of $\mathcal{T}_6$ isomorphic to $sl(2)$, the Lie algebra of 2×2 traceless matrices.[6] The elements p^+, p^-, p^3 generate a 3-dimensional Abelian ideal in $\mathcal{T}_6$.

Denote by T_6 the complex 6-parameter Lie group consisting of all elements $\{\mathbf{w}, g\}$,

$$\mathbf{w} = (\alpha, \beta, \gamma) \in \mathcal{C}^3, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2),$$
$$ad - bc = 1, \tag{1.2}$$

with group multiplication

$$\{\mathbf{w}, g\}\{\mathbf{w}', g\} = \{\mathbf{w} + g\mathbf{w}', gg'\}, \tag{1.3}$$

where "+" denotes vector addition in $\mathcal{C}^3$ and

$$g\mathbf{w} = (a^2\alpha - b^2\beta + ab\gamma,\ -c^2\alpha + d^2\beta - cd\gamma,\ 2ac\alpha - 2bd\beta + (bc + ad)\gamma). \tag{1.4}$$

Here $\mathbf{w}$ is a complex 3-vector and g is a complex 2×2 unimodular matrix. The identity element of

[1] W. Magnus, F. Oberhettinger, and R. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer-Verlag, New York, 1966), 3rd ed.
[2] N. Y. Vilenkin, Usp. Mat. Nauk. N.S., **11**, No. 3, 69 (1956).
[3] N. Y. Vilenkin, *Special Functions and Theory of Group Representations* (Izd. Nauka., Moscow, 1965).
[4] E. P. Wigner, *The Application of Group Theory to the Special Functions of Mathematical Physics,* Princeton Lecture Notes (1955) (unpublished).
[5] W. Miller, *On Lie Algebras and Some Special Functions of Mathematical Physics,* American Mathematical Society Memoir, No. 50 (Providence, 1964).
[6] W. Miller, *Lie Theory and Special Functions* (Academic Press Inc., New York, 1968), Chaps. 5, 6.

T_6 is $\{\mathbf{0}, \mathbf{e}\}$, where $\mathbf{0} = (0, 0, 0)$ and $\mathbf{e}$ is the identity element of $SL(2)$, and the inverse of an element $\{\mathbf{w}, g\}$ is given by

$$\{\mathbf{w}, g\}^{-1} = \{-g^{-1}\mathbf{w}, g^{-1}\}.$$

The set of all elements $\{\mathbf{0}, g\}$, $g \in SL(2)$, forms a subgroup of T_6 which can be identified with $SL(2)$. Similarly, the set of all elements $\{\mathbf{w}, \mathbf{e}\}$, $\mathbf{w} \in \mathcal{C}^3$, forms a subgroup of T_6 which can be identified with $\mathcal{C}^3$.

It is straightforward to show that $\mathcal{T}_6$ is the Lie algebra of T_6. Indeed, the generators of $\mathcal{T}_6$ can be chosen so that

$$\{\mathbf{w}, g\} = \exp(\alpha p^+ + \beta p^- + \gamma p^3) \exp[(-b/d)j^+] \times \exp(-cdj^-) \exp(-2 \ln dj^3), \qquad (1.5)$$

where $\{\mathbf{w}, g\}$ is defined by Eq. (1.2) and g is in a sufficiently small neighborhood of $\mathbf{e}$ [in the topology of $SL(2)$].[6] Here "exp" is the exponential map of a neighborhood of $\mathcal{O}$ in $\mathcal{T}_6$ onto a neighborhood of $\{\mathbf{o}, \mathbf{e}\}$ in T_6.[7]

The complex group T_6 is closely related to the real Euclidean group in 3-space[6]: the set of all pairs $(\mathbf{r}, R)$, $\mathbf{r}$ a real 3-vector, R a proper 3×3 orthogonal matrix, with group multiplication

$$(\mathbf{r}, R)(\mathbf{r}', R') = (\mathbf{r} + R\mathbf{r}', RR').$$

To see this we note that E_6, the real, simply connected covering group of the Euclidean group, can be defined as the set of all pairs $(\mathbf{r}, A)$, where $\mathbf{r} = (r_1, r_2, r_3)$ is a real column vector and A is an element of $SU(2)$ (the group of 2×2 unitary unimodular matrices). The group multiplication law is

$$(\mathbf{r}, A)(\mathbf{r}', A') = [\mathbf{r} + R(A)\mathbf{r}', AA'],$$

where $R(A)$ is a real 3×3 orthogonal matrix given explicitly by

$$R(A) = \begin{pmatrix} \frac{1}{2}(a^2-b^2+\bar{a}^2-\bar{b}^2), & \frac{i}{2}(\bar{a}^2+\bar{b}^2-a^2-b^2), & \bar{a}\bar{b}+ab \\ \frac{i}{2}(a^2-b^2-\bar{a}^2+\bar{b}^2), & \frac{1}{2}(\bar{a}^2+\bar{b}^2+a^2+b^2), & i(-\bar{a}\bar{b}+ab) \\ -(\bar{a}b+a\bar{b}), & i(-\bar{a}b+a\bar{b}), & a\bar{a}-b\bar{b} \end{pmatrix},$$

when

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2), \quad a\bar{a} + b\bar{b} = 1.$$

Now, E_6 can be considered as a real subgroup of T_6. Indeed, it is easy to show that the collection of all elements $\{\mathbf{w}, A\}$, where $\mathbf{w} = [\frac{1}{2}(-r_2 - ir_1), \frac{1}{2}(r_2 - ir_1), -ir_3]$ and $A \in SU(2)$ forms a subgroup of T_6 isomorphic to E_6. The isomorphism is given by $(\mathbf{r}, A) \leftrightarrow \{\mathbf{w}, A\}$, $\mathbf{r} = (r_1, r_2, r_3)$.

The real 6-dimensional Lie algebra $\mathcal{E}_6$ corresponding to E_6 is generated by elements j_k, p_k, $k = 1, 2, 3$, with commutation relations

$$[j_j, j_k] = \epsilon_{jkl} j_l, \quad [j_j, p_k] = \epsilon_{jkl} p_l,$$
$$[p_j, p_k] = \mathcal{O}, \qquad j, k, l = 1, 2, 3, \qquad (1.6)$$

where ϵ_{jkl} is the completely antisymmetric tensor such that $\epsilon_{123} = +1$. We choose these generators so that they are related to the finite group elements by

$$(\mathbf{r}, A) = \exp(r_1p_1 + r_2p_2 + r_3p_3) \exp \varphi_1 j_3 \times \exp \theta j_1 \exp \varphi_2 j_3,$$

where $\mathbf{r} = (r_1, r_2, r_3)$ and φ_1, θ, φ_2 are the Euler coordinates for A. The formal elements $p^\pm$, p^3, $j^\pm$, j^3, defined in terms of the generators (1.6) by

$$p^\pm = \mp p_2 + ip_1, \quad p^3 = ip_3,$$
$$j^\pm = \pm j_2 + ij_1, \quad j^3 = ij_3,$$

can easily be shown to satisfy the commutation relations (1.1) for the complex Lie algebra $\mathcal{T}_6$. Thus, we have explicitly determined $\mathcal{E}_6$ as a real form of $\mathcal{T}_6$ and $\mathcal{T}_6$ as a complexification of $\mathcal{E}_6$. In this sense we can say that T_6 is a complexification of the Euclidean group in 3-space.

Consider a complex vector space V (possibly infinite-dimensional) and a representation ρ of $\mathcal{T}_6$ by linear operators on V.[5,6] Set

$$\rho(p^\pm) = P^\pm, \quad \rho(p^3) = P^3,$$
$$\rho(j^\pm) = J^\pm, \quad \rho(j^3) = J^3.$$

Then the linear operators $P^\pm$, P^3, $J^\pm$, J^3 satisfy commutation relations on V analogous to Eq. (1.1), where now $[A, B] = AB - BA$ for operators A and B on V. We define two operators on V which are of special importance for the representation theory of $\mathcal{T}_6$. They are

$$\mathbf{P} \cdot \mathbf{P} = -P^+P^- - P^3P^3,$$
$$\mathbf{P} \cdot \mathbf{J} = \tfrac{1}{2}(P^+J^- + P^-J^+) - P^3J^3. \qquad (1.7)$$

It is easy to show that

$$[\mathbf{P} \cdot \mathbf{P}, \rho(\alpha)] = [\mathbf{P} \cdot \mathbf{J}, \rho(\alpha)] = 0$$

for all $\alpha \in \mathcal{T}_6$. Thus, if ρ is an irreducible representation of $\mathcal{T}_6$, we would expect $\mathbf{P} \cdot \mathbf{P}$ and $\mathbf{P} \cdot \mathbf{J}$ to be multiples of the identity operator on V.

The irreducible representations of $\mathcal{T}_6$ which are of interest in special function theory have been classified.[5,6] Among these representations we single out the following two classes related to Gegenbauer polynomials and Bessel functions:

(1) $\rho_0(\omega)$.

There is a countable basis $\{f_m^{(u)}\}$ for V such that $m = u, u-1, \cdots, -u+1, -u$, and $u = 0, 1, 2, \cdots$.

(2) $\rho_\mu(\omega)$, $(0 \le \operatorname{Re} \mu < 1$ and 2μ not an integer).

There is a countable basis $\{f_m^{(u)}\}$ for V such that $m = u, u-1, u-2, \cdots$, and $u = \mu + n$, where $n = 0, \pm 1, \pm 2, \cdots$.

These representations are defined for any nonzero complex number ω. Furthermore, corresponding to each representation, the action of the infinitesimal

[7] S. Helgason, *Differential Geometry and Symmetric Spaces* (Addison-Wesley Publ. Co., Inc., Reading, Mass., 1962), Chap. 2.

operators on the basis vectors $f_m^{(u)}$ is given by

$$J^3 f_m^{(u)} = m f_m^{(u)}, \quad J^+ f_m^{(u)} = (m-u) f_{m+1}^{(u)},$$
$$J^- f_m^{(u)} = -(m+u) f_{m-1}^{(u)}, \tag{1.8}$$

$$P^3 f_m^{(u)} = \frac{\omega}{2u+1} f_m^{(u+1)} + \frac{\omega(u+m)(u-m)}{2u+1} f_m^{(u-1)}, \tag{1.9}$$

$$P^+ f_m^{(u)} = \frac{\omega}{2u+1} f_{m+1}^{(u+1)} - \frac{\omega(u-m)(u-m-1)}{2u+1} f_{m+1}^{(u-1)}, \tag{1.10}$$

$$P^- f_m^{(u)} = \frac{-\omega}{2u+1} f_{m-1}^{(u+1)} + \frac{\omega(u+m)(u+m-1)}{2u+1} f_{m-1}^{(u-1)}, \tag{1.11}$$

$$\mathbf{P}\cdot\mathbf{P} f_m^{(u)} = -\omega^2 f_m^{(u)}, \quad \mathbf{P}\cdot\mathbf{J} f_m^{(u)} = 0. \tag{1.12}$$

[If a vector $f_m^{(u)}$ on the right-hand side of one of the expressions (1.8)–(1.12) does not belong to the representation space, we set this vector equal to zero.]

It is easy to verify directly that the infinitesimal operators given by these expressions (1.8)–(1.11) do satisfy the commutation relations (1.1) and define an irreducible representation of $\mathcal{C}_6$. Furthermore, the vectors $\{f_m^{(u)}\}$, corresponding to some fixed value of u, form a basis for an irreducible representation of the subalgebra $sl(2)$ of $\mathcal{C}_6$. Each such induced representation of $sl(2)$ associated with $\rho_0(\omega)$ has dimension $2u+1$ and is denoted by $D(2u)$. Each such induced representation of $sl(2)$ associated with $\rho_\mu(\omega)$ is infinite-dimensional and is denoted by $\downarrow u$. The representations $D(2u)$ and $\downarrow u$ have been studied in detail elsewhere.[5,6]

Our aim in this paper is to examine the relationship between the representations $\rho_0(\omega)$, $\rho_\mu(\omega)$ and special function theory. In particular, we shall be interested in the following two aspects of this relationship:

(1) We can look for models of the abstract representations $\rho_0(\omega)$, $\rho_\mu(\omega)$ such that the infinitesimal operators $\rho(\alpha)$, $\alpha \in \mathcal{C}_6$, are linear differential operators acting on a space V of analytic functions in n complex variables. In this case the basis vectors $f_m^{(u)}$ will be analytic functions and expressions (1.8)–(1.11) will yield differential recursion relations obeyed by these "special" functions. For $n = 1, 2$, all of the possible models have been constructed.[6] In particular, for $n = 1$ it is known that no models exist. For $n = 2$, there is Model A:

$$J^3 = t\frac{\partial}{\partial t}, \quad J^+ = -t\frac{\partial}{\partial z},$$
$$J^- = t^{-1}\left[(1-z^2)\frac{\partial}{\partial z} - 2zt\frac{\partial}{\partial t}\right], \tag{1.13}$$
$$P^+ = \omega t, \quad P^- = \omega(1-z^2)t^{-1}, \quad P^3 = \omega z.$$

Corresponding to this model, the basis vectors $f_m^{(u)}$ are uniquely defined by relations (1.8)–(1.12) up to an arbitrary multiplicative constant and may be given by

$$f_m^{(u)}(z,t) = \Gamma(u-m+1)\Gamma(m+\tfrac{1}{2})C_{u-m}^{m+\frac{1}{2}}(z)(2t)^m. \tag{1.14}$$

Here $\Gamma(x)$ is the gamma function and $C_n^\lambda(z)$ is a Gegenbauer polynomial defined by the generating function

$$(1 - 2\alpha z + \alpha^2)^{-\lambda} = \sum_{n=0} C_n^\lambda(z)\alpha^n.$$

If the representation under consideration is $\rho_0(\omega)$, then m takes the integer values $u, u-1, \cdots, -u$ and u runs over the nonnegative integers in Eq. (1.10). However, if the representation is $\rho_\mu(\omega)$, then $m = u$, $u-1$, $u-2, \cdots$, and u takes all values such that $u-\mu$ is an integer. Substitution of Eq. (1.13) and (1.14) into expressions (1.8)–(1.11) leads to some well-known recursion relations for the Gegenbauer polynomials:

$$\frac{d}{dz}C_n^\lambda(z) = 2\lambda C_{n-1}^{\lambda+1}(z),$$
$$\left[(1-z^2)\frac{d}{dz} - 2z\lambda + z\right]C_n^\lambda(z) = \frac{(n+1)(n+2\lambda-1)}{2(1-\lambda)}C_{n+1}^{\lambda-1}(z), \tag{1.8'}$$

$$zC_n^\lambda(z) = \frac{n+1}{2(\lambda+n)}C_{n+1}^\lambda(z) + \frac{(2\lambda+n-1)}{2(\lambda+n)}C_{n-1}^\lambda(z), \tag{1.9'}$$

$$C_n^\lambda(z) = \frac{\lambda}{\lambda+n}(C_n^{\lambda+1}(z) - C_{n-2}^{\lambda+1}(z)), \tag{1.10'}$$

$$2(\lambda-1)(1-z^2)C_n^\lambda(z) = \frac{-(n+2)(n+1)}{2(\lambda+n)}C_{n+2}^{\lambda-1}(z) + \frac{(n+2\lambda-1)(n+2\lambda-2)}{2(\lambda+n)}C_n^{\lambda-1}(z), \tag{1.11'}$$

valid for nonintegral $\lambda \in \mathbb{C}$, $n = 0, 1, 2, \cdots$.

There is another useful model of the representations $\rho_0(\omega)$, $\rho_\mu(\omega)$ which can be constructed in terms of differential operators in three complex variables. This model (Model B) is closely related to the separation of variables method for solution of the wave equation in spherical coordinates and is determined by the operators

$$J^3 = t\frac{\partial}{\partial t}, \quad J^+ = -t\frac{\partial}{\partial z},$$
$$J^- = t^{-1}\left((1-z^2)\frac{\partial}{\partial z} - 2zt\frac{\partial}{\partial t}\right),$$
$$P^3 = \omega\left[z\frac{\partial}{\partial r} + \frac{(1-z^2)}{r}\frac{\partial}{\partial z} - \frac{zt}{r}\frac{\partial}{\partial t}\right],$$
$$P^+ = \omega t\left(\frac{\partial}{\partial r} - \frac{z}{r}\frac{\partial}{\partial z} - \frac{t}{r}\frac{\partial}{\partial t}\right),$$
$$P^- = \omega t^{-1}\left[(1-z^2)\frac{\partial}{\partial r} - \frac{z(1-z^2)}{r}\frac{\partial}{\partial z} + \frac{(z^2+1)}{r}t\frac{\partial}{\partial t}\right]. \tag{1.15}$$

Notice that the J operators in expressions (1.13) and (1.15) coincide. Thus, to finish the construction of Model B based on operators (1.15), we look for basis vectors $f_m^{(u)}[r, z, t]$ of the form

$$f_m^{(u)}[r, z, t] = Z^{(u)}(r) f_m^{(u)}(z, t), \tag{1.16}$$

where the functions $f_m^{(u)}(z, t)$ are given by Eq. (1.14). A straightforward computation shows that the basis vectors (1.16) and infinitesimal operators (1.15) satisfy relations (1.8)–(1.12) if and only if the functions $Z^{(u)}(r)$ satisfy the recursion relations

$$\left(\frac{d}{dr} - \frac{u}{r}\right) Z^{(u)}(r) = Z^{(u+1)}(r),$$
$$\left(\frac{d}{dr} + \frac{u+1}{r}\right) Z^{(u)}(r) = Z^{(u-1)}(r), \tag{1.17}$$

for all values of u such that both sides of these expressions are defined. The solutions of these recursion relations are well known to be cylindrical functions.[1] For simplicity we shall primarily restrict ourselves to the solutions

$$Z^{(u)}(r) = r^{-\frac{1}{2}} I_{u+\frac{1}{2}}(r),$$

where $I_{u+\frac{1}{2}}(r)$ is a modified Bessel function

$$I_\lambda(r) = \sum_{k=0}^{\infty} \frac{(z/2)^{\lambda+2k}}{k!\,\Gamma(\lambda + k + 1)}.$$

Thus the basis vectors for Model B become

$$f_m^{(u)}[r, z, t] = (u - m)!\,\Gamma(m + \tfrac{1}{2}) r^{-\frac{1}{2}} I_{u+\frac{1}{2}}(r) C_{u-m}^{m+\frac{1}{2}}(z)(2t)^m. \tag{1.18}$$

As before, in the case of the representation $\rho_0(\omega)$, m takes the values $u, u - 1, \cdots, -u$ and u runs over the nonnegative integers, while, in the case of the representation $\rho_\mu(\omega)$, m takes values u, $u - 1$, $u - 2, \cdots, u - \mu$ is an integer, $0 \leq \operatorname{Re} \mu < 1$ and 2μ is not an integer. (Note that as far as special function theory is concerned, the above results are independent of ω. Hence, in the remainder of this paper, we shall always set $\omega = 1$.)

(2) Each of the representations $\rho_0(1)$, $\rho_\mu(1)$ of $\mathcal{T}_6$ induces a local representation of the Lie group T_6 defined by linear operators $\mathbf{T}(h)$, $h \in T_6$, acting on V.[6] These operators satisfy the group property $\mathbf{T}(h)\mathbf{T}(h') = \mathbf{T}(hh')$ for h and h' in a sufficiently small neighborhood of the identity. The general theory relating local representations of Lie groups to representations of Lie algebras will not be repeated here.[6] We shall limit ourselves to construction of the operators $\mathbf{T}(h)$ and computation of the matrix elements of these operators with respect to the basis $\{f_m^{(u)}\}$. The results when applied to Models A and B constructed in (1) yield addition theorems and other identities relating Gegenbauer polynomials and cylindrical functions.

2. COMPUTATIONAL IDENTITIES

In this section we collect together several computational results which will be needed later to extend the Lie algebra representations $\rho_0(1)$ and $\rho_\mu(1)$ ($\omega = 1$) to local group representations of T_6. Assume that the operators $J^\pm$, J^3, $P^\pm$, P^3 and the basis vectors $f_m^{(u)}$ satisfy relations (1.8)–(1.12) and that they define either of the irreducible representations $\rho_0(1)$ or $\rho_\mu(1)$. (Formally, the results for both representations look the same. The difference lies only in the allowable values of u and m.)

Lemma 1:

$$C_l^{m+\frac{1}{2}}(P^3) f_m^{(u)} = \sum_{k=0}^{\min(l, u-m)} A(m + \tfrac{1}{2}; l, u - m; k) f_m^{(u+l-2k)},$$

where

$$A(\lambda; l, s; k) = \frac{s!\,\Gamma(2\lambda + s + l - k)\Gamma(\lambda + l - k)\Gamma(\lambda + k)}{(s - k)!\,(l - k)!\,k!\,\Gamma(2\lambda + s + l - 2k) \times \Gamma(\lambda + s + l - k + 1)} \times \frac{\Gamma(\lambda + s - k)}{\Gamma^2(\lambda)} (\lambda + s + l - 2k),$$

if $0 \leq k \leq \min(l, s)$

$= 0$, otherwise.

Here, $\lambda \in \mathbb{C}$ and l, s, k are nonnegative integers.

Proof: Straightforward induction on l, using the recursion relations (1.9) and (1.9′).

Lemma 1 is a consequence merely of the abstract definition of the representations $\rho_0(1)$ and $\rho_\mu(1)$. Hence, the lemma must be valid for Models A and B. In Model A, $P^3 = z$ and $f_m^{(u)}$ is given by Eq. (1.14). We immediately obtain the known result:

Corollary 1:

$$C_l^\lambda(z) C_s^\lambda(z) = \sum_{k=0}^{\min(l,s)} A(\lambda; l, s; k) \frac{(s + l - 2k)!}{s!} \cdot C_{l+s-2k}^\lambda(z).$$

For Model B, we obtain

Corollary 2:

$$C_l^\lambda\left(z \frac{\partial}{\partial r} + \frac{(1 - z^2)}{r} \frac{\partial}{\partial z} - \frac{z(\lambda - \frac{1}{2})}{r}\right) \frac{I_{s+\lambda}(r)}{\sqrt{r}} C_s^\lambda(z) = \sum_{k=0}^{\min(l,s)} A(\lambda; l, s; k) \frac{(s + l - 2k)!}{s!} \times \frac{I_{l+s-2k+\lambda}(r)}{\sqrt{r}} C_{l+s-2k}^\lambda(z).$$

When $s = 0$, this expression simplifies to the identity

$$C_l^\lambda\left(z\frac{\partial}{\partial r}+\frac{(1-z^2)}{r}\frac{\partial}{\partial z}-\frac{z}{r}(\lambda-\tfrac{1}{2})\right)\frac{I_\lambda(r)}{\sqrt{r}} = \frac{I_{l+\lambda}(r)}{\sqrt{r}}C_l^\lambda(z).$$

Lemma 2: Let $\nu \in \mathcal{C}$ and l be a nonnegative integer. Then

$$C_l^\nu(P^3)f_u^{(u)} = \sum_{k=0}^{[l/2]} B(\nu, u+\tfrac{1}{2}; l, k) f_u^{(u+l-2k)},$$

where

$$B(\nu,\lambda; l, k) = \frac{(\lambda+l-2k)\Gamma(\nu+l-k)\Gamma(\nu-\lambda+k)\Gamma(\lambda)}{(l-2k)!\,k!\,\Gamma(\nu)\Gamma(\nu-\lambda)\Gamma(\lambda+l-k+1)}.$$

Proof: Induction of l using (1.9) and (1.9′).

In the remainder of this section, λ is any complex number not an integer, such that 2λ is not a negative integer.

Corollary 3: Let $\nu \in \mathcal{C}$. Then

$$C_l^\nu(z) = \sum_{k=0}^{[l/2]} B(\nu,\lambda; l, k)(l-2k)!\, C_{l-2k}^\lambda(z).$$

Corollary 4: Let $\nu, \lambda \in \mathcal{C}$. Then

$$C_l^\nu\left[z\frac{\partial}{\partial r}+\frac{(1-z^2)}{r}\frac{\partial}{\partial z}-\frac{z}{r}(\lambda-\tfrac{1}{2})\right]\frac{I_\lambda(r)}{\sqrt{r}} = \sum_{k=0}^{[l/2]} B(\nu,\lambda; l,k)(l-2k)!\,\frac{I_{\lambda+l-2k}(r)}{\sqrt{r}}C_{l-2k}^\lambda(z).$$

Lemma 3:

$$(P^3)^l f_u^{(u)} = \sum_{k=0}^{[l/2]} \frac{(u+l-2k+\tfrac{1}{2})}{2^l} \times \frac{\Gamma(u+\tfrac{1}{2})l!}{\Gamma(u+l-k+\tfrac{3}{2})k!\,(l-2k)!} f_u^{(u+l-2k)}.$$

Proof: Relation (1.9) and induction on l.

Corollary 5:

$$\frac{(2z)^l}{l!} = \sum_{k=0}^{[l/2]} \frac{(\lambda+l-2k)\Gamma(\lambda)}{\Gamma(\lambda+l-k+1)k!} C_{l-2k}^\lambda(z).$$

Corollary 6:

$$\left[z\frac{\partial}{\partial r}+\frac{(1-z^2)}{r}\frac{\partial}{\partial z}-\frac{z}{r}(\lambda-\tfrac{1}{2})\right]^l\frac{I_\lambda(r)}{\sqrt{r}} = \sum_{k=0}^{[l/2]} \frac{(\lambda+l-2k)\Gamma(\lambda)l!}{2^l\Gamma(\lambda+l-k+1)k!}\frac{I_{\lambda+l-2k}(r)}{\sqrt{r}}C_{l-2k}^\lambda(z).$$

Lemma 4:

$$(P^+)^l f_m^{(u)} = \sum_{k=0}^{l}\binom{l}{k} \times \frac{(-1)^k(u-m)!\,\Gamma(u+\tfrac{1}{2}-k)(u+\tfrac{1}{2}+l-2k)}{2\Gamma(u-m-2k+1)\Gamma(u+l-k+\tfrac{3}{2})} \times f_{m+l}^{(u+l-2k)}.$$

Proof: Relation (1.10) and induction on l.

Lemma 5:

$$(P^-)^l f_m^{(u)} = \sum_{k=0}^{l}\binom{l}{k} \times \frac{(-1)^{l+k}\Gamma(u+m+1) \times \Gamma(u-k+\tfrac{1}{2})(u+l-2k+\tfrac{1}{2})}{2^l\Gamma(u+m-2k+1)\Gamma(u+l-k+\tfrac{3}{2})} \times f_{m-l}^{(u+l-2k)}.$$

Proof: Relation (1.11) and induction on l.

We can use the above lemmas to compute the action of the operators $\exp(\alpha P^3)$, $\exp(\alpha P^+)$, and $\exp(\alpha P^-)$ on V. [If P is a linear operator on V and $\alpha_k \in \mathcal{C}$, we define $\exp(\alpha P)$ to be the formal sum $\sum_{k=0}^\infty (\alpha/k!)P^k$.] Although these results will be of only formal significance for the abstract representations $\rho_0(1)$ and $\rho_\mu(1)$, we will soon see that when applied to Models A and B they can be rigorously justified.

Lemma 6:

$$e^{\alpha P^3} f_u^{(u)} = \sum_{k=0}^\infty \frac{(u+k+\tfrac{1}{2})}{k!}\left(\frac{\alpha}{2}\right)^{\frac{1}{2}-u-} \times I_{u+k+\frac{1}{2}}(\alpha)\Gamma(u+\tfrac{1}{2})f_u^{(u+k)}.$$

Proof: This result follows directly from Lemma 3.

Assuming that Lemma 6 is valid when applied to Model A, we find:

Corollary 7: If $\alpha, \lambda \in \mathcal{C}$, then

$$e^{\alpha z} = \left(\frac{2}{\alpha}\right)^\lambda \Gamma(\lambda)\sum_{k=0}^\infty (\lambda+k)I_{\lambda+k}(\alpha)C_k^\lambda(z).$$

Corollary 8:

$$e^{\alpha P^3} = \left(\frac{2}{\alpha}\right)^\lambda \Gamma(\lambda)\sum_{k=0}^\infty (\lambda+k)I_{\lambda+k}(\alpha)C_k^\lambda(P^3).$$

3. DETERMINATION OF THE OPERATORS T(*h*)

The differential operators (1.13), which define Model A, satisfy the commutation relations of the Lie algebra $\mathcal{E}_6$. Hence, according to standard results in Lie theory,[8] these operators uniquely determine a

[8] H. W. Guggenheimer, *Differential Geometry* (McGraw-Hill Book Co., New York, 1963), Chap. 7.

local representation of T_6 by operators $\mathbf{T}(h)$, $h \in T_6$, acting on the space of analytic functions in two complex variables. The computation of $\mathbf{T}(h)$ is straightforward,[6,8] and we merely give the results. Due to the group multiplication law (1.3), we can write

$$\mathbf{T}(h) = \mathbf{T}(\mathbf{w}, g) = \mathbf{T}(\mathbf{w}; \mathbf{e})\mathbf{T}(\mathbf{0}; g),$$

where

$$h = \{\mathbf{w}, g\}, \quad \mathbf{w} = (\alpha, \beta, \gamma) \in \mathcal{C}^3,$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2).$$

If f is defined and analytic in a neighborhood of some point $(z, t) \in \mathcal{C}^2$ $(t \neq 0)$, then we have

$$\begin{aligned}[\mathbf{T}(\mathbf{w}; \mathbf{e})f](z, t) &= [\exp(\alpha P^+ + \beta P^- + \gamma P^3)f](z, t) \\ &= \exp[\alpha t + \beta(1 - z^2)t^{-1} + \gamma z] \\ &\quad \times f(z, t). \quad (3.1)\end{aligned}$$

Furthermore,

$$\begin{aligned}[\exp \alpha J^3 f](z, t) &= f(z, te^{\alpha}), \\ [\exp \alpha J^+ f](z, t) &= f(z - \alpha t, t), \\ [\exp \alpha J^- f](z, t) &= f\left(z + \frac{\alpha(1 - z^2)}{t},\right. \quad (3.2) \\ &\qquad \left. t - 2\alpha z - \alpha^2 \frac{(1 - z^2)}{t}\right).\end{aligned}$$

Combining these results, we obtain

$$\begin{aligned}&[\mathbf{T}(\mathbf{0}; g)f](z, t) \\ &= [\exp(-b/dJ^+) \exp(-cdJ^-) \exp(-2 \ln dJ^3)f](z, t) \\ &= f\left(z(1 + 2bc) + abt + \frac{cd(z^2 - 1)}{t},\right. \\ &\qquad \left. a^2 t + 2acz + c^2 \frac{(z^2 - 1)}{t}\right), \quad (3.3)\end{aligned}$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1.$$

By construction, the $\mathbf{T}$ operators satisfy the group multiplication property

$$\mathbf{T}(hh')f = \mathbf{T}(h)[\mathbf{T}(h')f], \quad (3.4)$$

whenever both sides of this expression are well defined.

In the same way, the differential operators (1.15), which define Model B, can be used to construct a local representation of T_6 by operators $\mathbf{T}(h)$ acting on the space of analytic functions in three complex variables. As before, we write $\mathbf{T}(h) = \mathbf{T}(\mathbf{w}; g) = \mathbf{T}(\mathbf{w}; \mathbf{e})\mathbf{T}(\mathbf{0}; g)$. Standard techniques in Lie theory[8] give

$$\begin{aligned}&[\exp \alpha P^+ f](r, z, t) \\ &= f\left(r\left(1 + \frac{2t\alpha}{r}\right)^{\frac{1}{2}}, z\left(1 + \frac{2t\alpha}{r}\right)^{-\frac{1}{2}}, t\left(1 + \frac{2t\alpha}{r}\right)^{-\frac{1}{2}}\right),\end{aligned}$$

$$\begin{aligned}&[\exp \beta P^- f](r, z, t) \\ &= f\left(r\left(1 + \frac{2\beta(1 - z^2)}{rt}\right)^{\frac{1}{2}}, z\left(1 + \frac{2\beta(1 - z^2)}{rt}\right)^{-\frac{1}{2}},\right. \\ &\qquad \left.\left(t + \frac{2\beta}{r}\right)\left(1 + \frac{2\beta(1 - z^2)}{rt}\right)^{-\frac{1}{2}}\right),\end{aligned}$$

$$\begin{aligned}&[\exp \gamma P^3 f](r, z, t) \\ &= f\left(r\left(1 + \frac{\gamma^2}{r^2} + \frac{2\gamma z}{r}\right)^{\frac{1}{2}}, \left(z + \frac{\gamma}{r}\right)\left(1 + \frac{\gamma^2}{r^2} + \frac{2\gamma z}{r}\right)^{-\frac{1}{2}},\right. \\ &\qquad \left. t\left(1 + \frac{\gamma^2}{r^2} + \frac{2\gamma z}{r}\right)^{-\frac{1}{2}}\right).\end{aligned}$$

Thus,

$$\begin{aligned}&[\mathbf{T}(\mathbf{w}, \mathbf{e})f](r, z, t) \\ &\quad = [\exp \alpha P^+ \exp \beta P^- \exp \gamma P^3 f](r, z, t) \\ &\quad = f[rQ, (z + \gamma/r)Q^{-1}, (t + 2\beta/r)Q^{-1}], \quad (3.5)\end{aligned}$$

where

$$Q = \left[1 + \frac{2\beta(1 - z^2)}{rt} + \frac{2\alpha}{r}\left(t + \frac{2\beta}{r}\right) + \frac{\gamma^2}{r^2} + \frac{2\gamma z}{r}\right]^{\frac{1}{2}}. \quad (3.6)$$

Here f is defined and analytic in some neighborhood of the point $(r, z, t) \in \mathcal{C}^3$. Exactly as in the computation (3.3) we find

$$\begin{aligned}&[\mathbf{T}(\mathbf{0}; g)f](r, z, t) \\ &= f\left(r, z(1 + 2bc) + abt + cd\frac{(z^2 - 1)}{t},\right. \\ &\qquad \left. a^2 t + 2acz + c^2 \frac{(z^2 - 1)}{t}\right). \quad (3.7)\end{aligned}$$

Again, we have the group multiplication property

$$\mathbf{T}(hh')f = \mathbf{T}(h)[\mathbf{T}(h')f],$$

whenever both sides of this expression are well defined as analytic functions of r, z, and t.

4. MATRIX ELEMENTS OF $\rho_0(1)$

We will now compute the matrix elements of the group representation of T_6 induced by the Lie algebra representation $\rho_0(1)$ of $\mathcal{T}_6$. The restriction of this group representation to the real subgroup E_6 of T_6 is well known (it is a member of the so-called principal series of representations of E_6) and the restricted matrix elements have been computed.[9–11] We carry out the computation for T_6 here to motivate the more complicated work to follow in the next section and also to point out the increased information about special functions obtained by studying the complex group.

In the remainder of this section, u and v will be nonnegative integers, while m and n will be integers ranging over values from $-u$ to u and $-v$

[9] N. Y. Vilenkin, E. L. Akim, and A. A. Levin, Dokl. Akad. Nauk SSSR **112**, 987 (1957).

[10] N. Y. Vilenkin, *Translations of the Moscow Mathematical Society for the Year 1963* (American Mathematical Society, Providence, 1965), English Transl., pp. 209–290.

[11] W. Miller, Commun. Pure Appl. Math. **17**, 527 (1964).

to v, respectively. We define the matrix elements $\{v, n\,|\mathbf{w}, g|\, u, m\}$ of the representation $\rho_0(1)$ by

$$\mathbf{T}(\mathbf{w}, g)f_m^{(u)} = \sum_{v=0}^{\infty}\sum_{n=-v}^{v}\{v, n|\,\mathbf{w}, g\,|u, m\}f_n^{(v)}, \quad (4.1)$$

where the operator $\mathbf{T}(\mathbf{w}, g)$ and the functions $f_m^{(u)}$ refer either to Model A or to Model B. It is known[12] that the functions $f_m^{(u)}$ for both Models A and B form an analytic basis for the representation space in the sense of Ref. 6, Chap. 2. In particular, the functions $\mathbf{T}(\mathbf{w}, g)f_m^{(u)}$ can be expressed uniquely as linear combination of basis functions uniformly convergent in suitable domains. The coefficients in this expansion are bounded linear functionals of the argument $\mathbf{T}(\mathbf{w}, g)f_m^{(u)}$ (in the topology of uniform convergence on compact sets). Since these conditions are satisfied, it can be shown that the matrix elements $\{v, n|\,\mathbf{w}, g\,|u, m\}$ are model-independent: They are determined uniquely by the infinitesimal operators (1.8)–(1.11) and are the same for every model of $\rho_0(1)$ for which the functions $f_m^{(u)}$ form an analytic basis.[6] Thus the matrix elements can be computed directly from expressions (1.8)–(1.11) and they will be valid for both Models A and B.

Furthermore, the group property

$$\mathbf{T}(\mathbf{w}, g)\mathbf{T}(\mathbf{w}', g') = \mathbf{T}(\mathbf{w} + g\mathbf{w}', gg')$$

leads immediately to the addition theorem

$$\sum_{v'=0}^{\infty}\sum_{n'=-v'}^{v'}\{v, n|\,\mathbf{w}, g\,|v', n'\}\{v', n'|\,\mathbf{w}', g'\,|u, m\} = \{v, n|\,\mathbf{w} + g\mathbf{w}', gg'\,|u, m\} \quad (4.2)$$

for the matrix elements.[6]

Matrix elements of the form $\{v, n|\,\mathbf{0}, g\,|u, m\}$ are determined completely by the J operators (1.8) and depend only on the representation theory of $SL(2)$. In fact, for fixed u, the functions $f_m^{(u)}$ form a basis for the $(2u + 1)$-dimensional irreducible representation of $sl(2)$. The matrix elements of these irreducible representations are well-known.[6] We quote the results:

$$\begin{aligned}\{v, n|\,\mathbf{0}, g\,|u, m\} &= \frac{d^{u-n}a^{u+m}b^{n-m}(u-m)!}{(u-n)!} \times \frac{F(n-u, -m-u; n-m+1; bc/ad)}{\Gamma(n-m+1)}\delta_{v,u} \\ &= \frac{d^{u-m}a^{u+n}c^{m-n}(u+m)!}{(u+n)!} \times \frac{F(m-u, -n-u; m-n+1; bc/ad)}{\Gamma(m-n+1)}\delta_{v,u},\end{aligned} \quad (4.3)$$

where

$$g = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2), \quad ad - bc = 1.$$

These expressions make sense even when the gamma function in the denominator has a singularity, since

$$\lim_{c\to -n}\frac{F(a, b; c; x)}{\Gamma(c)} = \frac{a(a+1)\cdots(a+n)b(b+1)\cdots(b+n)}{(n+1)!} \times x^{n+1}F(a+n+1, b+n+1; n+2; x), \quad n = 0, 1, 2, \cdots. \quad (4.4)$$

The hypergeometric functions in Eq. (4.3) are Jacobi polynomials.

It follows immediately that the identity

$$\mathbf{T}(\mathbf{0}; g)f_m^{(u)} = \sum_{n=-u}^{u}\{u, n|\,\mathbf{0}, g\,|u, m\}f_n^{(u)} \quad (4.5)$$

must be valid for both Models A and B. Substituting expressions (1.14) and (3.3) for Model A into (4.5) and simplifying, we easily obtain the identity

$$\begin{aligned}&\frac{k!\,\Gamma(u-k+\frac12)}{(2u-k)!}\left(\frac{x^2}{2}\right)^k C_k^{u-k+\frac12} \times [z^2 - z - 1 + (2z-1)/x + 1/x^2] \times (1 + 2xz + x^2(z^2-1))^{u-k} \\ &= \sum_{l=0}^{2u}\frac{l!}{(2u-l)!}\Gamma(u-l+\tfrac12)\left(\frac{x}{2}\right)^l \times \frac{F(-k, -2u+l; l-k+1; 1-x)}{\Gamma(l-k+1)}C_l^{u-l+\frac12}(z).\end{aligned} \quad (4.6)$$

When $k = 0$, this identity reduces to a simple generating function

$$[1 + 2xz + x^2(z^2-1)]^u = \sum_{l=0}^{2u}\binom{2u}{l}\binom{u-\frac12}{l}^{-1}\left(\frac{x}{2}\right)^l C_l^{u-l+\frac12}(z)$$

for the basis vectors (1.14). Model B gives no new results.

Combining Lemma 1 and Corollary 8 we find

$$\begin{aligned}\mathbf{T}(0, 0, \gamma; \mathbf{e})f_m^{(u)} &= \exp(\gamma P^3)f_m^{(u)} \\ &= \left(\frac{2}{\gamma}\right)^{m+\frac12}\Gamma(m+\tfrac12)\sum_{l=0}^{\infty}(m+l+\tfrac12) \times I_{m+l+\frac12}(\gamma)C_l^{m+\frac12}(P^3)f_m^{(u)} \\ &= \left(\frac{2}{\gamma}\right)^{m+\frac12}\Gamma(m+\tfrac12)\sum_{j=-\infty}^{\infty}f_m^{(u+j)} \times \sum_{k=0}^{\infty}A(m+\tfrac12; j+2k, u-m; k) \times (m+j+2k+\tfrac12\, I_{m+j+2k+\frac12}(\gamma).\end{aligned}$$

[12] F. W. Schäfke, *Einführung in die Theorie der Speziellen Funktionen der Mathematischen Physik* (Springer-Verlag, Berlin, 1963), Chap. 8.

Therefore,

$$\begin{aligned}\{v, n| o, o, \gamma; \mathbf{e} |u, m\} &= \delta_{n,m}(2/\gamma)^{m+\frac{1}{2}}\Gamma(m+\tfrac{1}{2}) \\ &\times \sum_{k=0}^{\infty} A(m+\tfrac{1}{2}; v-u+2k, u-m; k) \\ &\times (m+v-u+2k+\tfrac{1}{2})I_{m+v-u+2k+\frac{1}{2}}(\gamma). \quad (4.7)\end{aligned}$$

[Due to the properties of the symbol $A(\ )$, defined by Lemma 1, this sum is actually finite.] When $m = u$, we have the special case

$$\begin{aligned}\{v, n| 0, 0, \gamma; \mathbf{e} |u, u\} &= \delta_{n,u}(2/\gamma)^{u+\frac{1}{2}} \frac{\Gamma(u+\frac{1}{2})(v+\frac{1}{2})}{(v-u)!} I_{v+\frac{1}{2}}(\gamma), \quad \text{if} \quad v \geq u; \\ &= 0, \quad \text{if} \quad v < u, \quad (4.8)\end{aligned}$$

which also follows directly from Lemma 6.

The matrix element

$$\{u, m| \alpha, \beta, \gamma; \mathbf{e} |0, 0\}$$

can be computed by making use of the identity

$$\begin{aligned}\{u, m| ab\xi, -cd\xi, (1+2bc)\xi; \mathbf{e} |0, 0\} \\ = \{u, m| \mathbf{0}; g |u, 0\}\{u, 0| 0, 0, \xi; \mathbf{e} |0, 0\}, \quad (4.9)\end{aligned}$$

where $g \in SL(2)$. This identity is a special case of the addition theorem (4.2). In terms of new variables $\alpha = ab\xi$, $\beta = -cd\xi$, $\gamma = (1+2bc)\xi$, and $\rho^2 = \gamma^2 + 4\alpha\beta$, the matrix elements on the right-hand side of Eq. (4.9) are given by

$$\begin{aligned}\{u, m| \mathbf{0}; g |u, 0\} &= \frac{\Gamma(|m|+\frac{1}{2})u!}{\sqrt{\pi}(u+|m|)!}\left(\frac{4}{\rho}\right)^{|m|} \alpha^{(|m|+m)/2}(-\beta)^{(|m|-m)/2} \\ &\times C_{u-|m|}^{|m|+\frac{1}{2}}(\gamma/\rho), \\ \{u, 0| 0, 0, \xi; \mathbf{e} |0, 0\} &= (2/\rho)^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2})}{u!}(u+\tfrac{1}{2})I_{u+\frac{1}{2}}(\xi). \quad (4.10)\end{aligned}$$

Therefore,

$$\begin{aligned}\{u, m| \alpha, \beta, \gamma; \mathbf{e} |0, 0\} &= (2/\rho)^{\frac{1}{2}}(4/\rho)^{|m|}\alpha^{(|m|+m)/2}(-\beta)^{(|m|-m)/2} \\ &\times \frac{\Gamma(|m|+\frac{1}{2})(u+\frac{1}{2})}{(u+|m|)!} C_{u-|m|}^{|m|+\frac{1}{2}}(\gamma/\rho)I_{u+\frac{1}{2}}(\rho). \quad (4.11)\end{aligned}$$

There is an ambiguity in the signs of expressions (4.10) since $\rho = \pm[\gamma^2 + 4\alpha\beta]^{\frac{1}{2}}$. However, a close inspection of (4.11) reveals that the final matrix element is a function of ρ^2 so the ambiguity in sign causes no harm. Furthermore, the matrix element is an entire function of α, β, and γ.

Applying the identity

$$\mathbf{T}(\alpha, \beta, \gamma; \mathbf{e})f_0^{(0)} = \sum_{u=0}^{\infty} \sum_{m=-u}^{u} \{u, m |\alpha, \beta, \gamma; \mathbf{e}| 0, 0\} f_m^{(u)}$$

to Model A, we obtain

$$\begin{aligned}&\exp[\alpha t + \beta(1-z^2)/t + \gamma z] \\ &= \left(\frac{2}{\pi\rho}\right)^{\frac{1}{2}} \sum_{u=0}^{\infty} \sum_{m=0}^{u} (8t\alpha/\rho)^m(u+\tfrac{1}{2})(u-m)! \frac{\Gamma^2(m+\frac{1}{2})}{(u+m)!} \\ &\times I_{u+\frac{1}{2}}(\rho)C_{u-m}^{m+\frac{1}{2}}(\gamma/\rho)C_{u-m}^{m+\frac{1}{2}}(z) \\ &+ \left(\frac{2\pi}{\rho}\right)^{\frac{1}{2}} \sum_{u=0}^{\infty} \sum_{m=0}^{u} (2\beta/t\rho)^m(u+\tfrac{1}{2}) \\ &\times I_{u+\frac{1}{2}}(\rho)C_{u-m}^{m+\frac{1}{2}}(\gamma/\rho)C_{u+m}^{-m+\frac{1}{2}}(z). \quad (4.12)\end{aligned}$$

This formula is the complex generalization of the well-known formula

$$e^{i\mathbf{p}\cdot\mathbf{r}} = \left(\frac{8\pi^3}{pr}\right)^{\frac{1}{2}} \sum_{l=0}^{\infty} \sum_{k=-l}^{l} i^l J_{l+\frac{1}{2}}(pr)Y_l^k(\Theta_r, \varphi_r)Y_l^k(\Theta_p, \varphi_p)$$

for the expansion of plane waves into spherical waves. Since the left-hand side of Eq. (4.12) is an entire function of the variables αt, β/t, γ, and z, it follows from standard expansion theorems for Gegenbauer polynomials[12] that the right-hand side must converge for all values of these variables. Furthermore, the expansion coefficients $\{u, m |\alpha, \beta, \gamma; \mathbf{e}| 0, 0\}$ on the right-hand side must be entire functions of α, β, and γ.

At this point, we can fill a gap in our derivation of Eq. (4.11). This derivation was valid only for $\rho \neq 0$. However, using Model A, we have seen that the required matrix element is an entire function of α, β, and γ. Thus to compute $\{u, m |\alpha, \beta, \gamma; \mathbf{e}| 0, 0\}$ for $\gamma^2 + 4\alpha\beta = 0$ we need only find the value of Eq. (4.11) as $\rho \to 0$. The result is

$$\begin{aligned}\{u, m| \alpha, \beta, \gamma; \mathbf{e} |0, 0\} &= \frac{(2\alpha)^m\gamma^{u-m}}{(u+m)!(u-m)!}, \quad \text{if} \quad m \geq 0, \quad \rho = 0, \\ &= \frac{(-2\beta)^{-m}\gamma^{u+m}}{(u-m)!(u+m)!}, \quad \text{if} \quad m \leq 0, \quad \rho = 0. \quad (4.13)\end{aligned}$$

We are now in a position to calculate the general matrix element $\{v, n |\alpha, \beta, \gamma; \mathbf{e}| u, m\}$. Using Model A, we find

$$\begin{aligned}&\mathbf{T}(\alpha, \beta, \gamma; \mathbf{e})f_m^{(u)} \\ &= (u-m)!\,\Gamma(m+\tfrac{1}{2})(2t)^m \\ &\times \exp[\alpha t + (1-z^2)\beta/t + \gamma z]C_{u-m}^{m+\frac{1}{2}}(z) \\ &= (u-m)!\,\Gamma(m+\tfrac{1}{2}) \sum_{r=0}^{\infty} \sum_{k=-r}^{r} \{r, k| \alpha, \beta, \gamma; \mathbf{e} |0, 0\} \\ &\times C_{u-m}^{m+\frac{1}{2}}(z)C_{r-k}^{k+\frac{1}{2}}(z)(2t)^{m+k}(r-k)!\,\Gamma(k+\tfrac{1}{2}).\end{aligned}$$

From the connection between Gegenbauer polynomials and the representation theory of $SL(2)$, it follows that

$$(u-m)!\,(r-k)!\,\Gamma(m+\tfrac{1}{2})\Gamma(k+\tfrac{1}{2})C_{u-m}^{m+\frac{1}{2}}(z)C_{r-k}^{k+\frac{1}{2}}(z)$$
$$=[\pi(u-m)!\,(r-k)!\,(u+m)!\,(r+k)!]^{\frac{1}{2}}$$
$$\times\,\Gamma(m+k+\tfrac{1}{2})\sum_{s=0}^{2\min(u,r)}\left[\frac{(u+r-s-m-k)!}{(u+r-s+m+k)!}\right]^{\frac{1}{2}}$$
$$\times\,C(u,0;r,0\,|\,u+r-s,0)$$
$$\times\,C(u,m;r,k\,|\,u+r-s,m+k)C_{u+r-s-m-k}^{m+k+\frac{1}{2}}(z), \tag{4.14}$$

where the $C(.\,;.\,|\,.)$ are ordinary Clebsch–Gordan coefficients.[6,13]

Thus,

$$\mathbf{T}(\alpha,\beta,\gamma;\mathbf{e})f_m^{(u)}=\sum_{v=0}^{\infty}\sum_{n=-v}^{v}\{v,n\,|\,\alpha,\beta,\gamma;\mathbf{e}\,|u,m\}f_n^{(v)},$$

where

$$\{v,n\,|\,\alpha,\beta,\gamma,\mathbf{e}\,|u,m\}$$
$$=\sum_s\left[\frac{\pi(u-m)!\,(u+m)!\,(v-u+s+n-m)!}{(v-n)!\,(v+n)!}\times(v-u+s+m-n)!\right]^{\frac{1}{2}}$$
$$\times\,C(u,0;v-u+s,0\,|\,v,0)$$
$$\times\,C(u,m;v-u+s,n-m\,|\,v,n)$$
$$\times\,\{v-u+s,n-m\,|\,\alpha,\beta,\gamma;\mathbf{e}\,|0,0\}, \tag{4.15}$$

and s ranges over the finite set of nonnegative integer values for which the summand is defined.

Now that all of the matrix elements of the representation $\rho_0(1)$ have been computed, it is a simple task to substitute these expressions into the addition theorem (4.2) and obtain identities for special functions. This will be left to the reader.

5. MATRIX ELEMENTS OF $\rho_\mu(1)$

The task of computing the matrix elements of the representation $\rho_\mu(1)$ is analogous to that for $\rho_0(1)$ but somewhat more complicated. In this section, u and v will be arbitrary complex numbers such that $2u$ and $2v$ are not integers and such that $u-v$ is an integer. The variables m, n will take values $m=u,\ u-1,\ u-2,\cdots;\ n=v,\ v-1,\ v-2,\cdots$.

As in Sec. 4, we define the matrix elements $\{v,n\,|\mathbf{w};g|\,u,m\}$ of $\rho_\mu(1)$ by

$$\mathbf{T}(\mathbf{w};g)f_m^{(u)}=\sum_v\sum_n\{v,n\,|\,\mathbf{w};g\,|u,m\}f_n^{(v)}, \tag{5.1}$$

where the operator $\mathbf{T}(\mathbf{w};g)$ and the basis functions $f_m^{(u)}$ refer either to Model A or Model B. Again, it follows that the functions $\{f_m^{(u)}\}$ for both Models A and B form an analytic basis for the representation space.[12] Thus the matrix elements are well defined and are uniquely determined by the infinitesimal operators (1.8)–(1.11).

[13] G. Y. Lyubarskii, *The Application of Group Theory to Physics* (Pergamon Press, Inc., Oxford, 1960), English Transl., Chap. 10.

Under the action of J^+, J^-, J^3, the vectors $\{f_m^{(u)}\}$ for fixed u, $m=u,\ u-1,\ u-2,\cdots$, form a basis for an irreducible representation of $sl(2)$. This representation, denoted by $\downarrow u$, was studied in Ref. 6, Chap. 5, and the matrix elements were computed to be

$$\{v,n\,|\,\mathbf{0},g\,|u,m\}$$
$$=\frac{d^{u-n}a^{u+m}b^{n-m}(u-m)!}{(u-n)!}\times\frac{F(n-u,-m-u;n-m+1;bc/ad)}{\Gamma(n-m+1)}\delta_{v,u}$$
$$=\frac{d^{u-m}a^{u+n}c^{m-n}\Gamma(u+m+1)}{\Gamma(u+n+1)}\times\frac{F(m-u,-n-u;m-n+1;bc/ad)}{\Gamma(m-n+1)}\delta_{v,u}, \tag{5.2}$$

where

$$g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2),\quad ad-bc=1.$$

These matrix elements define a local representation of $SL(2)$: they are well defined and satisfy the group representation property only in a sufficiently small neighborhood of $\mathbf{e}$. Note, for example, in Eq. (5.2) that $(e^{2\pi i}a)^{u+m}\neq e^{2\pi i(u+m)}a^{u+m}$. A precise definition of this representation is worked out in Ref. 6 and will not be repeated here.

The identity

$$\mathbf{T}(\mathbf{0},g)f_m^{(u)}=\sum_{n\le u}\{u,n\,|\,\mathbf{0},g\,|u,m\}f_n^{(u)}$$

is valid for both Models A and B when g is in a sufficiently small neighborhood of $\mathbf{e}$. Substituting expressions (1.14) and (3.3) for Model A into this identity and simplifying, we obtain

$$\frac{k!\,\Gamma(u-k+\frac{1}{2})}{\Gamma(2u-k+1)}\left(\frac{x^2}{2}\right)^k$$
$$\times\,C_k^{u-k+\frac{1}{2}}[z^2-z-1+(2z-1)/x+1/x^2]$$
$$\times\,(1+2xz+x^2(z^2-1))^{u-k}$$
$$=\sum_{l=0}^{\infty}\frac{l!\,\Gamma(u-l+\frac{1}{2})}{\Gamma(2u-l+1)}\left(\frac{x}{2}\right)^l\times\frac{F(-k,-2u+l;l-k+1;1-x)}{\Gamma(l-k+1)}C_l^{u-l+\frac{1}{2}}(z),$$
$$|2xz+x^2(z^2-1)|<1. \tag{5.3}$$

The computation of the matrix element

$$\{v,n\,|0,0,\gamma;\mathbf{e}|\,u,m\}$$

is carried out exactly as for the corresponding element

(4.7) of $\rho_0(1)$:

$$\{v, n|0, 0, \gamma; \mathbf{e}|u, m\} = \delta_{n,m} I_m^{v,u}(\gamma) = \delta_{n,m}(2/\gamma)^{m+\frac{1}{2}} \frac{(u-m)!\,(v+\frac{1}{2})}{\Gamma(m+\frac{1}{2})\Gamma(v+m+1)}$$
$$\times \sum_{k=0}^{\infty} \frac{(m+v-u+2k+\frac{1}{2})\Gamma(v+m+k+1)\Gamma(v-u+m+k+\frac{1}{2})}{(u-m-k)!\,(v-u+k)!\,k!\,\Gamma(v+k+\frac{3}{2})}$$
$$\times \Gamma(m+k+\tfrac{1}{2})\Gamma(u-k+\tfrac{1}{2}) I_{m+v-u+2k+\frac{1}{2}}(\gamma). \tag{5.4}$$

The difference is solely the domain of definition of u, v, m, and n. Note that the sum in Eq. (5.4) contains only a finite number of nonzero terms and that the matrix element is an entire function of γ.

The functions $I_m^{v,u}(\gamma)$ form a natural generalization of the ordinary modified Bessel function.[10] In fact, if $m = u$, we have

$$I_u^{v,u}(\gamma) = (2/\gamma)^{u+\frac{1}{2}} \frac{\Gamma(u+\frac{1}{2})(v+\frac{1}{2})}{(v-u)!} I_{v+\frac{1}{2}}(\gamma), \quad \text{if } v-u \geq 0,$$
$$= 0, \quad \text{if } v-u < 0. \tag{5.5}$$

The addition theorem

$$\{v, m|0, 0, \gamma+\gamma'; \mathbf{e}|u, m\} = \sum_{k=-\infty}^{\infty} \{v, m|0, 0, \gamma; \mathbf{e}|u+k, m\} \times \{u+k, m|0, 0, \gamma'; \mathbf{e}|u, m\}$$

implies the identity

$$I_m^{v,u}(\gamma+\gamma') = \sum_{k=-\infty}^{\infty} I_m^{v,u+k}(\gamma) I_m^{u+k,u}(\gamma'). \tag{5.6}$$

Moreover, the identity

$$\mathbf{T}(0, 0, \gamma; \mathbf{e}) f_m^{(u)} = \sum_{k=-\infty}^{\infty} \{u+k, m|0, 0, \gamma; \mathbf{e}|u, m\} f_m^{(u+k)}$$

applied to Model A yields

$$l!\, e^{\gamma z} C_l^{m+\frac{1}{2}}(z) = \sum_{k=0}^{\infty} k!\, I_m^{m+k,m+l}(\gamma) C_k^{m+\frac{1}{2}}(z). \tag{5.7}$$

The right-hand side of this expression converges for all $\gamma, z \in \mathbb{C}$.

Using standard techniques from special function theory, we can apply relations (1.8′)–(1.10′) to the generating function (5.7) and derive recursion relations for the generalized Bessel functions. Among the results which can be obtained in this way are

$$\frac{(k+1)}{\gamma} I_m^{m+k+1,m+l}(\gamma) = \frac{1}{2m+2k+1} I_m^{m+k,m+l}(\gamma) - \frac{(k+1)(k+2)}{2m+2k+5} \times I_m^{m+k+2,m+l}(\gamma) + \frac{l}{\gamma} I_{m+1}^{m+k+1,m+l}(\gamma),$$

$$\frac{d}{d\gamma} I_m^{m+k,m+l}(\gamma) = \frac{1}{2m+2l+1} I_m^{m+k,m+l+1}(\gamma) + \frac{l(2m+l)}{2m+2l+1} I_m^{m+k,m+l-1}(\gamma)$$
$$= \frac{1}{2m+2k-1} I_m^{m+k-1,m+l}(\gamma) + \frac{(k+1)(2m+k+1)}{2m+2k+3} I_m^{m+k+1,m+l}(\gamma), \quad k, l = 0, 1, 2, \cdots.$$

Rather than compute directly an expression for the general matrix element $\{v, n|\alpha, \beta, \gamma; \mathbf{e}|u, m\}$ of $\rho_\mu(1)$, we will derive this result indirectly by determining a relation between the matrix elements of two different representations $\rho_\mu(1)$ and $\rho_{\mu'}(1)$. Denote the matrix elements of $\rho_{\mu'}(1)$ by $\{v', n'|\alpha, \beta, \gamma; \mathbf{e}|u', m'\}'$ to distinguish them from those of $\rho_\mu(1)$. (Our results will be valid even if $\mu' = 0$ or $\mu = 0$.)

Using Model A and Corollary 3, we find

$$f_m^{(u)}(z, t) = (2t)^{m-m'} \frac{(u-m)!}{\Gamma(m-m')} \times \sum_{k=0}^{[(u-m)/2]} \frac{(u+m'-m-2k+\frac{1}{2})}{k!\,(u-m-2k)!} \times \frac{\Gamma(u-k+\frac{1}{2})\Gamma(m-m'+k)}{\Gamma(m'-m+u-k+\frac{3}{2})} f_{m'}^{(m'+u-m-2k)}(z, t)$$
$$= (2t)^{m-m'} \sum_k D(u, m, m', k) f_{m'}^{(m'+u-m-2k)}(z, t),$$

where the basis functions $f_m^{(u)}(z, t)$ are given by Eq. (1.14). Applying the operator

$$\mathbf{T}(\alpha, \beta, \gamma; \mathbf{e}) = \exp[\alpha t + \beta(1-z^2)/t + \gamma z]$$

to both sides of this equation and using Eq. (5.1) to expand each side in terms of its corresponding basis functions, we obtain the identity

$$\sum_{v,n} \{v, n|\alpha, \beta, \gamma; \mathbf{e}|u, m\} f_n^{(v)}(z, t) = (2t)^{m-m'} \times \sum_{k,v',n'} \{v', n'|\alpha, \beta, \gamma; \mathbf{e}|m'+u-m-2k, m'\}' \times D(u, m, m', k) f_{n'}^{(v')}(z, t).$$

Finally, using Corollary 3, again, to express the functions $f_n^{(v')}(z,t)$ as linear combinations of functions $f_n^{(v)}(z,t)$, $(n = m - m' + n')$, and equating coefficients of $f_n^{(v)}(z,t)$ on both sides of the identity, we derive the equality

$$\{v, n|\,\alpha,\beta,\gamma;\mathbf{e}\,|u,m\} = \frac{(u-m)!\,(v+\frac{1}{2})}{\Gamma(m-m')\Gamma(m'-m)(v-n)!} \times \sum_{k=0}^{[(u-m)/2]}\sum_{s}\frac{(m'+u-m-2k+\frac{1}{2})(v-n+2s)!\,\Gamma(u-k+\frac{1}{2})}{k!\,s!\,(u-m-2k)!\,\Gamma(m'+u-m-k+\frac{3}{2})} \times \frac{\Gamma(m-m'+k)\Gamma(m'+v-m+s+\frac{1}{2})\Gamma(m'-m+s)}{\Gamma(v+s+\frac{3}{2})} \times \{m'+v-m+2s, m'+n-m|\,\alpha,\beta,\gamma;\mathbf{e}\,|m'+u-m-2k, m'\}'. \tag{5.8}$$

Here s ranges over all nonnegative integral values such that the summand is well defined.

Formula (5.8) can be employed to evaluate the matrix elements of $\rho_\mu(1)$. For example, set $m = u$, $m' = 0$, and use expression (4.11) for the primed elements on the right-hand side of Eq. (5.8). The result is

$$\{v, n|\,\alpha,\beta,\gamma;\mathbf{e}\,|u,u\} = \left(\frac{2}{\rho\pi}\right)^{\frac{1}{2}}\left(\frac{4}{\rho}\right)^{|n-u|}\frac{(v+\frac{1}{2})\Gamma(u+\frac{1}{2})}{(v-m)\Gamma(-u)} \times \Gamma(|n-u|+\tfrac{1}{2})\alpha^{(|n-u|+u-n)/2}(-\beta)^{(|n-u|+u-n)/2} \times \sum_{s}\frac{(v-u+2s+\frac{1}{2})(v-n+2s)!\,\Gamma(v-u+s+\frac{1}{2})\Gamma(-u+s)}{s!\,\Gamma(v+s+\frac{3}{2})(|n-u|+v-u+2s)!} \times C^{|n-u|+\frac{1}{2}}_{v-u-|n-u|+2s}(\gamma/\rho)I_{v-u+2s+\frac{1}{2}}(\rho),\quad \rho^2=\gamma^2+4\alpha\beta. \tag{5.9}$$

For the case $\alpha = \beta = 0$, $n = m = u$, Eqs. (5.8) and (5.5) yield

$$(\gamma/2)^{\nu-\lambda}I_\lambda(\gamma) = \sum_{s=0}^{\infty}\frac{\Gamma(\nu+s)\Gamma(\nu-\lambda+s)(\nu+2s)}{s!\,\Gamma(\nu-\lambda)\Gamma(\lambda+s+1)}I_{\nu+2s}(\gamma),\quad \lambda,\nu\in\mathcal{C}. \tag{5.10}$$

In addition to the general result (5.8), we list two special classes of matrix elements whose forms follow immediately from Lemmas 4 and 5:

$$\{v, n|\,\alpha, 0, 0;\mathbf{e}\,|u,m\} = \frac{(\alpha/2)^{n-m}}{(v-n)!}\frac{(u-m)!}{\left(\frac{u-m+n-v}{2}\right)!} \times \frac{(-1)^{(u-m+n-v)/2}\Gamma\left(\frac{1+m-n+v+u}{2}\right)(v+\frac{1}{2})}{\left(\frac{v+n-u-m}{2}\right)!\,\Gamma\left(\frac{-m+u+n+v+3}{2}\right)},$$

if $n - m - |v-u|$ is a nonnegative even integer,

$$=0,\quad \text{otherwise.} \tag{5.11}$$

$$\{v, n|\,0,\beta,0;\mathbf{e}\,|u,m\} = \left(\frac{\beta}{2}\right)^{m-n}\frac{(-1)^{(m-n-u+v)/2}}{\left(\frac{m-n+u-v}{2}\right)!\left(\frac{m-n-u+v}{2}\right)!} \times \frac{\Gamma(u+m+1)\Gamma\left(\frac{n-m+u+v+1}{2}\right)(v+\frac{1}{2})}{\Gamma(v+n+1)\Gamma\left(\frac{u+m-n+v+3}{2}\right)},$$

if $m - n - |u-v|$ is a nonnegative even integer,

$$=0,\quad \text{otherwise.} \tag{5.12}$$

By construction, the matrix elements of $\rho_\mu(1)$ satisfy the addition theorem:

$$\{v, n|\,\mathbf{w}+g\mathbf{w}'; gg'\,|u,m\} = \sum_{k=-\infty}^{\infty}\sum_{l=0}^{\infty}\{v,n|\,\mathbf{w},g\,|u+k,u+k-l\} \times \{u+k,u+k-l|\,\mathbf{w}',g'\,|u,m\}, \tag{5.13}$$

for all $\mathbf{w}, \mathbf{w}' \in \mathcal{C}^3$ and for g, g' in a sufficiently small neighborhood of $\mathbf{e}\in SL(2)$. (In any given example the restriction on g and g' can usually be determined by inspection.) We will list a few special cases of this theorem.

When $m = u$, $g = g' = \mathbf{e}$, $\mathbf{w} = (\alpha, \beta, \gamma)$, and $\mathbf{w}' = (\alpha', o, o)$, relation (5.13) simplifies to

$$\{v, n|\, \alpha + \alpha', \beta, \gamma; \mathbf{e}\,|u, u\} = \sum_{k=0}^{\infty}\left(\frac{\alpha'}{2}\right)^k \frac{\Gamma(u+\frac{1}{2})}{k!\,\Gamma(u+k+\frac{1}{2})} \times \{v, n|\, \alpha, \beta, \gamma; \mathbf{e}\,|u+k, u+k\}, \quad (5.14)$$

where the matrix elements on both sides of this expression are defined by Eq. (5.9).

The relation

$$\{v, n|\, ab\rho, -cd\rho, (1+2bc)\rho; \mathbf{e}\,|u, m\} = \sum_s \{v, n|\, \mathbf{0}, g\,|v, s\}\{v, s|\, 0, 0, \rho; \mathbf{e}\,|u, s\} \times \{u, s|\, \mathbf{0}, g^{-1}\,|u, m\}$$

leads to the identity

$$\{v, n|\, \alpha, \beta, \gamma; \mathbf{e}\,|u, m\} = \sum_{k=0}^{\infty}\left(\frac{1+\gamma/\rho}{2}\right)^{2u+v-n-k}\left(\frac{1-\gamma/\rho}{2}\right)^{m-u+k}(\alpha/\rho)^{n-m} \times \frac{\Gamma(u+m+1)(v-u+k)!}{\Gamma(2u-k+1)(v-n)!} \times \frac{F\left(n-v, -u-v+k; n-u+k; \dfrac{\gamma-\rho}{\gamma+\rho}\right)}{\Gamma(n-u+k+1)} \times \frac{F\left(m-u, -2u+k; m-u+k+1; \dfrac{\gamma-\rho}{\gamma+\rho}\right)}{\Gamma(m-u+k+1)} \times I^{v,u}_{u-k}(\rho), \quad (5.15)$$

valid for $|1 - z/\rho| < 2$. Here, $\rho = z[1 + 4xy/z^2]^{\frac{1}{2}}$. In case $\gamma = 0$, the identity becomes

$$\{v, n|\, \alpha, \beta, 0; \mathbf{e}\,|u, m\} = \sum_{k=0}^{\infty}(\tfrac{1}{2})^{u+v}(\alpha/\beta)^{(n-m)/2}\Gamma(u+m+1)(v-u+k)! \times \frac{F(m-u, -2u+k; m-u+k+1; -1) \times F(n-v, -u-v+k; n-u+k; -1)}{\Gamma(2u-k+1)(v-n)!\,\Gamma(m-u+k+1) \times \Gamma(n-u+k+1)} \times I^{v,u}_{u-k}(2\sqrt{\alpha\beta}), \quad \text{if } v+n-u-m \text{ is even},$$

$= 0$ otherwise.

Finally, we note the result

$$\{v, n|\, g\boldsymbol{\gamma}; g\,|u, m\} = \{v, n|\, \mathbf{0}, g\,|v, m\}\{v, m|\, \boldsymbol{\gamma}, \mathbf{e}\,|u, m\} = \sum_s \{v, n|\, g\boldsymbol{\gamma}; \mathbf{e}\,|u, s\}\{u, s|\, \mathbf{0}; g\,|u, m\},$$

where

$$\boldsymbol{\gamma} = (0, 0, \gamma), \quad g\boldsymbol{\gamma} = [ab\gamma, -cd\gamma, (1+2bc)\gamma],$$
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \rho = z(1 + 4xy/z^2)^{\frac{1}{2}}.$$

If $u \geq n$ and $u = m$, this implies

$$\frac{\Gamma(u+v+1)\Gamma(u+\frac{1}{2})(v+\frac{1}{2})}{\Gamma(v+n+1)(v-u)!\,(u-n)!} \times \left(\frac{z+\rho}{2\rho}\right)^{v-u}(-y)^{u-n}\left(\frac{2}{\rho}\right)^{u+\frac{1}{2}} \times F\left(u-v, -n-v; u-n+1; \frac{z-\rho}{z+\rho}\right) I_{v+\frac{1}{2}}(\rho) = \sum_{k=0}^{\infty}\{v, n|\, \alpha, \beta, \gamma; \mathbf{e}\,|u, u-k\} \times \left(\frac{z+\rho}{2}\right)^{u-n-k}\frac{(-y)^k}{k!}\,\frac{\Gamma(2u+1)}{\Gamma(2u-k+1)}.$$

There is a similar result for $n \geq u$.

6. APPLICATIONS TO MODEL B

Now that we have succeeded in computing matrix elements of the representations $\rho_0(1)$ and $\rho_\mu(1)$ we can apply our results to any model of these representations and obtain identities for special functions. As an illustration, consider Model B.

According to the work of Sec. 1, the basis vectors for Model B take the form

$$f^{(u)}_m[r, z, t] = Z^{(u)}(r)(u-m)!\,\Gamma(m+\tfrac{1}{2})C^{m+\frac{1}{2}}_{u-m}(z)(2t)^m,$$

where the $Z^{(u)}(r)$ satisfy recursion relations (1.17). Both the functions

$$Z^{(u)}(r) = r^{-\frac{1}{2}}I_{u+\frac{1}{2}}(r) \quad \text{and} \quad Z^{(u)}(r) = r^{-\frac{1}{2}}I_{-u-\frac{1}{2}}(r) \quad (6.1)$$

separately satisfy Eq. (1.17). Similarly, any linear combination of these functions satisfies Eq. (1.17). For purposes of illustration, we will use only the first of solutions (6.1). Recall that corresponding to the representation $\rho_0(1)$: $u = 0, 1, 2, \cdots$; $m = u, u-1, \cdots, -u$; while corresponding to $\rho_\mu(1)$: $u = \mu + k$; $k = 0, \pm 1, \pm 2, \cdots$; $m = u, u-1, u-2, \cdots$; $0 \leq \operatorname{Re}\mu < 1$ and 2μ is not an integer.

Since the functions $f^{(u)}_m[r, z, t]$ form an analytic basis for the representation space, we have immediately

$$[\mathbf{T}(\mathbf{w}; g)f^{(u)}_m][r, z, t] = \sum_{v,n}\{v, n|\, \mathbf{w}; g\,|u, m\}f^{(v)}_n[r, z, t], \quad (6.2)$$

where the operators $\mathbf{T}(\mathbf{w}; g)$ are given by Eqs. (3.5)–(3.7) and the matrix elements $\{v, n|\, \mathbf{w}; g\,|u, m\}$ are those computed in Secs. 4 and 5. The operators $\mathbf{T}(\mathbf{0}, g)$ yield no information which could not have been obtained from Model A. Therefore, we restrict ourselves to operators $\mathbf{T}(\mathbf{w}, \mathbf{e})$. In this case, Eq. (6.2) yields

$$(u-m)!\,\Gamma(m+\tfrac{1}{2})I_{u+\frac{1}{2}}(rQ)$$
$$\times\, C_{u-m}^{m+\frac{1}{2}}((z+\gamma/r)Q^{-1})Q^{-m-\frac{1}{2}}[2(t+2\beta/r)]^m$$
$$=\sum_{v,n}\{v,n\,|\,\mathbf{w};\mathbf{e}\,|u,m\}(v-n)!\,\Gamma(n+\tfrac{1}{2})$$
$$\times\, I_{v+\frac{1}{2}}(r)C_{v-n}^{n+\frac{1}{2}}(z)(2t)^n, \tag{6.3}$$

where

$$Q=\left[1+\frac{2\beta(1-z)^2}{rt}+\frac{2\alpha}{r}\left(t+\frac{2\beta}{r}\right)+\frac{\gamma^2}{r^2}+\frac{2\gamma z}{r}\right]^{\frac{1}{2}}.$$

When applied to the representation $\rho_0(1)$, Eq. (6.3) constitutes a generalization of the so-called addition theorem for spherical waves.[14] We will list a few special cases of Eq. (6.3), treating the representations $\rho_0(1)$ and $\rho_\mu(1)$ simultaneously.

If $\alpha=\beta=0$, Eq. (6.3) yields

$$(u-m)!\,I_{u+\frac{1}{2}}(rR)C_{u-m}^{m+\frac{1}{2}}[(z+\gamma/r)R^{-1}]R^{-m-\frac{1}{2}}$$
$$=\sum_{k=m-u}^{\infty}(u+k-m)!\,I_m^{u+k,u}(\gamma)I_{u+k+\frac{1}{2}}(r)C_{u-m+k}^{m+\frac{1}{2}}(z), \tag{6.4}$$

where

$$R=(1+2\gamma z/r+\gamma^2/r^2)^{\frac{1}{2}},\quad |2\gamma z/r+\gamma^2/r^2|<1.$$

When $m=u$, this expression simplifies to the well-known addition theorem of Gegenbauer:

$$I_{u+\frac{1}{2}}(rR)(2R)^{-u-\frac{1}{2}}$$
$$=\Gamma(u+\tfrac{1}{2})\sum_{k=0}^{\infty}(u+k+\tfrac{1}{2})I_{u+k+\frac{1}{2}}(\gamma)I_{u+k+\frac{1}{2}}(r)C_k^{u+\frac{1}{2}}(z).$$

[14] B. Friedman and J. Russek, Quart. Appl. Math. 12, 13 (1954).

There is an interesting special form of Eq. (6.4), obtained by setting $z=1$:

$$(1+\gamma/r)^{-m-\frac{1}{2}}I_{u+\frac{1}{2}}(r+\gamma)$$
$$=\sum_{k=m-u}^{\infty}\frac{\Gamma(u+m+k+1)}{\Gamma(u+m+1)}I_m^{u+k,u}(\gamma)I_{u+k+\frac{1}{2}}(r),$$
$$|\gamma/r|<1.$$

When $m=u$, the above identity simplifies to

$$(1+\gamma/r)^{-u-\frac{1}{2}}I_{u+\frac{1}{2}}(\cdot\ +\gamma)$$
$$=(2/\gamma)^{u+\frac{1}{2}}\frac{\Gamma(u+\frac{1}{2})}{\Gamma(2u+1)}$$
$$\times\sum_{k=0}^{\infty}\frac{\Gamma(2u+k+1)(u+k+\frac{1}{2})}{k!}I_{u+k+\frac{1}{2}}(\gamma)I_{u+k+\frac{1}{2}}(r).$$

If $\beta=\gamma=0$, Eqs. (6.3) and (5.11) give

$$I_{u+\frac{1}{2}}(rS)C_{u-m}^{m+\frac{1}{2}}(zS^{-1})S^{-m-\frac{1}{2}}$$
$$=\sum_{k=0}^{\infty}\sum_{j=0}^{[(u-m)/2]}\frac{(\alpha t)^k(-1)^j\Gamma(u-j+\frac{1}{2})(u+k-2j+\frac{1}{2})}{(u-m-2j)!\,j!\,(k-j)!\,\Gamma(u+k-j+\frac{3}{2})}$$
$$\times\frac{(u-m-2j)!\,\Gamma(m+k+\frac{1}{2})}{\Gamma(m+\frac{1}{2})}I_{u+k-2j+\frac{1}{2}}(r)$$
$$\times\, C_{u-m-2j}^{m+k+\frac{1}{2}}(z), \tag{6.5}$$

where

$$S=(1+2\alpha t/r)^{\frac{1}{2}},\quad |2\alpha t/r|<1.$$

When $m=u$, Eq. (6.5) reduces to

$$I_{u+\frac{1}{2}}[r(1+2\alpha/r)^{\frac{1}{2}}](1+2\alpha/r)^{-u-\frac{1}{2}}=\sum_{k=0}^{\infty}\frac{\alpha^k}{k!}I_{u+k+\frac{1}{2}}(r),$$
$$|2\alpha/r|<1.$$

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Special Functions and the Complex Euclidean Group in 3-Space. II

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This paper is the second in a series devoted to the derivation of identities for special functions which can be obtained from a study of the local irreducible representations of the Euclidean group in 3-space. A number of identities obeyed by Jacobi polynomials and Whittaker functions are derived and their group-theoretic meaning is discussed.

INTRODUCTION

Much of the theory of special functions, as it is applied in mathematical physics, is a disguised form of Lie group theory. The symmetry groups, which are built into the foundations of modern physics, determine many of the special functions which can arise in physics, as well as the principal properties of these functions. It is the author's opinion that a detailed analysis of this relationship between Lie theory and special functions is of importance for a good understanding of both special function theory and the laws of physics.