yields

$$(u-m)!\,\Gamma(m+\tfrac{1}{2})I_{u+\frac{1}{2}}(rQ)\times C_{u-m}^{m+\frac{1}{2}}((z+\gamma/r)Q^{-1})Q^{-m-\frac{1}{2}}[2(t+2\beta/r)]^m = \sum_{v,n}\{v,n|\,\mathbf{w};\mathbf{e}\,|u,m\}(v-n)!\,\Gamma(n+\tfrac{1}{2})\times I_{v+\frac{1}{2}}(r)C_{v-n}^{n+\frac{1}{2}}(z)(2t)^n, \quad (6.3)$$

where

$$Q=\left[1+\frac{2\beta(1-z)^2}{rt}+\frac{2\alpha}{r}\left(t+\frac{2\beta}{r}\right)+\frac{\gamma^2}{r^2}+\frac{2\gamma z}{r}\right]^{\frac{1}{2}}.$$

When applied to the representation $\rho_0(1)$, Eq. (6.3) constitutes a generalization of the so-called addition theorem for spherical waves.[14] We will list a few special cases of Eq. (6.3), treating the representations $\rho_0(1)$ and $\rho_\mu(1)$ simultaneously.

If $\alpha=\beta=0$, Eq. (6.3) yields

$$(u-m)!\,I_{u+\frac{1}{2}}(rR)C_{u-m}^{m+\frac{1}{2}}[(z+\gamma/r)R^{-1}]R^{-m-\frac{1}{2}} = \sum_{k=m-u}^{\infty}(u+k-m)!\,I_m^{u+k,u}(\gamma)I_{u+k+\frac{1}{2}}(r)C_{u-m+k}^{m+\frac{1}{2}}(z), \quad (6.4)$$

where

$$R=(1+2\gamma z/r+\gamma^2/r^2)^{\frac{1}{2}}, \quad |2\gamma z/r+\gamma^2/r^2|<1.$$

When $m=u$, this expression simplifies to the well-known addition theorem of Gegenbauer:

$$I_{u+\frac{1}{2}}(rR)(2R)^{-u-\frac{1}{2}} = \Gamma(u+\tfrac{1}{2})\sum_{k=0}^{\infty}(u+k+\tfrac{1}{2})I_{u+k+\frac{1}{2}}(\gamma)I_{u+k+\frac{1}{2}}(r)C_k^{u+\frac{1}{2}}(z).$$

[14] B. Friedman and J. Russek, Quart. Appl. Math. **12**, 13 (1954).

There is an interesting special form of Eq. (6.4), obtained by setting $z=1$:

$$(1+\gamma/r)^{-m-\frac{1}{2}}I_{u+\frac{1}{2}}(r+\gamma) = \sum_{k=m-u}^{\infty}\frac{\Gamma(u+m+k+1)}{\Gamma(u+m+1)}I_m^{u+k,u}(\gamma)I_{u+k+\frac{1}{2}}(r), \quad |\gamma/r|<1.$$

When $m=u$, the above identity simplifies to

$$(1+\gamma/r)^{-u-\frac{1}{2}}I_{u+\frac{1}{2}}(\cdot+\gamma) = (2/\gamma)^{u+\frac{1}{2}}\frac{\Gamma(u+\frac{1}{2})}{\Gamma(2u+1)}\times\sum_{k=0}^{\infty}\frac{\Gamma(2u+k+1)(u+k+\frac{1}{2})}{k!}I_{u+k+\frac{1}{2}}(\gamma)I_{u+k+\frac{1}{2}}(r).$$

If $\beta=\gamma=0$, Eqs. (6.3) and (5.11) give

$$I_{u+\frac{1}{2}}(rS)C_{u-m}^{m+\frac{1}{2}}(zS^{-1})S^{-m-\frac{1}{2}} = \sum_{k=0}\sum_{j=0}^{[(u-m)/2]}\frac{(\alpha t)^k(-1)^j\Gamma(u-j+\frac{1}{2})(u+k-2j+\frac{1}{2})}{(u-m-2j)!\,j!\,(k-j)!\,\Gamma(u+k-j+\frac{3}{2})}\times\frac{(u-m-2j)!\,\Gamma(m+k+\frac{1}{2})}{\Gamma(m+\frac{1}{2})}I_{u+k-2j+\frac{1}{2}}(r)\times C_{u-m-2j}^{m+k+\frac{1}{2}}(z), \quad (6.5)$$

where

$$S=(1+2\alpha t/r)^{\frac{1}{2}}, \quad |2\alpha t/r|<1.$$

When $m=u$, Eq. (6.5) reduces to

$$I_{u+\frac{1}{2}}[r(1+2\alpha/r)^{\frac{1}{2}}](1+2\alpha/r)^{-u-\frac{1}{2}} = \sum_{k=0}^{\infty}\frac{\alpha^k}{k!}I_{u+k+\frac{1}{2}}(r), \quad |2\alpha/r|<1.$$

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Special Functions and the Complex Euclidean Group in 3-Space. II

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This paper is the second in a series devoted to the derivation of identities for special functions which can be obtained from a study of the local irreducible representations of the Euclidean group in 3-space. A number of identities obeyed by Jacobi polynomials and Whittaker functions are derived and their group-theoretic meaning is discussed.

INTRODUCTION

Much of the theory of special functions, as it is applied in mathematical physics, is a disguised form of Lie group theory. The symmetry groups, which are built into the foundations of modern physics, determine many of the special functions which can arise in physics, as well as the principal properties of these functions. It is the author's opinion that a detailed analysis of this relationship between Lie theory and special functions is of importance for a good understanding of both special function theory and the laws of physics.

This paper is the second in a series analyzing the special function theory related to T_6, the complex Euclidean group in 3-space. In the first paper[1] (which we shall refer to as I), it was shown that an important class of identities relating Bessel functions and Gegenbauer polynomials had a simple interpretation in terms of certain local irreducible representations of T_6. In the present paper, which generalizes the results of I, a similar interpretation will be given for identities relating Whittaker functions and Jacobi polynomials.

Most of the identities for special functions derived in this paper are well known. We will be more interested in systematically deriving and uncovering the group-theoretic meaning of known identities than in the derivation of new identities.

Just as in I, the special functions obtained in this paper will arise in two ways: as matrix elements corresponding to local representations of T_6 and as basis vectors in a model of such a representation. Once the matrix elements of an abstract representation have been computed, they remain valid for any model of the representation. Only two models will be considered here, but the results of this paper can easily be extended to any other model which occurs in modern physical theories.

Finally, the reader will note that the algebraic and group-theoretic aspects of special function theory are emphasized at the expense of the analytic aspects. In particular, the order of summation of an infinite series will often be changed without explicit justification, and the convergence of the infinite series will not be verified. Such justification exists, however, and can be found in Ref. 2.

1. REPRESENTATIONS OF $\mathcal{T}_6$

Just as in I, we study irreducible representations of the 6-dimensional complex Lie algebra $\mathcal{T}_6$. This Lie algebra is defined by the commutation relations

$$\begin{aligned}
&[j^3, j^\pm] = \pm j^\pm, \quad [j^+, j^-] = 2j^3,\\
&[j^3, p^\pm] = [p^3, j^\pm] = \pm p^\pm,\\
&[j^+, p^+] = [j^-, p^-] = [j^3, p^3] = \mathcal{O},\\
&[j^+, p^-] = [p^+, j^-] = 2p^3,\\
&[p^3, p^\pm] = [p^+, p^-] = \mathcal{O}.
\end{aligned} \tag{1.1}$$

Here, the elements j^+, j^-, j^3 generate a subalgebra of $\mathcal{T}_6$ isomorphic to $sl(2)$, while p^+, p^-, p^3 generate a 3-dimensional Abelian ideal of $\mathcal{T}_6$.

[1] W. Miller, J. Math. Phys. **9**, 1163 (1968) (preceding paper).

[2] F. W. Schäfke, *Einführung in die Theorie der Speziellen Funktionen der Mathematischen Physik* (Springer-Verlag, Berlin, 1963), Chap. 8.

The 6-parameter Lie group T_6 consists of elements $\{\mathbf{w}, g\}$,

$$\mathbf{w} = (\alpha, \beta, \gamma) \in \mathbb{C}^3,$$
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2), \quad ad - bc = 1,$$

with group multiplication

$$\{\mathbf{w}, g\}\{\mathbf{w}', g'\} = \{\mathbf{w} + g\mathbf{w}', gg'\}, \tag{1.2}$$

where

$$g\mathbf{w} = (a^2\alpha - b^2\beta + ab\gamma, -c^2\alpha + d^2\beta - cd\gamma, 2ac\alpha - 2bd\beta + (bc + ad)\gamma). \tag{1.3}$$

The identity element of T_6 is $\{\mathbf{0}, \mathbf{e}\}$, where $\mathbf{0} = (0, 0, 0)$ and $\mathbf{e}$ is the 2×2 identity matrix. As mentioned in I, $\mathcal{T}_6$ is the Lie algebra of T_6 and a neighborhood of $\mathcal{O}$ in $\mathcal{T}_6$ can be mapped diffeomorphically onto a neighborhood of $\{\mathbf{0}, \mathbf{e}\}$ in T_6 by means of the relation

$$\{\mathbf{w}, g\} = \exp(\alpha p^+ + \beta p^- + \gamma p^3) \exp(-b/d j^+) \times \exp(-cdj^-) \exp(-2 \ln d j^3). \tag{1.4}$$

Here "exp" is the exponential map from $\mathcal{T}_6$ to T_6.

Let V be a complex abstract vector space and ρ a representation of $\mathcal{T}_6$ by linear operators on V. Set

$$\rho(p^\pm) = P^\pm, \quad \rho(p^3) = P^3,$$
$$\rho(j^\pm) = J^\pm, \quad \rho(j^3) = J^3.$$

The linear operators $P^\pm$, P^3, $J^\pm$, J^3 satisfy commutation relations analogous to Eqs. (1.1), where $[A, B] = AB - BA$ for linear operators A and B on V. The operators

$$\mathbf{P} \cdot \mathbf{P} = -P^+P^- - P^3P^3,$$
$$\mathbf{P} \cdot \mathbf{J} = -\tfrac{1}{2}(P^+J^- + P^-J^+) - P^3J^3$$

on V are of special interest, since they have the property

$$[\mathbf{P} \cdot \mathbf{P}, \rho(\alpha)] = [\mathbf{P} \cdot \mathbf{J}, \rho(\alpha)] = 0$$

for all $\alpha \in \mathcal{T}_6$. These two operators turn out to be multiples of the identity operator whenever ρ is one of the irreducible representations of $\mathcal{T}_6$ to be studied in this paper.

Let $\omega \neq 0$ and q be complex numbers. Among the known irreducible representations of $\mathcal{T}_6$,[3,4] we shall examine the following:

(1) $\uparrow_3(\omega, q)$

There is a countable basis $\{f_m^{(u)}\}$ for V such that $m = u, u - 1, u - 2, \cdots; u = -q, -q + 1, -q + 2, \cdots$; and $2q$ is not an integer.

[3] W. Miller, *On Lie Algebras and Some Special Functions of Mathematical Physics*, American Mathematical Society Memoir, No. 50 (Providence, 1964).

[4] W. Miller, *Lie Theory and Special Functions* (Academic Press Inc., New York, 1968), Chaps. 5, 6.

(2) $\uparrow_4(\omega, q)$

There is a countable basis $\{f_m^{(u)}\}$ for V such that $m = u, u-1, \cdots, -u+1, -u$; $u = -q, -q+1, \cdots$; and $-2q$ is a nonnegative integer.

(3) $R_3(\omega, q, u_0)$

Here q and u_0 are complex numbers such that $0 \le \operatorname{Re} u_0 < 1$, and none of $u_0 \pm q$ or $2u_0$ is an integer. There is a countable basis $\{f_m^{(u)}\}$ for V such that $m = u, u-1, u-2, \cdots$, and $u = u_0, u_0 \pm 1, u_0 \pm 2, \cdots$.

Corresponding to each of the above representations, the action of the infinitesimal operators on the basis vectors $f_m^{(u)}$ is given by

$$J^3 f_m^{(u)} = m f_m^{(u)}, \quad J^+ f_m^{(u)} = (m-u) f_{m+1}^{(u)}, \quad J^- f_m^{(u)} = -(m+u) f_{m-1}^{(u)}, \tag{1.5}$$

$$P^3 f_m^{(u)} = \frac{\omega(u-q+1)}{(2u+1)(u+1)} f_m^{(u+1)} + \frac{m\omega q}{u(u+1)} f_m^{(u)} - \frac{\omega(u+q)(u+m)(u-m)}{u(2u+1)} f_m^{(u-1)}, \tag{1.6}$$

$$P^+ f_m^{(u)} = \frac{\omega(u-q+1)}{(2u+1)(u+1)} f_{m+1}^{(u+1)} - \frac{(u-m)\omega q}{u(u+1)} f_{m+1}^{(u)} - \frac{\omega(u+q)(u-m)(u-m-1)}{(2u+1)u} f_{m+1}^{(u-1)}, \tag{1.7}$$

$$P^- f_m^{(u)} = -\frac{\omega(u-q+1)}{(2u+1)(u+1)} f_{m-1}^{(u+1)} - \frac{(u+m)\omega q}{u(u+1)} f_{m-1}^{(u)} + \frac{\omega(u+q)(u+m)(u+m-1)}{(2u+1)u} f_{m-1}^{(u-1)}, \tag{1.8}$$

$$\mathbf{P}\cdot\mathbf{P} f_m^{(u)} = -\omega^2 f_m^{(u)} \not\equiv 0, \quad \mathbf{P}\cdot\mathbf{J} f_m^{(u)} = -\omega q f_m^{(u)}. \tag{1.9}$$

[If a vector $f_m^{(u)}$ on the right-hand side of one of the expressions (1.5)–(1.9) does not belong to the representation space, we set this vector equal to zero.]

The reader can verify that the operators defined by expressions (1.5)–(1.8) do satisfy the commutation relations (1.1) and determine the irreducible representations of $\mathcal{E}_6$ listed above. Corresponding to a fixed value of u, the vectors $\{f_m^{(u)}\}$ form a basis for an irreducible representation of the subalgebra $sl(2)$ of $\mathcal{E}_6$. Each such representation of $sl(2)$ induced by $\uparrow_4(\omega, q)$ has dimension $2u+1$ and is denoted by $D(2u)$. Each irreducible representation of $sl(2)$, induced by $\uparrow_3(\omega, q)$ or $R_3(\omega, q, u_0)$, is infinite-dimensional and is denoted by $\downarrow_u$. The notation for the representations in classes (1)–(3) is taken from Ref. 4. A detailed analysis of the representation $D(2u)$ and $\downarrow_u$ is also given in this reference. Note that the representations $\rho_0(\omega)$, $\rho_\mu(\omega)$, studied in I, are identical with the representations $\uparrow_4(\omega, 0)$, $R_3(\omega, 0, \mu)$ presented here.

In analogy with the procedure carried out in I, we will analyze the relationship between the representations in classes (1)–(3) and the special functions of mathematical physics. That is, we will look for models of these abstract representations ρ such that the infinitesimal operators $\rho(\alpha)$, $\alpha \in \mathcal{E}_6$, are linear differential operators acting on a space V of analytic functions in n complex variables. The basis vectors $\{f_m^{(u)}\}$ are then analytic functions and expressions (1.5)–(1.8) are differential recursion relations for these "special" functions. In addition, we will extend each of our Lie-algebra representations of $\mathcal{E}_6$ to a local group representation of T_6. Each such local representation is defined by linear operators $\mathbf{T}(h)$, $h \in T_6$, acting on V and satisfying the group property $\mathbf{T}(h)\mathbf{T}(h') = \mathbf{T}(hh')$ for h, h' in a sufficiently small neighborhood of the identity. We will compute the matrix elements of $\mathbf{T}(h)$ with respect to the basis $\{f_m^{(u)}\}$. The group property then immediately yields addition theorems for these matrix elements. The addition theorems so obtained provide identities relating Bessel functions, Whittaker functions, and Jacobi polynomials.

2. MODELS OF THE REPRESENTATIONS

All possible models of the Lie-algebra representations in classes (1)–(3) are known in which the basis space consists of functions of one or two complex variables.[4] In fact, there is only one such model ($n = 2$):

$$\text{Model A} \quad J^3 = t\frac{\partial}{\partial t}, \quad J^+ = -t\frac{\partial}{\partial z}, \quad J^- = t^{-1}\left((1-z^2)\frac{\partial}{\partial z} - 2zt\frac{\partial}{\partial t} + 2q\right), \quad P^+ = \omega t, \quad P^- = \omega(1-z^2)t^{-1}, \quad P^3 = \omega z. \tag{2.1}$$

Here z, t are complex variables, and ω, q are fixed complex constants. It is easy to verify that operators (2.1) satisfy the commutation relations (1.1). Furthermore, we have

$$\mathbf{P}\cdot\mathbf{P} \equiv -\omega^2, \quad \mathbf{P}\cdot\mathbf{J} \equiv -\omega q.$$

Corresponding to this model, the basis vectors $f_m^{(u)}$ are defined up to a multiplicative constant by expressions (1.5)–(1.8), and may be given by

$$f_m^{(u)}(z, t) = \frac{(u-m)!\,\Gamma(u+m+1)}{\Gamma(u-q+1)2^m} P_{u-m}^{(m-q, m+q)}(z) t^m, \tag{2.2}$$

where $\Gamma(x)$ is the gamma function and $P_n^{(\alpha,\beta)}$ is a

Jacobi polynomial.[4] The possible values of u, m, q, ω depend on the representation in classes (1)–(3) with which we are concerned, and these values have been listed in Sec. 1.

By substituting the Model A operators and basis vectors into expressions (1.5)–(1.8), we obtain the following well-known recursion relations obeyed by Jacobi polynomials:

$$\frac{d}{dz}P_n^{(\alpha,\beta)}(z)=\tfrac{1}{2}(\alpha+\beta+n+1)P_{n-1}^{(\alpha+1,\beta+1)}(z),$$

$$\left[(1-z^2)\frac{d}{dz}-(\alpha+\beta)z+\beta-\alpha\right]P_n^{(\alpha,\beta)}(z)=-2(n+1)P_{n+1}^{(\alpha-1,\beta-1)}(z),\quad(1.5')$$

$$\begin{aligned}zP_n^{(\alpha,\beta)}(z)&=\frac{2(n+1)(\alpha+\beta+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}P_{n+1}^{(\alpha,\beta)}(z)\\&+\frac{(\beta^2-\alpha^2)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}P_n^{(\alpha,\beta)}(z)\\&+\frac{2(n+\alpha)(n+\beta)}{(\alpha+\beta+2n)(\alpha+\beta+2n+1)}P_{n-1}^{(\alpha,\beta)}(z),\quad(1.6')\end{aligned}$$

$$\begin{aligned}P_n^{(\alpha,\beta)}(z)&=\frac{(\alpha+\beta+n+1)(\alpha+\beta+n+2)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}P_n^{(\alpha+1,\beta+1)}(z)\\&+\frac{(\alpha-\beta)(\alpha+\beta+n+1)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}P_{n-1}^{(\alpha+1,\beta+1)}(z)\\&-\frac{(\alpha+n)(\beta+n)}{(\alpha+\beta+2n)(\alpha+\beta+2n+1)}P_{n-2}^{(\alpha+1,\beta+1)}(z),\end{aligned}\quad(1.7')$$

$$\begin{aligned}\tfrac{1}{4}(1-z^2)P_n^{(\alpha,\beta)}(z)&=-\frac{(n+2)(n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}P_{n+2}^{(\alpha-1,\beta-1)}(z)\\&+\frac{(\alpha-\beta)(n+1)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}P_{n+1}^{(\alpha-1,\beta-1)}(z)\\&+\frac{(\alpha+n)(\beta+n)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}P_n^{(\alpha-1,\beta-1)}(z),\end{aligned}\quad(1.8')$$

valid for $n=0,1,2,\cdots$, and $\alpha,\beta\in\mathcal{C}$.

Those representations of $\mathcal{G}_6$, for which $q=0$, have a model (Model B) in terms of differential operators in three complex variables. Model B was constructed and studied in I. If $q\neq 0$, there is no model in three complex variables. However, in Sec. 8 we will construct a model (Model C) in terms of differential operators acting on spinor-valued functions in three complex variables. The special functions obtained from Model C are closely related to the spinor-valued solutions of the wave equation in 3-space.

3. ANALYSIS OF THE MODELS

The following section contains several auxilliary lemmas which will enable us to extend the representations $\uparrow_3(\omega,q)$, $\uparrow_4(\omega,q)$, and $R_3(\omega,q,u_0)$ of $\mathcal{G}_6$ to local group representations of T_6. Throughout this section it is assumed that the operators $J^\pm$, J^3, $P^\pm$, P^3 and the basis vectors $f_m^{(u)}$ correspond to one of the irreducible Lie-algebra representations listed above. The results will be formally the same for all of these representations, the only difference being the allowable values of u, m, q, and ω.

Lemma 1: Let I be the identity operator on V:

$$\begin{aligned}(\omega I-P^3)^k f_u^{(u)}&=\frac{(2\omega)^k k!\,\Gamma(u-q+k+1)\Gamma(2u+1)}{\Gamma(u-q+1)}\\&\times\sum_{n=0}^{k}\frac{(-1)^n(2u+2n+1)}{n!\,(k-n)!\,\Gamma(2u+n+k+2)}f_u^{(u+n)}.\end{aligned}$$

Proof: Use of expression (1.6) and induction on k.

Corollary 1: Let $\alpha,\beta\in\mathcal{C}$ and k a nonnegative integer:

$$\begin{aligned}\left(\frac{1-z}{2}\right)^k&=k!\,\Gamma(k+\alpha+1)\\&\times\sum_{n=0}^{k}\frac{\Gamma(\alpha+\beta+n+1)(\alpha+\beta+2n+1)(-1)^n}{(k-n)!\,\Gamma(n+\alpha+1)\Gamma(\alpha+\beta+n+k+2)}\\&\times P_n^{(\alpha,\beta)}(z).\end{aligned}$$

Proof: This is the content of Lemma 1 when it is applied to Model A.

As is well known,[5] the Jacobi polynomials are related to the Gauss hypergeometric functions by the formula

$$P_n^{(\gamma,\delta)}(z)=\binom{n+\gamma}{n}\,{}_2F_1\left(-n,\gamma+\delta+n+1;\gamma+1;\frac{1-z}{2}\right).\quad(3.1)$$

From this expression and Corollary 1 it is a straightforward computation to obtain the identity

$$\begin{aligned}P_n^{(\gamma,\delta)}(z)=\sum_{k=0}^{n}&\frac{\Gamma(\gamma+\delta+n+k+1)\Gamma(\alpha+\beta+k+1)\Gamma(\gamma+n+1)}{\Gamma(\alpha+\beta+2k+1)\Gamma(\gamma+\delta+n+1)\Gamma(\gamma+k+1)(n-k)!}\\&\times{}_3F_2(k-n,\gamma+\delta+n+k+1,\alpha+k+1;\gamma+k+1,\alpha+\beta+2k+2;1)P_k^{(\alpha,\beta)}(z),\quad(3.2)\end{aligned}$$

expressing an arbitrary Jacobi polynomial $P_n^{(\gamma,\delta)}(z)$ as a linear combination of the polynomials $P_k^{(\alpha,\beta)}(z)$.

[5] W. Magnus, F. Oberhettinger, and R. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer-Verlag, New York, 1966), 3rd ed.

Passing from Model A back to our abstract representation, we obtain:

Lemma 2: Let $\gamma, \delta \in \mathcal{C}$ and n a nonnegative integer.

$$\begin{aligned}
&P_n^{(\gamma,\delta)}(\omega^{-1}P^3)f_u^{(u)} \\
&= \sum_{k=0}^{n} \frac{\Gamma(2u+1)\Gamma(u-q+k+1)}{k!\,(n-k)!\,\Gamma(u-q+1)} \\
&\times \frac{\Gamma(\gamma+\delta+n+k+1)\Gamma(\gamma+n+1)}{\Gamma(2u+2k+1)\Gamma(\gamma+\delta+n+1)\Gamma(\gamma+k+1)} \\
&\times {}_3F_2(k-n, \gamma+\delta+n+k+1, u-q+k+1; \\
&\qquad \gamma+k+1, 2u+2k+2; 1)f_u^{(u+k)}.
\end{aligned}$$

Although the function ${}_3F_2(1)$ appears complicated, it can be explicitly evaluated in several interesting special cases. If $\alpha = \gamma$ in Eq. (3.2), then ${}_3F_2(1)$ reduces to the form ${}_2F_1(1)$. Using the well-known formula[5]

$$ {}_2F_1(a, -n; c; 1) = \frac{\Gamma(c-a+n)\Gamma(c)}{\Gamma(c-a)\Gamma(c+n)}, \qquad n = 0, 1, 2, \cdots, $$

we find

$$\begin{aligned}
P_n^{(\alpha,\delta)}(z) = &\sum_{k=0}^{n} \frac{\Gamma(\alpha+\delta+n+k+1)\Gamma(\alpha+\beta+k+1)\Gamma(\alpha+n+1)}{\Gamma(\alpha+\delta+n+1)\Gamma(\alpha+k+1)\Gamma(\alpha+\beta+k+n+2)} \\
&\times \frac{(\alpha+\beta+2k+1)}{(n-k)!} \frac{\Gamma(\beta-\delta+1)}{\Gamma(\beta-\delta+k-n+1)} \\
&\times P_n^{(\alpha,\beta)}(z).
\end{aligned} \tag{3.3}$$

If $\beta = \delta$, then ${}_3F_2$ is Saalschutzian[6] and ${}_3F_2(1)$ can be explicitly evaluated to yield

$$\begin{aligned}
P_n^{(\gamma,\beta)}(z) = &\sum_{k=0}^{n} \frac{\Gamma(\gamma+\beta+n+k+1)\Gamma(\alpha+\beta+k+1)\Gamma(\beta+n+1)}{\Gamma(\gamma+\beta+n+1)\Gamma(\alpha+\beta+k+n+2)} \\
&\times \frac{(\alpha+\beta+2k+1)(-1)^{n-k}\Gamma(\alpha-\gamma+1)}{\Gamma(\beta+k+1)(n-k)!\,\Gamma(\alpha-\gamma+k-n+1)} \\
&\times P_k^{(\alpha,\beta)}(z).
\end{aligned} \tag{3.4}$$

Finally, if $\alpha = \beta$ and $\gamma = \delta$, we can use Watson's theorem[6]

$$ {}_3F_2\left(a, b, c; \frac{1}{2}+\frac{a}{2}+\frac{b}{2}, 2c; 1\right) = \frac{\Gamma(\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(\frac{1-a-b+2c}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(\frac{1-a+2c}{2}\right)\Gamma\left(\frac{1-b+2c}{2}\right)}, $$

with the result

$$\begin{aligned}
P_n^{(\gamma,\gamma)}(z) = &\sum_{k=0}^{n} \frac{\Gamma(2\gamma+n+k+1)\Gamma(\gamma+\alpha+k+1)\Gamma(\gamma+n+1)\Gamma(\frac{1}{2})}{\Gamma(2\alpha+2k+1)\Gamma(2\gamma+n+1)\Gamma(\gamma+k+1)(n-k)!} \\
&\times \frac{\Gamma(\alpha+k+\frac{1}{2})\Gamma(\gamma+k+1)\Gamma(\alpha-\gamma+1)P_k^{(\alpha,\alpha)}(z)}{\Gamma\left(\frac{k-n}{2}+\frac{1}{2}\right)\Gamma\left(\frac{2\gamma+n+k+2}{2}\right)\Gamma\left(\frac{2\alpha+n+k+3}{2}\right)\Gamma\left(\frac{2\alpha-2\gamma+k-n+2}{2}\right)}.
\end{aligned} \tag{3.5}$$

Since $\Gamma[(k-n)/2+\frac{1}{2}]$ occurs in the denominator of the right-hand side of Eq. (3.5), the coefficient of $P_k^{(\alpha,\alpha)}(z)$ is nonzero only if $n-k$ is an even integer. Because of the well-known identity[5]

$$ C_n^\lambda(z) = \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(2\lambda+n)}{\Gamma(2\lambda)\Gamma(\lambda+n+\frac{1}{2})} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(z), $$

expression (1.16) is readily seen to be equivalent to Corollary 3 of I.

[6] L. J. Slater, *Generalized Hypergeometric Functions* (Cambridge University Press, Cambridge, England, 1966), Chap. 2.

In the following sections we shall find it useful to expand the product $P_n^{(\alpha,\beta)}(z)P_l^{(\alpha,\beta)}(z)$ as a linear combination of Jacobi polynomials $P_k^{(\alpha,\beta)}(z)$:

$$ P_n^{(\alpha,\beta)}(z)P_l^{(\alpha,\beta)}(z) = \sum_{k=0}^{n+l} E^{\alpha,\beta}(n, l; k)P_k^{(\alpha,\beta)}(z). \tag{3.6}$$

The coefficient $E^{\alpha,\beta}(\cdot)$ can be obtained by first using Eq. (3.1) to express the left-hand side of Eq. (3.6) as a polynomial in $(1-z)$ and then using Corollary 1 to write the resulting polynomial as a linear combination of $P_k^{(\alpha,\beta)}(z)$. The result is

$$E^{\alpha,\beta}(n,l;k)=\sum_{r=0}^{n}\sum_{s=0}^{l}\frac{\Gamma(\alpha+n+1)\Gamma(\alpha+l+1)\Gamma(\alpha+\beta+k+1)}{\Gamma(\alpha+\beta+n+1)\Gamma(\alpha+\beta+l+1)}$$
$$\times\frac{\Gamma(\alpha+\beta+n+r+1)\Gamma(\alpha+\beta+l+s+1)\Gamma(\alpha+r+s+1)}{\Gamma(\alpha+k+1)\Gamma(\alpha+r+1)\Gamma(\alpha+s+1)\Gamma(\alpha+\beta+r+s+k+2)}\times\frac{(-1)^k(\alpha+\beta+2k+1)(r+s)!}{r!(n-r)!s!(l-s)!(r+s-k)!}. \tag{3.7}$$

This is not a very enlightening expression. However, in certain special cases, the coefficients can be evaluated very simply. For example, as was shown in I, if $\alpha=\beta=\lambda-\frac{1}{2}$, then Eq. (3.6) becomes

$$C_n^\lambda(z)C_l^\lambda(z)=\sum_{k=0}^{\min(n,l)}\frac{(n+l-2k)!(\lambda+n+l-2k)}{k!(n-k)!(l-k)!}$$
$$\times\frac{\Gamma(2\lambda+n+l-k)\Gamma(\lambda+l-k)\Gamma(\lambda+k)\Gamma(\lambda+n-k)}{\Gamma(2\lambda+n+l-2k)\Gamma(\lambda+n+l-k+1)\Gamma^2(\lambda)}\times C_{n+l-2k}^\lambda(z).$$

The reader can undoubtedly derive other formulas for $E^{\alpha,\beta}(\cdot)$, some of which are more transparent than Eq. (3.7). In particular, it is not difficult to show [by means of the recursion relation (1.6)] that $E^{\alpha,\beta}(n,l;k)=0$, unless $n+l\geq k\geq|n-l|$. Here we will merely point out the connection between these coefficients and the representation theory of $\mathcal{G}_6$.

Expression (3.6) has been established by direct computation for Model A, but it implies the existence of a similar expression obeyed by the abstract representations of $\mathcal{G}_6$ and by any model of these representations.

Lemma 3:

$$P_n^{(m-q,m+q)}(\omega^{-1}P^3)f_m^{(m+l)}=\sum_{k=0}^{n+l}\frac{l!\,\Gamma(2m+l+1)\Gamma(m-q+k+1)}{k!\,\Gamma(2m+k+1)\Gamma(m-q+l+1)}\times E^{m-q,m+q}(n,l;k)f_m^{(m+k)},\quad n,l=0,1,2,\cdots.$$

Corollary 2:

$$P_n^{(m-q,m+q)}(\omega^{-1}P^3)f_m^{(m)}=\frac{\Gamma(2m+1)\Gamma(m-q+n+1)}{n!\,\Gamma(2m+n+1)\Gamma(m-q+1)}f_m^{(m+n)}.$$

Lemma 4:

$$(\tfrac{1}{2}P^+)^l f_m^{(m+n)}=\omega^l\sum_{k=\max(n-2l,0)}^{n}\frac{n!\,\Gamma(m-q+l+k+1)\Gamma(2m+n+k+1)}{k!\,(n-k)!\,\Gamma(m-q+k+1)\Gamma(2m+2l+2k+1)}$$
$$\times\,{}_3F_2(k-n,2m+n+k+1,m-q+l+k+1;m-q+k+1,2m+2l+2k+2;1)f_{m+l}^{(m+l+k)}.$$

Proof: It follows from Eq. (3.2) that the lemma is true for Model A. Hence, it must be true for any model.

Let P be a linear operator on V and $a\in\mathcal{C}$. Define $\exp(aP)$ as the formal sum $\sum_{k=0}^\infty(a^k/k!)(P)^k$. We will use our lemmas to compute the operators $\exp(aP^3)$, $\exp(aP^+)$, and $\exp(aP^-)$ on V. These results are purely formal when applied to the abstract representations of $\mathcal{G}_6$. However, when applied to models of these representations, they have a rigorous justification.

Lemma 5:

$$\exp(a(P^3-\omega I))f_u^{(u)}=\sum_{n=0}^\infty\frac{\Gamma(2u+1)\Gamma(u+n-q+1)}{n!\,\Gamma(u-q+1)\Gamma(2u+2n+1)}(2a\omega)^n\times{}_1F_1(m-q+n+1;2m+2n+2;-2a\omega)f_u^{(u+n)}.$$

Proof: This is a direct consequence of Lemma 1.

It will be shown later that Lemma 5 is valid for Model A. Thus, we have:

Corollary 3: Let $\alpha,\beta,a\in\mathcal{C}$. Then

$$e^{az}=(2a)^{-1-(\alpha+\beta)/2}\times\sum_{n=0}^\infty\frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)}M_{\chi,\mu}(2a)P_n^{(\alpha,\beta)}(z),$$

where $\chi=(\alpha-\beta)/2$, $\mu=n+(\alpha+\beta+1)/2$, and

$$M_{\chi,\mu}(a)=e^{a/2}a^{\mu+\frac{1}{2}}{}_1F_1(\mu+\chi+\tfrac{1}{2};1+2\mu;-a)$$

is a Whittaker function.[5]

Corollary 4:

$$\exp(aP^3)f_u^{(u)} = (2a\omega)^{-1-u}\sum_{n=0}^{\infty}\frac{\Gamma(2u+1)\Gamma(u-q+n+1)}{n!\,\Gamma(u-q+1)\Gamma(2u+2n+1)} \times M_{-q,u+n+\frac{1}{2}}(2a\omega)f_u^{(u+n)}.$$

Corollary 5:

$$\exp(a\omega P^3) = (2a\omega)^{-1-(\alpha+\beta)/2}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)}M_{\chi,\mu}(2a\omega) \times P_n^{(\alpha,\beta)}(P^3),$$

$$\chi = (\alpha-\beta)/2,\quad \mu = n+(\alpha+\beta+1)/2.$$

4. LOCAL REPRESENTATIONS OF T_6

Since the Model A operators (1.10) satisfy the commutation relations of $\mathcal{T}_6$, they induce a local representation of T_6 by operators $\mathbf{T}(h)$, $h \in T_6$, acting on the space of analytic functions in two complex variables. The details necessary for the computation of $\mathbf{T}(h)$ have been listed elsewhere.[4,7] We present only the results. According to the group multiplication law, it follows that

$$\mathbf{T}(h) = \mathbf{T}(\mathbf{w}; g) = \mathbf{T}(\mathbf{w}; \mathbf{e})\mathbf{T}(\mathbf{0}; g),$$

where

$$h = \{\mathbf{w}, g\},\quad \mathbf{w} = (\alpha, \beta, \gamma) \in \mathbb{C}^3,$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2).$$

Let f be an analytic function defined in a neighborhood of some point $(z, t) \in \mathbb{C}^2$ $(t \neq 0)$. Then

$$[\mathbf{T}(\mathbf{w}; \mathbf{e})f](z,t) = [\exp(\alpha P^+ + \beta P^- + \gamma P^3)f](z,t) = \exp\omega[\alpha t + \beta(1-z^2)t^{-1} + \gamma z]f(z,t), \quad (4.1)$$

$$[\mathbf{T}(\mathbf{0}; g)f](z,t) = [\exp(-b/dJ^+)\exp(-cdJ^-) \times \exp(-2\ln dJ^3)f](z,t)$$
$$= \left(\frac{at+c(z-1)}{at+c(z+1)}\right)^q f\left(z(1+2bc)+abt + \frac{cd}{t}(z^2-1),\ a^2t + 2acz + \frac{c^2}{t}(z^2-1)\right). \quad (4.2)$$

These operators satisfy the group property

$$\mathbf{T}(hh')f = \mathbf{T}(h)[\mathbf{T}(h')f], \quad (4.3)$$

whenever both sides of this expression are well defined.

[7] H. W. Guggenheimer, *Differential Geometry* (McGraw-Hill Book Co., Inc., New York, 1963), Chap. 7.

5. MATRIX ELEMENTS FOR $\uparrow_4(\omega, q)$

We are now able to compute the matrix elements of the group representations of T_6 induced by the representations $\uparrow_4(\omega, q)$. The restrictions of these representations to the real Euclidean group in 3-space are known to be unitary and irreducible, and have been studied in detail elsewhere.[4,8,9]

Throughout this section, $u, v = -q, -q+1, \cdots$; and $-2q$ is a nonnegative integer. Furthermore, m and n will range over the values $m = -u, -u+1, \cdots, u-1, u$; and $n = -v, -v+1, \cdots, v-1, v$. The matrix elements $\{v, n|\,\mathbf{w}, g\,|u, m\}$ of $\uparrow_4(\omega, q)$ are defined by

$$\mathbf{T}(\mathbf{w}, g)f_m^{(u)} = \sum_{2v=-2q}^{\infty}\sum_{n=-v}^{v}\{v, n|\,\mathbf{w}, g\,|u, m\}f_n^{(v)}, \quad (5.1)$$

where the operator $\mathbf{T}(\mathbf{w}, g)$ and basis functions $f_m^{(u)}$ refer to Model A. According to Ref. 2, the Jacobi polynomials (2.2) form an analytic basis for the representation space. That is, the functions $\mathbf{T}(\mathbf{w}, g)f_m^{(u)}$ can be expressed uniquely as a linear combination of basis functions $f_n^{(v)}$ uniformly convergent in a suitable domain. The coefficients in the expansion are bounded linear functionals of the argument $\mathbf{T}(\mathbf{w}, g)f_m^{(u)}$ in the topology of uniform convergence on compact sets.

Under these conditions, the matrix elements (5.1) are model-independent: They are uniquely determined by relations (1.5)–(1.8) and are the same for every model of $\uparrow_4(\omega, q)$ which has an analytic basis.[4] We can compute the matrix elements using either (1.5)–(1.8) or Model A and our results will automatically be valid for any other model of $\uparrow_4(\omega, q)$. Moreover, the relation

$$\mathbf{T}(\mathbf{w}, g)\mathbf{T}(\mathbf{w}', g') = \mathbf{T}(\mathbf{w} + g\mathbf{w}', gg')$$

implies the addition theorem[4]

$$\sum_{2v'=-2q}^{\infty}\sum_{n'=-v'}^{v'}\{v, n|\,\mathbf{w}, g\,|v', n'\}\{v', n'|\,\mathbf{w}', g'\,|u, m\} = \{v, n|\,\mathbf{w} + g\mathbf{w}', gg'\,|u, m\}. \quad (5.2)$$

The matrix elements $\{v, n|\,\mathbf{0}, g\,|u, m\}$ are uniquely determined by the J operators (1.5) and depend entirely on the representation theory of $Sl(2)$. Indeed, for fixed u the vectors $f_m^{(u)}$ form a basis for the $(2u+1)$-dimensional irreducible representation of $Sl(2)$. The matrix elements of these finite-dimensional

[8] N. Y. Vilenkin, E. L. Akim, and A. A. Levin, Dokl. Akad. Nauk SSSR **112**, 987 (1957).

[9] W. Miller, Commun. Pure Appl. Math. **17**, 527 (1964).

representations are well known[4]:

$$\{v, n|\, \mathbf{0}, g\,|u, m\} = \frac{d^{u-n}a^{u+m}b^{n-m}(u-m)!}{(u-n)!} \times {}_2F_1\frac{(n-u, -m-u; n-m+1; bc/ad)}{\Gamma(n-m+1)}\delta_{v,u}$$
$$= d^{u-m}a^{u+n}c^{m-n}\frac{(u+m)!}{(u+n)!} \times \frac{{}_2F_1(m-u, -n-u; m-n+1; bc/ad)}{\Gamma(m-n+1)}\delta_{v,u},$$
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2), \quad ad - bc = 1. \quad (5.3)$$

Because of the relation

$$\lim_{c\to -k}\frac{{}_2F_1(a, b; c; z)}{\Gamma(c)} = \frac{a(a+1)\cdots(a+k)b(b+1)\cdots(b+k)}{(k+1)!} \times z^{k+1}{}_2F_1(a+k+1, b+k+1; k+2; z), \quad k = 0, 1, 2, \cdots,$$

expressions (5.3) make sense for all permissible values of m and n. Note that the hypergeometric functions can be expressed in terms of Jacobi polynomials.

In terms of Model A, the identity

$$\mathbf{T}(\mathbf{0}; g)f_m^{(u)} = \sum_{n=-u}^{u}\{u, n|\, \mathbf{0}, g\,|u, m\}f_n^{(u)} \quad (5.4)$$

implies

$$[at - c(z-1)]^{m+q}[at + c(z+1)]^{m-q}P_{u-m}^{(m-q,m+q)} \times \left[z(1+2bc) + abt + \frac{cd}{t}(z^2-1)\right]$$
$$= \sum_{n=-u}^{u} d^{u-m}a^{u+n}(2c)^{m-n}\frac{(u-n)!}{(u-m)!} \times \frac{{}_2F_1(m-u, -n-u; m-n+1; bc/ad)}{\Gamma(m-n+1)} \times P_{u-n}^{(n-q,n+q)}(z)t^{n+m},$$
$$\left|\frac{c(z \pm 1)}{at}\right| < 1, \quad ad - bc = 1. \quad (5.5)$$

When $u = m$, Eq. (5.5) simplifies to

$$[1 - c(z-1)]^{u+q}[1 + c(z+1)]^{u-q} = \sum_{n=-u}^{u}(2c)^{u-n}P_{u-n}^{(n-q,n+q)}(z), \quad |c(z \pm 1)| < 1.$$

Since the Model A functions $f_m^{(u)}(z, t)$ form an analytic basis, Lemma 5 and its corollaries are rigorously true for Model A. Thus,

$$\mathbf{T}(0, 0, \gamma; \mathbf{e})f_m^{(u)} = \exp(\gamma P^3)f_m^{(u)}$$
$$= (2\gamma)^{-1-m}\sum_{k=0}^{\infty}\frac{\Gamma(2m+k+1)}{\Gamma(2m+2k+1)}M_{-q,m+k+\frac{1}{2}}(2\gamma) \times P_k^{(m-q,m+q)}(\omega^{-1}P^3)f_m^{(u)}$$
$$= (2\gamma)^{-1-m}\sum_{j=-\infty}^{\infty}f_m^{(u+j)}\sum_{k=0}^{\infty}\frac{\Gamma(2m+k+1)}{\Gamma(2m+2k+1)} \times M_{-q,m+k+\frac{1}{2}}(2\gamma)\frac{(u-m)!\,(u+m)!\,(u-q+j)!}{(u-m+j)!\,(u+m+j)!\,(u-q)!} \times E^{m-q,m+q}(k, u-m; u-m+j)$$

and

$$\{v, n|\, 0, 0, \gamma; \mathbf{e}\,|u, m\} = \delta_{n,m}(2\gamma)^{-1-m} \times \sum_k \frac{(2m+k)!\,(u-m)!\,(u+m)!\,(v-q)!}{(2m+2k)!\,(v-m)!\,(v+m)!\,(u-q)!} \times E^{m-q,m+q}(k, u-m; v-m)M_{-q,m+k+\frac{1}{2}}(2\gamma), \quad (5.6)$$

where the sum is taken over the finite number of values of k such that the summand is defined. In the special case $m = u$ we obtain

$$\{v, n|\, 0, 0, \gamma; \mathbf{e}\,|u, u\} = \delta_{n,u}(2\gamma)^{-1-u}\frac{(2u)!\,(v-q)!}{(2v)!\,(v-u)!\,(u-q)!}M_{-q,v+\frac{1}{2}}(2\gamma), \quad \text{if } v \geq u,$$
$$= 0, \quad \text{if } v < u. \quad (5.7)$$

To compute the general matrix element

$$\{v, n|\, \alpha, \beta, \gamma; \mathbf{e}\,|u, m\},$$

we make use of the identity

$$\exp \omega[\alpha t + \beta(1-z^2)t^{-1} + \gamma z] = \sum_{j=0}^{\infty}\sum_{k=-j}^{j}(\pi\omega\rho/2)^{-\frac{1}{2}}(4/\rho)^{|k|}(\alpha)^{(|k|+k)/2}(-\beta)^{(|k|-k)/2} \times \frac{\Gamma(|k|+\frac{1}{2})\Gamma(k+\frac{1}{2})(j-k)!\,(j+\frac{1}{2})}{(j+|k|)!} \times I_{j+\frac{1}{2}}(\omega\rho)C_{j-k}^{|k|+\frac{1}{2}}(\gamma/\rho)C_{j-k}^{k+\frac{1}{2}}(z)(2t)^k, \quad (5.8)$$

which was derived in I. Here $\rho^2 = \gamma^2 + 4\alpha\beta$, $C_j^\lambda(z)$ is a Gegenbauer polynomial and

$$I_\lambda(z) = \frac{(z/2)^\lambda}{\Gamma(1+\lambda)}{}_0F_1(\lambda+1; z^2/4) = \frac{(2z)^{-\frac{1}{2}}2^{-2\lambda}}{\Gamma(1+\lambda)}M_{0,\lambda}(2z)$$

is a modified Bessel function. The right-hand side of Eq. (5.8) is an entire function of αt, β/t, γ, and z. Furthermore, it is a function of ρ^2.

The second identity we will need is related to the representation theory of $SL(2)$:

$$P_{u-m'}^{m'-m,m'+m}(z)P_{v-n'}^{n'-n,n'+n}(z) = \sum_{s=0}^{2\min(u,v)}D(u, m, m'; v, n, n'; s) \times C(u, m; v, n\,|\,u+v-s, m+n) \times C(u, m'; v, n'\,|\,u+v-s, m'+n') \times P_{u+v-s-m'-n'}^{m'+n'-m-n,m'+n'+m+n}(z). \quad (5.9)$$

Here,

$$D(u, m, m'; v, n, n'; s) = \left[\frac{(u-m)!\,(u+m)!\,(v-n)!\,(v+n)!}{(u-m')!\,(u+m')!\,(v-n')!\,(v+n')!} \times \frac{(u+v-s-m'-n')!\,(u+v-s+m'+n')!}{(u+v-s-m-n)!\,(u+v-s+m+n)!}\right]$$

and the $C(\cdot\,;\cdot\,|\,\cdot)$ are Clebsch–Gordan coefficients. (For a group-theoretic proof of this result see Refs. 4, 10, 11.)

Now, making use of Model A, we have

$$\mathbf{T}(\alpha, \beta, \gamma; \mathbf{e}) f_m^{(u)}(z, t) = \sum_{v,n} \{v, n \,|\, \alpha, \beta, \gamma; \mathbf{e} \,|\, u, m\} f_n^{(v)}(z, t) = \exp\left[\omega(\alpha t + \beta(1 - z^2)/t + \gamma z)\right] \times \frac{(u-m)!\,(u+m)!}{(u-q)!\,2^m} P_{u-m}^{(m-q,m+q)}(z) t^m. \quad (5.10)$$

Applying the two identities to the right-hand side of this expression, we obtain

$$\{v, n \,|\, \alpha, \beta, \gamma; \mathbf{e} \,|\, u, m\} = (\pi\omega\rho/2)^{-\frac{1}{2}} (4/\rho)^{|n-m|} \times (4\alpha)^{(|n-m|+n-m)/2} \left(\frac{-\beta}{4}\right)^{(|n-m|+m-n)/2} \times \frac{\Gamma(|n-m|+\frac{1}{2})\Gamma(n-m+\frac{1}{2})(n-m)!}{(2n-2m)!} \times \sum_s G(u, m; v, n; q, s) C(v-u+s; u, q \,|\, v, q) \times C(v-u+s, n-m; u, m \,|\, v, n) \times I_{v-u+s+\frac{1}{2}}(\omega\rho) C_{v-u+s-|n-m|}^{|n-m|+\frac{1}{2}}(\gamma/\rho), \quad (5.11)$$

where

$$G(u, m; v, n; q, s) = \frac{(v-u+s)}{(v-u+s+|n-m|)!} \times \left[\frac{(u+q)!\,(v-u+s-n+m)!\,(v-u+s+n-m)!\,(u-m)!\,(u+m)!\,(v-q)!}{(u-q)!\,(v-k-m)!\,(v+k+m)!\,(v+q)!}\right]^{\frac{1}{2}}.$$

The sum is taken over the finite set of nonnegative integral values for which the summand is defined. These matrix elements are entire functions of α, β, and γ.

By substituting expressions (5.3) and (5.11) for the matrix elements of $\uparrow_4(\omega, q)$ into the addition theorem (5.2), the reader can derive a number of identities relating spherical Bessel functions and Gegenbauer polynomials.

6. MORE MATRIX ELEMENTS

The expressions for matrix elements of $\uparrow_4(\omega, q)$ were rather complicated, and the expressions for matrix elements of $\uparrow_3(\omega, q)$ and $R_3(\omega, q, u_0)$ are even more complicated. Nonetheless, these representations are closely related to a number of important identities in special function theory. In order to keep the computations as simple as possible, we compute directly only a few interesting special cases of the matrix elements of $\uparrow_3(\omega, q)$ and $R_3(\omega, q, u_0)$. (In Sec. 7, however, we obtain expressions for the general matrix elements by relating them to matrix elements of other representations of $\mathcal{G}_6$.)

The matrix elements $\{v, n \,|\, \mathbf{w}, g \,|\, u, m\}$ of $\uparrow_3(\omega, q)$ and $R_3(\omega, q, u_0)$ are defined by

$$\mathbf{T}(\mathbf{w}; g) f_m^{(u)}(z, t) = \sum_v \sum_n \{v, n \,|\, \mathbf{w}; g \,|\, u, m\} f_n^{(v)}(z, t), \quad (6.1)$$

where the operators and basis functions refer to Model A. [Corresponding to $\uparrow_3(\omega, q)$, the variables assume values u, $v = -q, -q+1, -q+2, \cdots$; $m = u, u-1, u-2, \cdots, n = v, v-1, v-2, \cdots$, where $2q \in \mathcal{C}$ is not an integer. Corresponding to $R_3(\omega, q, u_0)$, $u, v = u_0, \; u_0 \pm 1, \; u_0 \pm 2, \cdots$; $m = u, u-1, u-2, \cdots; n = v, v-1, v-2, \cdots$, where q, u_0 are complex numbers such that $0 \leq \mathrm{Re}\, u_0 < 1$, and none of u_0, $\pm q$, or $2u_0$ is an integer. The formal expressions giving the matrix elements are identical for both classes of representations; the difference between them is merely the different range of values assumed by the variables $u, v, m, n, q, u_0, \omega$.]

It is well known[2] that the Model A functions $f_m^{(u)}(z, t)$ form an analytic basis for the representation space. Hence, the matrix elements are well defined and uniquely determined by the Lie-algebra relations (1.5)–(1.8). Moreover, Lemma 5 and its corollaries are valid.

The action of the operators $J^{\pm}$, J^3 on the basis vectors $\{f_m^{(u)}\}$ for fixed u, $m = u, u-1, u-2, \cdots$ defines an irreducible representation $\downarrow_u$ on $sl(2)$. This infinite-dimensional representation was studied in Ref. 4, Chap. 5. Its matrix elements are

$$\{v, n \,|\, \mathbf{0}, g \,|\, u, m\} = d^{u-n} a^{u+m} b^{n-m} \frac{(u-m)!}{(u-n)!} \times \frac{{}_2F_1(n-u, -m-u; n-m+1; bc/ad)}{\Gamma(n-m+1)} \delta_{v,u}$$

[10] G. Y. Lyubarskii, *The Application of Group Theory to Physics* (Pergamon Press, Oxford, 1960), English Transl., Chap. 10.

[11] N. Y. Vilenkin, *Special Functions and Theory of Group Representations* (Izd. Nauka., Moscow, 1965).

$$= d^{u-m}a^{u+n}c^{m-n}\frac{\Gamma(u+m+1)}{\Gamma(u+n+1)}\times\frac{{}_2F_1(m-u,-n-u;m-n+1;bc/ad)}{\Gamma(m-n+1)}\delta_{v,u},$$

$$g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL(2),\quad ad-bc=1. \qquad (6.2)$$

The matrix elements define a local representation of the group $SL(2)$. That is, they are defined and satisfy the group representation property only in suitably small neighborhoods of **e**. These neighborhoods have been determined elsewhere[4] and are usually evident by inspection.

Substituting expression (6.2) into the identity

$$\mathbf{T}(\mathbf{0},g)f_m^{(u)}(z,t)=\sum_{k=0}^{\infty}\{u,u-k\,|\,\mathbf{0},g\,|u,m\}f_{u-k}^{(u)}(z,t),$$

we find

$$t^{m-u}\left(a-\frac{c(z-1)}{t}\right)^{m+q}\left(a+\frac{c(z+1)}{t}\right)^{m-q}\times P_{u-m}^{(m-q,m+q)}\left[z(1+2bc)+abt+\frac{cd}{t}(z^2-1)\right]$$
$$=\sum_{k=0}^{\infty}d^{u-m}a^{2u-k}(2c)^{m-u+k}\frac{k!}{(u-m)!}\times\frac{{}_2F_1(m-u,-2u+k;m-u+k+1;bc/ad)}{\Gamma(m-u+k+1)}\times P_k^{(u-k-q,u-k+q)}(z)t^{-k},$$
$$\left|\frac{c(z\pm1)}{at}\right|<1,\quad ad-bc=1. \qquad (6.3)$$

When $u=m$, this relation becomes

$$[1-c(z-1)]^{u+q}[1+c(z-1)]^{u-q}=\sum_{k=0}^{\infty}(2c)^kP_k^{(u-k-q,u-k+q)}(z),\quad |c(z\pm1)|<1.$$

Matrix elements of the form $\{v,n\,|\,0,0,\gamma;\mathbf{e}\,|u,m\}$ can be computed directly from Corollary 5 and Lemma 3. The result is

$$\{v,n\,|\,0,0,\gamma;\mathbf{e}\,|u,m\}=\delta_{n,m}M_{m;q}^{v,u}(2\gamma\omega)$$
$$=(2\gamma\omega)^{-1-m}\frac{\Gamma(u+m+1)(u-m)!\,\Gamma(v-q+1)}{\Gamma(v+m+1)(v-m)!\,\Gamma(u-q+1)}\times\sum_{n=0}^{\infty}\frac{\Gamma(2m+n+1)}{\Gamma(2m+2n+1)}\times E^{m-q,m+q}(n,u-m,v-m)M_{-q,m+n+\frac12}(2\gamma\omega). \qquad (6.4)$$

(The sum actually contains only a finite number of nonzero terms.)

The functions $M_{m;q}^{v,u}(\gamma)$, defined by Eq. (6.4), form a generalization of the Whittaker functions $M_{\chi,\mu}(\gamma)$, since

$$M_{u;q}^{v,u}(\gamma)=\frac{\gamma^{-1-u}}{(v-u)!}\frac{\Gamma(2u+1)\Gamma(v-q+1)}{\Gamma(2v+1)\Gamma(u-q+1)}M_{-q,v+\frac12}(\gamma)\quad\text{if}\quad v-u\geq0,$$
$$=0,\quad\text{if}\quad v-u<0. \qquad (6.5)$$

Furthermore,

$$M_{m;0}^{v,u}(2\gamma)=I_m^{v,u}(\gamma), \qquad (6.6)$$

where $I_m^{v,u}(\gamma)$ is the generalized Bessel function defined in I.

We list a few properties of the generalized Whittaker functions. The relations

$$\mathbf{T}(0,0,\gamma;\mathbf{e})f_m^{(u)}=\sum_{k=-\infty}^{\infty}\{u+k,m\,|\,0,0,\gamma,\mathbf{e}\,|u,m\}f_m^{(u+k)}$$

$$\{v,m\,|\,0,0,\gamma+\gamma';\mathbf{e}\,|u,m\}=\sum_{k=-\infty}^{\infty}\{v,m\,|\,0,0,\gamma;\mathbf{e}\,|u+k,m\}\times\{u+k,m\,|\,0,0,\gamma';\mathbf{e}\,|u,m\}$$

imply the identities

$$\frac{\Gamma(2m+l+1)l!}{\Gamma(m+l-q+1)}e^{\gamma z}P_l^{(m-q,m+q)}(z)=\sum_{k=0}^{\infty}\frac{k!\,\Gamma(k+2m+1)}{\Gamma(m+k-q+1)}M_{m;q}^{m+k,m+l}(2\gamma)P_k^{(m-q,m+q)}(z), \qquad (6.7)$$

$$M_{m;q}^{u,v}(\gamma+\gamma')=\sum_{k=-\infty}^{\infty}M_{m;q}^{u+k,v}(\gamma)M_{m;q}^{u,u+k}(\gamma'), \qquad (6.8)$$

convergent for all values of γ,γ'.

By applying the recursion relations (1.5′)–(1.8′) to expression (6.7), we can derive recursion relations for the generalized Whittaker functions:

$$\frac{(k+1)}{\gamma}M_{m;q}^{m+k+1,m+l}(\gamma)-\frac{l}{\gamma}M_{m+1;q}^{m+k+1,m+l}(\gamma)$$
$$=\frac{(m+k-q+1)}{(2m+2k+1)(2m+2k+2)}M_{m;q}^{m+k,m+l}(\gamma)-\frac{2(k+1)q}{(2m+2k+2)(2m+2k+4)}M_{m;q}^{m+k+1,m+l}(\gamma)-\frac{(k+1)(k+2)(m+q+k+2)}{(2m+2k+4)(2m+2k+5)}M_{m;q}^{m+k+2,m+l}(\gamma), \qquad (6.9)$$

$$\frac{d}{d\gamma}M_{m;q}^{m+k,m+l}(\gamma)=\frac{(m+l-q+1)}{2(m+l+1)(2m+2l+1)}M_{m;q}^{m+k,m+l+1}(\gamma)-\frac{mq}{2(m+l)(m+l+1)}M_{m;q}^{m+k,m+l}(\gamma)+\frac{l(l+m+q)(2m+l)}{(2m+2l+1)(2m+2l)}M_{m;q}^{m+k,m+l-1}(\gamma)$$

$$= \frac{(m+k-q)}{(2m+2k-1)(2m+2k)} M^{m+k-1,m+l}_{m;q}(\gamma) - \frac{mq}{2(m+k)(m+k+1)} M^{m+k,m+l}_{m;q}(\gamma) + \frac{(k+1)(k+2m+1)(k+m+q+1)}{(2m+2k+2)(2m+2k+3)} \times M^{m+k+1,m+l}_{m;q}(\gamma), \quad k, l = 0, 1, 2, \cdots.$$

The matrix elements $\{v, n|\, \alpha, 0, 0; \mathbf{e}\,|u, m\}$ can be determined easily from Lemma 4:

$$\{v, n|\, \alpha, 0, 0; \mathbf{e}\,|u, m\} = \frac{(2\alpha\omega)^{n-m}(u-m)!\,\Gamma(v-q+1)}{(n-m)!\,(u-m+n-v)!} \times \frac{\Gamma(u+m+v-n+1)}{\Gamma(m-q+v-n+1)\Gamma(2v+1)} \times {}_3F_2(v-n+m-u, u+m+v-n+1, v-q+1; m-q+v-n+1, 2v+2; 1), \quad \text{if} \quad n-m \geq |v-u|,$$
$$= 0, \quad \text{otherwise.} \tag{6.10}$$

The addition theorem

$$\{v, n|\, \alpha+\alpha', 0, 0; \mathbf{e}\,|u, m\} = \sum_{u',m'} \{v, n|\, \alpha, 0, 0; \mathbf{e}\,|u', m'\}\{u', m'|\, \alpha', 0, 0; \mathbf{e}\,|u, m\}$$

leads to an identity for the functions ${}_3F_2(1)$ which the reader can derive for himself.

The matrix elements of the operators $\exp(\beta P^+)$ and $\exp(\beta P^-)$ are very similar. In fact, if we rewrite expression (1.8) in terms of basis vectors $f^{(u)\prime}_m = (-1)^u f^{(u)}_{-m}$, we see that Eqs. (1.7) and (1.8) become formally identical. Therefore, the matrix elements of $\exp(\beta P^-)$ can be obtained from the matrix elements (6.10) by formally replacing α, n, m by β, $-n$, $-m$, respectively, and multiplying the resulting expression by $(-1)^{v-u}$:

$$\{v, n|\, 0, \beta, 0; \mathbf{e}\,|u, m\} = \frac{(2\beta\omega)^{m-n}\Gamma(u+m+1)(-1)^{v-u}}{(m-n)!\,(u+m-n-v)!} \times \frac{\Gamma(u-m+v+n+1)\Gamma(v-q+1)}{\Gamma(n-q+v-m+1)\Gamma(2v+1)} \times {}_3F_2(v-m+n-u, u-m+v+n+1, v-q+1; n-q+v-m+1, 2v+2; 1), \quad \text{if} \quad m-n \geq |v-u|,$$
$$= 0, \quad \text{otherwise.} \tag{6.11}$$

Another identity for the functions ${}_3F_2(1)$ can be derived from the addition theorem:

$$\sum_{u',m'} \{v, n|\, \alpha, 0, 0; \mathbf{e}\,|u', m'\}\{u', m'|\, 0, \beta, 0; \mathbf{e}\,|u, m\} = \sum_{u',m'} \{v, n|\, 0, \beta, 0; \mathbf{e}\,|u', m'\}\{u', m'|\, \alpha, 0, 0; \mathbf{e}\,|u, m\}.$$

In general, the matrix elements satisfy an addition theorem

$$\{v, n|\, \mathbf{w}+g\mathbf{w}'; gg'\,|u, m\} = \sum_{u',m'} \{v, n|\, \mathbf{w}, g\,|u', m'\}\{u', m'|\, \mathbf{w}', g'\,|u, m\}, \tag{6.12}$$

for g, g' in a sufficiently small neighborhood of $\mathbf{e} \in SL(2)$. This theorem can be used to derive identities relating the special functions constructed above. Several of these identities have been proved in I.

7. RELATIONS BETWEEN MATRIX ELEMENTS

We now investigate the relationship (for $g = \mathbf{e}$) between corresponding matrix elements of two different representations of $\mathcal{C}_6$, ρ, and ρ'. We suppose, first of all, that ρ and ρ' are distinct representations in the list (1)–(3), Sec. 1, except that they have the same ω. The representations may lie in different classes and may have different parameters $q, u_0; q', u_0'$. Denote the matrix elements of ρ by $\{v, n|\, \alpha, \beta, \gamma; \mathbf{e}\,|u, m\}$ and those of ρ' by $\{v', n'|\, \alpha, \beta, \gamma; \mathbf{e}\,|u', m'\}'$.

Making use of Model A and expression (3.2), we can express the ρ basis vectors as linear combinations of the ρ' basis vectors:

$$f^{(u)}_m(z, t) = \sum_{k=0}^{u-m} H(u, m, q; m', q', k) t^{m-m'} f^{(m'+k)}_{m'}(z, t)', \tag{7.1}$$

$$H(u, m, q; m', q'; k) = \frac{2^{m'-m}(u-m)!\,\Gamma(u+m+k+1)\Gamma(m'-q'+k+1)}{k!\,(u-m-k)!\,\Gamma(2m'+2k+1)\Gamma(m-q+k+1)} \times {}_3F_2(m-u+k, u+m+k+1, m'-q'+k+1; m-q+k+1, 2m'+2k+2; 1). \tag{7.2}$$

Application of the operator

$$\mathbf{T}(\alpha, \beta, \gamma; \mathbf{e}) = \exp\,[\omega(\alpha t + \beta(1-z^2)/t + \gamma z)]$$

to both sides of Eq. (7.1) yields the identity

$$\sum_{v,n} \{v, n|\, \alpha, \beta, \gamma; \mathbf{e}\,|u, m\} f^{(v)}_n(z, t) = t^{m-m'} \sum_{k=0}^{u-m} H(u, m, q; m', q'; k) \times \sum_{v',n'} \{v', n'|\, \alpha, \beta, \gamma; \mathbf{e}\,|m'+k, m'\}' f^{(v')}_{n'}(z, t)'.$$

The vectors $f^{(v')}_{n'}(z, t)'$ in this last expression can be expanded as linear combinations of vectors $f^{(v)}_n(z, t)$, where $n = m - m' + n'$ [use (7.1), interchanging primed and unprimed quantities]. Equating coefficients of $f^{(v)}_n(z, t)$ on both sides of the resulting identity, we find

$$\{v, n|\alpha,\beta,\gamma;\mathbf{e}|u,m\} = \frac{(u-m)!\,\Gamma(v-q+1)}{(v-n)!\,\Gamma(2v+1)\Gamma(v-m+m'-q'+1)}$$
$$\times \sum_{k=0}^{u-m}\sum_{l=v-m}^{\infty} \frac{\Gamma(u+m+k+1)\Gamma(m'-q'+k+1)}{k!\,(u-m-k)!\,\Gamma(2m'+2k+1)\Gamma(m-q+k+1)}$$
$$\times \frac{(m-n+l)!\,\Gamma(2m'+v-m+l+1)}{(m-v+l)!}$$
$$\times\, {}_3F_2(m-u+k, u+m+k+1, m'-q'+k+1; m-q+k+1, 2m'+2k+2; 1)$$
$$\times\, {}_3F_2(v-m-l, 2m'+v-m+l+1, v-q+1; m'+v-m-q'+1, 2v+2; 1)$$
$$\times \{m'+l, m'+n-m|\alpha,\beta,\gamma;\mathbf{e}|m'+k, m'\}', \; l, k, \quad \text{integers.} \qquad (7.3)$$

Equation (7.3) is a generalization of a number of important identities in special function theory. For example, if $\alpha = \beta = 0$; $n = m = u = v$, then this equation becomes

$$\gamma^{m'-v} M_{-q,v+\frac{1}{2}}(\gamma) = \sum_{l=0}^{\infty} \frac{(m'-q'+1)_l}{l!\,(2m'+l+1)_l} \times {}_3F_2(-l, 2m'+l+1, v-q+1; m'-q'+1, 2v+2; 1) M_{-q',m'+l+\frac{1}{2}}(\gamma), \quad (7.4)$$

where

$$(\mu)_l = \Gamma(\mu+l)/\Gamma(\mu).$$

In case $q' = 0$, $m' = v - q$, identity (7.4) simplifies to

$$\frac{\gamma^{-q-\frac{1}{2}}}{\Gamma(2v+2)} M_{-q,v+\frac{1}{2}}(\gamma) = 2^{2v-2q+1}\Gamma(v-q+\tfrac{1}{2}) \times \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \frac{(v-q+l+\frac{1}{2})}{\Gamma(2v+l+2)} (2v-2q+1)_l \times (v-q)_l(-2q)_l I_{v-q+l+\frac{1}{2}}(\gamma/2). \quad (7.5)$$

Next, the most general case, we will determine a relationship between the matrix elements of the representations ρ and ρ' when these representations correspond to different values of the nonzero parameter ω. The representation ρ has parameters q, u_0, ω, while ρ' has parameters q', u_0', ω'. There will be no loss of generality, if we assume $\omega' = 1$. As before, the matrix elements of ρ will be denoted by

$$\{v, n|\alpha,\beta,\gamma;\mathbf{e}|u,m\}$$

and those of ρ' by $\{v', n'|\alpha,\beta,\gamma;\mathbf{e}|u',m'\}'$.

If $\beta = \gamma = 0$, it is obvious from expression (6.10) that the matrix elements of ρ depend on ω according to the multiplicative factor ω^{n-m}. If $\alpha = \gamma = 0$, it follows from Eq. (6.11) that the matrix element varies as ω^{m-n}. However, if $\alpha = \beta = 0$, $\gamma \neq 0$, the ω dependence of the matrix elements is much more complicated.

To uncover the ω dependence we need a slight generalization of the identity (3.2):

Lemma 6: If $1 - x = \omega(1 - z)$, then

$$P_n^{(\gamma,\delta)}(z) = \sum_{k=0}^{n} \frac{\Gamma(\gamma+\delta+n+k+1)\Gamma(\alpha+\beta+k+1)\Gamma(\gamma+n+1)}{\Gamma(\alpha+\beta+2k+1)\Gamma(\gamma+\delta+n+1)\Gamma(\gamma+k+1)(n-k)!} \times {}_3F_2(k-n, \gamma+\delta+n+k+1, \alpha+k+1; \gamma+k+1, \alpha+\beta+2k+2; \omega^{-1}) P_k^{(\alpha,\beta)}(x).$$

This lemma is proved in exactly the same way as the identity (3.2). Making use of Model A again, we observe that

$$\mathbf{T}(0,0,\gamma;\mathbf{e}) f_m^{(u)}(z,t) = e^{\omega\gamma z} f_m^{(u)}(z,t) = \exp\,[(\omega-1)\gamma + \gamma x] f_m^{(u)}(z,t) = \exp\,[(\omega-1)\gamma+\gamma x] \sum_{k=0}^{u-m} H^{\omega}(u,m,q;m',q';k) \times t^{m-m'} f_{m'}^{(m'+k)}(x,t)', \quad (7.6)$$

where $H^{\omega}(\cdot)$ is defined by Eq. (7.2), except that the function ${}_3F_2(1)$, occurring in Eq. (7.2), is replaced by ${}_3F_2(\omega^{-1})$. Thus,

$$\sum_{v,n} \{v, n|0,0,\gamma;\mathbf{e}|u,m\} f_n^{(v)}(z,t) = e^{(\omega-1)\gamma} t^{m-m'} \sum_{k=0}^{u-m} H^{\omega}(u,m,q;m',q';k) \times \sum_{v',n'} \{v', n'|0,0,\gamma;\mathbf{e}|m'+k, m'\}' f_{n'}^{(v')}(x,t)'.$$

Expanding the right-hand side of this expression in terms of the basis $f_n^{(v)}(z,t)$, $n = m - m' + n'$, and equating coefficients of the basis vectors, we obtain the identity

$$M^{v,u}_{m;q}(\gamma\omega) = \frac{(u-m)!\,\Gamma(v-q+1)\exp[\frac{1}{2}(\omega-1)\gamma]}{(v-m)!\,\Gamma(2v+1)\Gamma(v-m+m'-q'+1)} \times \sum_{k=0}^{u-m}\sum_{l=v-m}^{\infty} \frac{\Gamma(u+m+k+1)\Gamma(m'-q'+k+1)l!\,\Gamma(2m'+v-m+l+1)}{k!\,(u-m-k)!\,\Gamma(2m'+2k+1)\Gamma(m-q+k+1)(m-v+l)!}$$
$$\times\ {}_3F_2(m-u+k, u+m+k+1, m'-q'+k+1; m-q+k+1, 2m'+2k+2; \omega^{-1})$$
$$\times\ {}_3F_2(v-m-l, 2m'+v-m+l+1, v-q+1; m'+v-m-q'+1, 2v+2; \omega)M^{m'+l,m'+k}_{m';q'}(\gamma). \quad (7.7)$$

If $m = u = v$, this identity becomes

$$\gamma^{m'-v}M_{-q,v+\frac{1}{2}}(\gamma\omega) = \exp[\tfrac{1}{2}(\omega-1)\gamma]\omega^{v+1}\sum_{l=0}^{\infty}\frac{(m'-q'+1)_l}{l!\,(2m'+l+1)_l} \times {}_3F_2(-l, 2m'+l+1, v-g+1; m'-q'+1, 2v+2; \omega)M_{-q',m'+l+\frac{1}{2}}(\gamma). \quad (7.8)$$

When $v - q = m' - q'$, Eq. (7.8) simplifies to

$$\gamma^{m'-v}M_{-q'+m'-v,v+\frac{1}{2}}\left(\gamma\frac{1-\xi}{2}\right) = \exp\left[-\frac{\gamma(1+\xi)}{4}\right]\left(\frac{1-\xi}{2}\right)^{v+1} \times \sum_{l=0}^{\infty}\frac{(m'-q'+1)_l}{(2v+2)_l}\frac{1}{(2m'+l+1)_l} \times P_l^{(2v+1,2m'-2v-1)}(\xi)M_{-q',m'+l+\frac{1}{2}}(\gamma), \quad (7.9)$$

where $\xi = 2\omega - 1$.

8. MODELS IN THREE COMPLEX VARIABLES

It was shown in I that the representations $\uparrow_4(\omega, 0)$, $\uparrow_3(\omega, 0)$, and $R_3(\omega, 0, u_0)$ have models in terms of differential operators and analytic functions in three complex variables. Those representations for which $q \neq 0$, however, have no such models. On the other hand, we will show that the matrix elements

$$\{v, n|\ \alpha, \beta, \gamma, \mathbf{e}\ |u, m\}$$

of the representations $\uparrow_4(\omega, q)$, $\uparrow_3(\omega, q)$, and $R_3(\omega, q, u_0)$ themselves define models in terms of differential operators acting on vector-valued functions of three complex variables. To see this, we consider a representation ρ from one of the classes listed above and note the relation

$$\{\mathbf{w}, g\} = \{\mathbf{0}, g\}\{g^{-1}\mathbf{w}, \mathbf{e}\} = \{\mathbf{w}, \mathbf{e}\}\{\mathbf{0}, g\},$$

which leads to the addition theorem

$$\sum_{n'} U^v_{n,n'}(g)\{v, n'|\ g^{-1}\mathbf{w}, \mathbf{e}\ |u, m\} = \sum_{m'}\{v, n|\ \mathbf{w}, \mathbf{e}\ |u, m'\}U^u_{m',m}(g) \quad (8.1)$$

for the matrix elements of ρ. Here,

$$U^v_{n,n'}(g) = \{v, n|\ \mathbf{0}, g\ |v, n'\}$$

and g is in a small enough neighborhood of $\mathbf{e} \in SL(2)$ so that all terms in Eq. (8.1) make sense.

Fix v, and consider the vector-valued function

$$\mathrm{X}^\rho_{v;u,m}(\mathbf{w}) = (\{v, n|\ \mathbf{w}, \mathbf{e}\ |u, m\}). \quad (8.2)$$

Here, n runs over the values $n = -v, -v+1, \cdots, +v$, if $\rho = \uparrow_4(\omega, q)$ and $n = v, v-1, v-2, \cdots$ if $\rho = \uparrow_3(\omega, q)$ or $\rho = R_3(\omega, q, u_0)$. Define the action $\mathbf{T}$ of T_6 on $\mathrm{X}(\mathbf{w})$ by (in matrix notation)

$$[\mathbf{T}(\mathbf{a}, g)\mathrm{X}^\rho_{v;u,m}](\mathbf{w}) = U^v(g)\mathrm{X}^\rho_{v;u,m}(g^{-1}(\mathbf{w}+\mathbf{a})). \quad (8.3)$$

Clearly,

$$\mathbf{T}(g\mathbf{a}' + \mathbf{a}, gg') = \mathbf{T}(\mathbf{a}, g)\mathbf{T}(\mathbf{a}', g').$$

According to Eq. (8.1), the vector-valued function $\mathrm{X}^\rho_{v;u,m}(\mathbf{w})$ transforms like the basis vector $f^{(u)}_m$ under the operator $\mathbf{T}(\mathbf{0}, g)$. Furthermore, it is easy to verify the relation

$$[\mathbf{T}(\mathbf{a}, \mathbf{e})\mathrm{X}^\rho_{v;u,m}](\mathbf{w}) = \sum_{v',n'}\{v', n'|\ \mathbf{a}, \mathbf{e}\ |u, m\}\mathrm{X}^\rho_{v;v',n'}(\mathbf{w}). \quad (8.4)$$

It follows from these expressions that the operators $\mathbf{T}(\mathbf{a}, g)$ and the vectors $\mathrm{X}^\rho_{v;u,m}(\mathbf{w}) \equiv f^{(u)}_m$ define a model (Model C) of the abstract representation ρ. Standard methods in the theory of Lie transformation groups[4,7] can be used to compute the infinitesimal operators corresponding to this model. The results are

$$\begin{aligned} J^+ &= \gamma\frac{\partial}{\partial\alpha} + 2\beta\frac{\partial}{\partial\gamma} + S^+, \\ J^- &= -\gamma\frac{\partial}{\partial\beta} + 2\alpha\frac{\partial}{\partial\gamma} + S^-, \\ J^3 &= -\alpha\frac{\partial}{\partial\alpha} + \beta\frac{\partial}{\partial\beta} + S^3, \\ P^+ &= \frac{\partial}{\partial\alpha}, \quad P^- = \frac{\partial}{\partial\beta}, \quad P^3 = \frac{\partial}{\partial\gamma}, \end{aligned} \quad (8.5)$$

where

$$S^{\pm}\{v, n\ |\cdot|\ u, m\} = (\pm n - v)\{v, n \mp 1\ |\cdot|\ u, m\},$$
$$S^3\{v, n\ |\cdot|\ u, m\} = n\{v, n\ |\cdot|\ u, m\}.$$

It is an immediate consequence of these results that the vectors $\mathrm{X}^\rho_{v;u,m}(\mathbf{w}) \equiv f^{(u)}_m$ and the infinitesimal operators (8.5) satisfy the recursion relations (1.5)–(1.8). Lemmas 1–5 can now be used to provide additional information about the Model C basis vectors. For example, Corollary 2 yields the identity

$$P_l^{(m-q,m+q)}\left(\omega^{-1}\frac{\partial}{\partial\gamma}\right)\mathrm{X}^\rho_{v;m,m}(\mathbf{w}) = \frac{\Gamma(2m+1)\Gamma(m-q+l+1)}{l!\,\Gamma(2m+l+1)\Gamma(m-q+1)}\mathrm{X}^\rho_{v;m+l,m}(\mathbf{w}). \quad (8.6)$$

This identity, as well as all others obtained from Lemmas 1–5, constitute generalizations of the "Maxwell theory of poles" for solutions of the wave equation.[12]

[12] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Co., Inc., New York, 1953), Vol. 2, Chap. 11.