7. SUMMARY

In this paper we considered an orthogonal Gaussian ensemble of random matrices biased by a random matrix H_0, whose eigenvalues λ_n are given by a distribution $f(\lambda)$. The limiting cases of large and small γ were considered.

The results for the small-γ limit were found to approach those for an unbiased Gaussian distribution regardless of the form of $f(\lambda)$. On the other hand, the large-γ limit results were found to approach the corresponding results for $f(\lambda)$. In each limit a perturbation method was developed and a first-order correction calculated. In the small-γ limit we considered the nth-order spacing distribution, while in the large-γ limit we considered the single-eigenvalue and nearest-neighbor spacing distribution, as well as the application of the approximation to the thermodynamics of an incompletely specified system.

It was also pointed out that the methods developed for the large-γ limit are easily intended to include the unitary and symplectic cases. Thus, many interesting problems (such as mixtures of various ensembles) can be investigated using the formalism which was developed.

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Special Functions and the Complex Euclidean Group in 3-Space. III

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This paper is the third in a series analyzing identities for special functions which can be derived from a study of the local representations of the Euclidean group in 3-space. Here identities are derived which relate Gegenbauer polynomials, Whittaker functions, Jacobi polynomials, and Bessel functions. Among the results are generalizations of the addition theorems for solid-spherical harmonics and a group-theoretic interpretation of the Maxwell theory of poles.

INTRODUCTION

This paper is the third in a series analyzing the special function theory related to T_6, the complex Euclidean group in 3-space. In the first two papers[1,2] (which we shall refer to as I and II, respectively) it was shown that important identities relating Bessel functions, Gegenbauer polynomials, Whittaker functions, and Jacobi polynomials could be derived in a straightforward manner from the study of certain local *irreducible* representations of T_6. After a brief review of terminology (Sec. 1), this paper proceeds as follows: In Secs. 2–4 we study classes of local *reducible* representations of T_6. These representations, closely related to the solution of Laplace's equation in spherical coordinates, lead to identities for Gegenbauer polynomials, which are generalizations of the addition theorems for solid-spherical harmonics.[3,4] Also, the Maxwell pole theory for spherical harmonics appears as a byproduct of the analysis. Section 5 is devoted to an examination of a class of irreducible representations closely related to the type F factorizations of Infeld and Hull.[5] These representations yield new identities for the Whittaker functions. Finally, in Sec. 6 we apply a technique developed by Weisner[6] and use T_6 to derive identities for special functions which are not directly related to the local representations of T_6.

As usual with this kind of work, most of the special function identities that we derive are well known. Our primary interest is in systematically deriving and elucidating the group-theoretic meaning of these identities rather than in deriving new identities.

The special functions studied in this paper ordinarily arise in one of two ways: as matrix elements corresponding to a local representation of T_6, or as basis vectors in a model of such a representation. Once the matrix elements have been computed, they remain valid for any model of the representation which occurs in modern physical theories.

[1] W. Miller, J. Math. Phys. **9**, 1162 (1968).
[2] W. Miller, J. Math. Phys. **9**, 1175 (1968).
[3] R. A. Sack, J. Math. Phys. **5**, 252 (1964).
[4] Y. N. Chiu, J. Math. Phys. **5**, 283 (1964).
[5] L. Infeld and T. Hull, Rev. Mod. Phys. **23**, 21 (1951).
[6] L. Weisner, Pacific J. Math. **5**, 1033 (1955).

1. THE LIE ALGEBRA $\mathcal{T}_6$

The 6-dimensional complex Lie algebra $\mathcal{T}_6$ is defined by the commutation relations

$$\begin{aligned}
&[\mathfrak{J}^3, \mathfrak{J}^\pm] = \pm\mathfrak{J}^\pm, [\mathfrak{J}^+, \mathfrak{J}^-] = 2\mathfrak{J}^3,\\
&[\mathfrak{J}^3, \mathfrak{P}^\pm] = [\mathfrak{P}^3, \mathfrak{J}^\pm] = \pm\mathfrak{P}^\pm,\\
&[\mathfrak{J}^+, \mathfrak{P}^+] = [\mathfrak{J}^-, \mathfrak{P}^-] = [\mathfrak{J}^3, \mathfrak{P}^3] = \mathcal{O},\\
&[\mathfrak{J}^+, \mathfrak{P}^-] = [\mathfrak{P}^+, \mathfrak{J}^-] = 2\mathfrak{P}^3,\\
&[\mathfrak{P}^3, \mathfrak{P}^\pm] = [\mathfrak{P}^+, \mathfrak{P}^-] = \mathcal{O}. \qquad (1.1)
\end{aligned}$$

The 6-parameter complex Lie group T_6 consists of elements $\{\mathbf{w}, g\}$, $\mathbf{w} = (\alpha, \beta, \gamma) \in C^3$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2)$, $ad - bc = 1$ with group multiplication

$$\{\mathbf{w}, g\}\{\mathbf{w}', g'\} = \{\mathbf{w} + g\mathbf{w}', gg'\}, \qquad (1.2)$$

$$g\mathbf{w} = (a^2\alpha - b^2\beta + ab\gamma, -c^2\alpha + d^2\beta - cd\gamma, \times 2ac\alpha - 2bd\beta + (bc + ad)\gamma). \qquad (1.3)$$

$\mathcal{T}_6$ is the Lie algebra of T_6 and a neighborhood of $\mathcal{O} \in \mathcal{T}_6$ can be mapped diffeomorphically onto a neighborhood of the identity $\{\mathbf{0}, \mathbf{e}\} \in T_6$ ($\mathbf{e}$ is the 2×2 identity matrix) by means of the relation

$$\{\mathbf{w}, g\} = \exp(\alpha\mathfrak{P}^+ + \beta\mathfrak{P}^- + \gamma\mathfrak{P}^3)\exp(-b/d\mathfrak{J}^+) \times \exp(-cd\mathfrak{J}^-)\exp(-2\ln d\mathfrak{J}^3), \qquad (1.4)$$

where "exp" is the exponential map.

If V is a complex abstract vector space and ρ is a representation of $\mathcal{T}_6$ by linear operators on V, we set $\rho(\mathfrak{P}^\pm) = P^\pm$, $\rho(\mathfrak{P}^3) = P^3$, $\rho(\mathfrak{J}^\pm) = J^\pm$, $\rho(\mathfrak{J}^3) = J^3$. The linear operators $P^\pm$, P^3, $J^\pm$, J^3 satisfy commutation relations analogous to (1.1), where now $[A, B] = AB - BA$ for linear operators A, B on V. The invariant operators

$$\mathbf{P}\cdot\mathbf{P} = -P^+P^- - P^3P^3, \quad \mathbf{P}\cdot\mathbf{J} = -\tfrac{1}{2}(P^+J^- + P^-J^+) - P^3J^3$$

have the property

$$[\mathbf{P}\cdot\mathbf{P}, \rho(\alpha)] = [\mathbf{P}\cdot\mathbf{J}, \rho(\alpha)] = 0$$

for all $\alpha \in \mathcal{T}_6$.

2. SOME REDUCIBLE REPRESENTATIONS

We examine the following two classes of *reducible* representations of $\mathcal{T}_6$ on a complex vector space V: $R^+(u_0)$ and $\uparrow^+$.

A. $R^+(u_0)$

Here u_0 is a complex number of such that $0 \leq \operatorname{Re} u_0 < 1$ and $2u_0$ is not an integer. There is a countable basis $\{f_m^{(u)}\}$ for V such that $m = u, u - 1, u - 2, \cdots$, and $u = u_0, u_0 \pm 1, u_0 \pm 2, \cdots$. The action of the infinitesimal operators on the basis vectors is given by

$$J^3 f_m^{(u)} = m f_m^{(u)}, \quad J^\pm f_m^{(u)} = (-u \pm m) f_{m\pm1}^{(u)}, \qquad (2.1)$$

$$P^3 f_m^{(u)} = \frac{1}{2u+1} f_m^{(u+1)}, \quad P^+ f_m^{(u)} = \frac{1}{2u+1} f_{m+1}^{(u+1)},$$

$$P^- f_m^{(u)} = \frac{-1}{2u+1} f_{m-1}^{(u+1)}, \qquad (2.2)$$

$$\mathbf{P}\cdot\mathbf{P} f_m^{(u)} \equiv 0, \quad \mathbf{P}\cdot\mathbf{J} f_m^{(u)} \equiv 0. \qquad (2.3)$$

B. $\uparrow^+$

There is a countable basis $\{f_m^{(u)}\}$ for V such that $m = u, u - 1, \cdots, -u + 1, -u$; $u = 0, 1, 2, \cdots$. The action of the infinitesimal operators on the basis vectors is given by (2.1)–(2.3). [If a vector $f_m^{(u)}$ on the right-hand side of one of the expressions (2.1)–(2.3) does not belong to the representation space, we set this vector equal to zero.]

It is left to the reader to verify that $R^+(u_0)$ and $\uparrow^+$ do define reducible representations of $\mathcal{T}_6$. In fact these representations are degenerate cases of the irreducible representations $R_3(\omega, 0, u_0)$ and $\uparrow_4(\omega, 0)$, constructed in II, obtained formally by choosing a new basis $f_m'^{(u)} = \omega^u f_m^{(u)}$ and passing to the limit as $\omega \to 0$. Corresponding to a fixed value of u, the vectors $\{f_m^{(u)}\}$ form a basis for an irreducible representation of the subalgebra $sl(2)$ of $\mathcal{T}_6$. Each such representation induced by $\uparrow^+$ has dimension $2u + 1$ and is denoted by $D(2u)$, while each representation induced by $R^+(u_0)$ is infinite-dimensional and is denoted by $\downarrow u$. A detailed analysis of the representations $D(2u)$ and $\downarrow u$ is given by Miller.[7]

In accordance with the procedure developed in I and II, we search for models of these abstract representations ρ such that the infinitesimal operators $\rho(\alpha)$, $\alpha \in \mathcal{T}_6$, are linear-differential operators in n complex variables. The basis vectors $\{f_m^{(u)}\}$ are then certain functions in these variables and the relations (2.1)–(2.3) are differential equations and recursion relations for the "special" functions $\{f_m^{(u)}\}$. Furthermore, each of our Lie algebra representations of $\mathcal{T}_6$ can be extended to a local Lie group representation of T_6. Such a local representation is defined by linear operators $\mathbf{T}(h)$, $h \in T_6$, acting on V such that $\mathbf{T}(h)\mathbf{T}(h') = \mathbf{T}(hh')$ for h, h' in a sufficiently small neighborhood of the identity. Due to this group property of the $\mathbf{T}$ operators, the matrix elements of these operators with respect to the basis $\{f_m^{(u)}\}$ will satisfy a series of addition theorems.

[7] W. Miller, *Lie Theory and Special Functions* (Academic Press Inc., New York, 1968).

3. MODELS OF THE REPRESENTATIONS

To begin we look for all models of the representation $R^+(u_0)$ in $n = 1$, 2, or 3 complex variables. According to Ref. 7, no model exists for $n = 1$. For $n = 2$, there is exactly one model (a special case of the type F operators):

$$J^3 = -z\frac{\partial}{\partial z} + u_0, \quad J^- = z^2\frac{\partial}{\partial z} - zt\frac{\partial}{\partial t} - u_0 z,$$
$$J^+ = \frac{-\partial}{\partial z} - \frac{t}{z}\frac{\partial}{\partial t} + \frac{u_0}{z}, \quad P^- = -\tfrac{1}{2}zt,$$
$$P^+ = \frac{1}{2}\frac{t}{z}, \quad P^3 = \tfrac{1}{2}t. \qquad (3.1)$$

Here z, t are complex variables and u_0 is a fixed complex constant. The constant $\frac{1}{2}$ has been chosen for convenience in the computations to follow. Clearly the operators (3.1) satisfy the commutation relations (1.1). Furthermore, $\mathbf{P}\cdot\mathbf{P} \equiv 0$, $\mathbf{P}\cdot\mathbf{J} \equiv 0$. The basis vectors $f_m^{(u)}(z, t)$ for this model of $R^+(u_0)$ are defined up to a multiplicative constant by expressions (2.1) and (2.2) and may be chosen as follows:

$$f_m^{(u)} = \Gamma(u + \tfrac{1}{2})z^k t^u,$$
$$k = u_0 - m = u_0 - u, \quad u_0 - u + 1, \quad u_0 - u + 2, \cdots. \qquad (3.2)$$

The possible values of u_0, u, m depend on the representation $R^+(u_0)$ and are listed in Sec. 2.

Since the operators (3.1) satisfy the commutation relations of $\mathcal{T}_6$, they induce a local-multiplier representation of T_6 by operators $\mathbf{T}(h)$, $h \in T_6$, acting on the space of analytic functions in 2 complex variables. The operators $\mathbf{T}(h)$ can easily be computed from standard results in local Lie theory. We list only the results.

Clearly $\mathbf{T}(h) = \mathbf{T}(\mathbf{w}, g) = \mathbf{T}(\mathbf{w}, \mathbf{e})\mathbf{T}(\mathbf{0}, g)$, where $h = \{\mathbf{w}, g\}$ is defined by (1.3). If f is an analytic function defined in a neighborhood of the point $(z, t) \in \mathbb{C}^2$, $(t \neq 0)$, then

$$[\mathbf{T}(\mathbf{w}, \mathbf{e})f](z, t) = \exp\left[\frac{t}{2}\left(\frac{\alpha}{z} - \beta z + \gamma\right)\right]f(z, t),$$
$$\mathbf{w} = (\alpha, \beta, \gamma), \qquad (3.3)$$

$$[\mathbf{T}(\mathbf{0}, g)f](z, t) = (a + cz)^{u_0}\left(d + \frac{b}{z}\right)^{-u_0} \times f\left[\frac{dz + b}{cz + a}, t(a + cz)\left(d + \frac{b}{z}\right)\right],$$
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2), \quad ad - bc = 1. \qquad (3.4)$$

As the reader can verify, these operators satisfy the property

$$\mathbf{T}(hh')f = \mathbf{T}(h)[T(h')f] \qquad (3.5)$$

whenever both sides of the expression are well defined.

The matrix elements $\{v, n|\ \mathbf{w}, g\ |u, m\}$ of our model are defined by

$$\mathbf{T}(\mathbf{w}, g)f_m^{(u)} = \sum_v \sum_n \{v, n|\ \mathbf{w}, g\ |u, m\} f_n^{(v)}, \qquad (3.6)$$

where v is summed over the values u_0, $u_0 \pm 1$, $u_0 \pm 2, \cdots$, and n over the values v, $v - 1$, $v - 2, \cdots$. It is clear that the functions (3.2) form an analytic basis for the representation space in the sense of Ref. 7, Chap. 2. Therefore, the matrix elements (3.6) are uniquely determined by the Lie-algebra relations (2.1) and (2.2), and are independent of our model.

Substituting (3.2) and (3.4) into (3.6), we find

$$(a + cz)^{u+m}\left(d + \frac{b}{z}\right)^{u-m} = \sum_{k=0}^{\infty}\{u, u - k|\ \mathbf{0}, g\ |u, m\}z^k,$$
$$\left|\frac{cz}{a}\right| < 1, \quad \left|\frac{b}{dz}\right| < 1, \qquad (3.7)$$

or

$$\{v, n|\ \mathbf{0}, g\ |u, m\} = \frac{d^{u-n}a^{u+m}b^{n-m}(u - m)!}{(u - n)!} \times \frac{F(n - u, -m - u; n - m + 1; bc/ad)}{\Gamma(n - m + 1)}\delta_{v,u}$$
$$= \frac{d^{u-m}a^{u+n}c^{m-n}\Gamma(u + m + 1)}{\Gamma(u + n + 1)} \times \frac{F(m - u, -n - u; m - n + 1; bc/ad)}{\Gamma(m - n + 1)}\delta_{v,u}, \qquad (3.8)$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2)$, $ad - bc = 1$. Clearly, these matrix elements are defined only in a suitably small neighborhood of $\mathbf{e}$. Substituting (3.3) into (3.6), we find

$$\{v, n|\ \mathbf{w}, \mathbf{e}\ |u, m\} = \frac{\Gamma(u + \frac{1}{2})(-\beta)^{m-n}(\gamma)^{v-u+n-m}}{2^{v-u}(v - u)!\Gamma(v + \frac{1}{2})} \times \sum_a \frac{(-\alpha b/\gamma^2)^a}{a!(m - n + a)!(v - u + n - m - 2a)!} \quad \text{if } v - u \geq 0,$$
$$= 0 \quad \text{otherwise}. \qquad (3.9)$$

Here the sum is taken over all integral values of a such that the summand is defined.

By construction the matrix elements satisfy the addition theorem

$$\{v, n|\, \mathbf{w} + g\mathbf{w}', gg'\,|u, m\} = \sum_{l=-\infty}^{\infty} \sum_{k=0}^{\infty} \{v, n|\, \mathbf{w}, g\,|u + l, u + l - k\} \times \{u + l, u + l - k|\, \mathbf{w}', g'\,|u, m\}, \quad (3.10)$$

valid for g, g' in a suitably small neighborhood of $\mathbf{e} \in SL(2)$.

Now that we have computed the matrix elements of $R^+(u_0)$, we look for a model of this representation in three complex variables. There is only one such model:

$$J^3 = t\frac{\partial}{\partial t}, \quad J^+ = -t\frac{\partial}{\partial z},$$
$$J^- = t^{-1}\left((1 - z^2)\frac{\partial}{\partial z} - 2zt\frac{\partial}{\partial t}\right),$$
$$P^3 = z\frac{\partial}{\partial r} + \frac{(1 - z^2)}{r}\frac{\partial}{\partial z} - \frac{zt}{r}\frac{\partial}{\partial t},$$
$$P^+ = t\left(\frac{\partial}{\partial r} - \frac{z}{r}\frac{\partial}{\partial z} - \frac{t}{r}\frac{\partial}{\partial t}\right),$$
$$P^- = t^{-1}\left((1 - z^2)\frac{\partial}{\partial r} - \frac{z(1 - z^2)}{r}\frac{\partial}{\partial z} + \frac{(z^2 + 1)}{r}t\frac{\partial}{\partial t}\right) \quad (3.11)$$

(the Model B operators constructed in I). The basis vectors $f_m^{(u)}[r, z, t]$ for this model are determined up to a multiplicative constant by relations (2.1)–(2.3) and may be chosen as follows:

$$f_m^{(u)}[r, z, t] = \frac{(r/2)^{-u-1}(u - m)!}{\Gamma(-u + \frac{1}{2})\sqrt{2}}\Gamma(m + \tfrac{1}{2})C_{u-m}^{m+\frac{1}{2}}(z)(2t)^m. \quad (3.12)$$

[Note: The relation $\mathbf{P} \cdot \mathbf{J} \equiv 0$ is satisfied identically by the operators (3.11), while the requirement $\mathbf{P} \cdot \mathbf{P} f_m^{(v)} = 0$ is closely related to Laplace's equation in spherical coordinates.] In fact, substitution of (3.11) and (3.12) into (2.1) and (2.2) leads to the following identities for the Gegenbauer polynomials $C_n^\lambda(z)$:

$$\frac{d}{dz}C_n^\lambda(z) = 2\lambda C_{n-1}^{\lambda+1}(z),$$

$$\left[(1 - z^2)\frac{d}{dz} - 2z\lambda + z\right]C_n^\lambda(z) = \frac{(n + 1)(n + 2\lambda - 1)}{2(1 - \lambda)}C_{n+1}^{\lambda-1}(z), \quad (2.1')$$

$$\left[(z^2 - 1)\frac{d}{dz} + (2\lambda + n + \tfrac{1}{2})z\right]C_n^\lambda(z) = (n + 1)C_{n+1}^\lambda(z),$$

$$\left[z\frac{d}{dz} + (2\lambda + n)z\right]C_n^\lambda(z) = 2\lambda C_n^{\lambda+1}(z),$$

$$2(1 - \lambda)\left[-(2\lambda + n)z^2 + (n + 1) + z(1 - z^2)\frac{d}{dz}\right]C_n^\lambda(z) = (n + 1)(n + 2)C_{n+2}^{\lambda-1}(z), \quad (2.2')$$

valid for $2\lambda \in \mathbb{C}$ not an integer and $n = 0, 1, 2, \cdots$.

Using the type F operators and the basis vectors (3.2), we find

$$f_m^{(u)} = \frac{\Gamma(u + \frac{1}{2})}{\Gamma(m + \frac{1}{2})}(2P^3)^{u-m} f_m^{(m)}. \quad (3.13)$$

Clearly, this relation must hold for any model of the representations $R^+(u_0)$ or $\uparrow^+$. In terms of the operators (3.11) and basis functions (3.12) it reads

$$k!\,r^{-\lambda-k-\frac{1}{2}}C_k^\lambda(z) = \left(z\frac{\partial}{\partial r} + \frac{(1 - z^2)}{r}\frac{\partial}{\partial z} - \frac{z}{r}(\lambda - \tfrac{1}{2})\right)^k (r^{-\lambda-\frac{1}{2}}), \quad k = 0, 1, 2, \cdots.$$

Using the type F operators, the reader can easily derive other similar identities for the Gegenbauer polynomials. The study of identities of this form constitutes the Maxwell theory of poles.[8]

The differential operators (3.11) which define model B can be used to construct a local representation of T_6 by operators $\mathbf{T}(h)$, $h \in T_6$, acting on the space of analytic functions in 3 complex variables. The operators $\mathbf{T}(h)$ have been computed in I:

$$[\mathbf{T}(\mathbf{0}, g)f](r, z, t) = f\left(r, z(1 + 2bc) + abt + cd\frac{(z^2 - 1)}{t}, a^2t + 2acz + c^2\frac{(z^2 - 1)}{t}\right),$$
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2), \quad (3.14)$$

$$[\mathbf{T}(\mathbf{w}, \mathbf{e})f](r, z, t) = f(rQ, (z + \gamma/r)Q^{-1}, (t + 2\beta/r)Q^{-1}),$$
$$Q = \left[1 + \frac{2\beta(1 - z^2)}{rt} + \frac{2\alpha}{r}\left(t + \frac{2\beta}{r}\right) + \frac{\gamma^2}{r^2} + \frac{2\gamma z}{r}\right]^{\frac{1}{2}},$$
$$\mathbf{w} = (\alpha, \beta, \gamma). \quad (3.15)$$

Here f is defined and analytic in some neighborhood of the point $(r, z, t) \in \mathbb{C}^3$. We have the group multiplication property

$$\mathbf{T}(hh')f = \mathbf{T}(h)[\mathbf{T}(h')f]$$

whenever both sides of this expression are well defined as analytic functions of r, z, and t.

It is easy to verify that the basis functions $f_m^{(u)}[r, z, t]$, Eq. (3.12), form an analytic basis for the representation space V. Therefore, we immediately have the identity

$$[\mathbf{T}(\mathbf{w}, g)f_m^{(u)}][r, z, t] = \sum_v \sum_n \{v, n|\, \mathbf{w}, g\,|u, m\} f_n^{(v)}[r, z, t], \quad (3.16)$$

[8] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, New York, 1953), Vol. 2, Chap. 11.

where the operators $\mathbf{T}(\mathbf{w}, g)$ are given by (3.14), (3.15); the matrix elements are given by (3.8), (3.9). We examine some special cases of this identity.

If $\mathbf{w} = \mathbf{0}$, (3.16) reduces to

$$\frac{k!\Gamma(u-k+\frac{1}{2})}{\Gamma(2u-k+1)}\left(\frac{x^2}{2}\right)^k C_k^{u-k+\frac{1}{2}} \times \left[z^2 - z - 1 + \frac{2z-1}{x} + \frac{1}{x^2}\right] \times (1 + 2xz + x^2(z^2-1))^{u-k} = \sum_{l=0}^{\infty} \frac{l!\Gamma(u-l+\frac{1}{2})}{\Gamma(2u-l+1)}\left(\frac{x}{2}\right)^l \times \frac{F(-k, -2u+l; l-k+1; 1-x)}{\Gamma(l-k+1)} C_l^{u-l+\frac{1}{2}}(z),$$
$$|2xz + x^2(z^2-1)| < 1, \quad k = 0, 1, 2, \cdots, \quad (3.17)$$

which was already derived in I. If $g = \mathbf{e}$, $\alpha = \beta = 0$, we obtain

$$[1 + 2\gamma z + \gamma^2]^{-\lambda-k/2} C_k^\lambda[(z+\gamma)(1 + 2\gamma z + \gamma^2)^{-1/2}] = \sum_{l=0}^{\infty} (-\gamma)^l \binom{l+k}{l} C_{k+l}^\lambda(z), \quad |2\gamma z + \gamma^2| < 1, \quad (3.18)$$

which, when $k = 0$, simplifies to the well-known generating function

$$[1 + 2\gamma z + \gamma^2]^{-\lambda} = \sum_{l=0}^{\infty} (-\gamma)^l C_l^\lambda(z).$$

If $g = \mathbf{e}$, $\alpha = \gamma = 0$, we obtain

$$[1 + \beta(1-z^2)]^{-\lambda-k/2} \times C_k^\lambda[z(1 + \beta(1-z^2))^{-1/2}](1+\beta)^{\lambda-1/2} = \sum_{l=0}^{\infty} (\beta/4)^l \frac{(k+2l)!}{k!\,l!} \frac{\Gamma(\lambda-l)}{\Gamma(\lambda)} C_{k+2l}^{\lambda-l}(z),$$
$$k = 0, 1, 2, \cdots, \quad |\beta(1-z^2)|, \quad |\beta| < 1. \quad (3.19)$$

Finally, if $g = \mathbf{e}$, $\beta = \gamma = 0$, (3.16) reduces to

$$[1+\alpha]^{-\lambda-k/2} C_k^\lambda[z(1+\alpha)^{-1/2}] = \sum_{l=0}^{\infty} \frac{(-\alpha)^l}{l!} \frac{\Gamma(\lambda+l)}{\Gamma(\lambda)} C_k^{\lambda+l}(z),$$
$$k = 0, 1, 2, \cdots, \quad |\alpha| < 1. \quad (3.20)$$

If we restrict ourselves to consideration of the representation $\uparrow^+$, we can be somewhat more specific. First of all, the matrix element

$$\{u, m|\,\mathbf{w}, \mathbf{e}\,|0, 0\} = \{u, m|\,\alpha, \beta, \gamma; \mathbf{e}\,|0, 0\}$$

can be computed by making use of the identity

$$\{u, m|\,ab\xi, -cd\xi, (1+2bc)\xi; \mathbf{e}\,|0, 0\} = \{u, m|\,\mathbf{0}, g\,|u, 0\}\{u, 0|\,0, 0, \xi; \mathbf{e}\,|0, 0\}, \quad (3.21)$$

where $g \in SL(2)$. In terms of the new variables [$\alpha = \beta b\xi$, $\beta = -cd\xi$, $\gamma = (1+2bc)\xi$, $\rho^2 = \gamma^2 + 4\alpha\beta = \xi^2$] the matrix elements on the right-hand side of (3.21) are

$$\{u, m|\,\mathbf{0}, g\,|u, 0\} = \frac{\Gamma(|m|+\frac{1}{2})u!}{\pi^{\frac{1}{2}}(u+|m|)!}\left(\frac{4}{\rho}\right)^{|m|} \alpha^{(|m|+m)/2} \times (-\beta)^{(|m|-m)/2} C_{u-|m|}^{|m|+\frac{1}{2}}(\gamma/\rho),$$

$$\{u, 0|\,0, 0, \xi; \mathbf{e}\,|0, 0\} = \left(\frac{\xi}{2}\right)^u \frac{\Gamma(\frac{1}{2})}{u!\,\Gamma(u+\frac{1}{2})}.$$

Hence

$$\{u, m|\,\alpha, \beta, \gamma; \mathbf{e}\,|0, 0\} = \frac{\Gamma(|m|+\frac{1}{2})}{(u+|m|)!\,\Gamma(u+\frac{1}{2})}\left(\frac{\rho}{2}\right)^u \left(\frac{4}{\rho}\right)^{|m|} \times \alpha^{(|m|+m)/2}(-\beta)^{(|m|-m)/2} C_{u-|m|}^{|m|+\frac{1}{2}}(\gamma/\rho). \quad (3.22)$$

Note that this matrix element is a polynomial function of α, β, γ, and ρ^2. Thus, even though our derivation was valid only if $\rho^2 \neq 0$, (3.22) is also correct in the limit as $\rho \to 0$.

Applying the identity

$$\mathbf{T}(\alpha, \beta, \gamma; \mathbf{e}) f_0^{(0)}[r, z, t] = \sum_{u=0}^{\infty} \sum_{m=-u}^{u} \{u, m\,|\,\alpha, \beta, \gamma; \mathbf{e}\,|0, 0\} f_m^{(u)}[r, z, t]$$

to the Model B operators and simplifying, we obtain

$$[1 + 2\beta(1-z^2) + 2\alpha + \rho^2 + 2\gamma z]^{-\frac{1}{2}} = \sum_{u=0}^{\infty} \sum_{m=-u}^{u} \frac{\Gamma(|m|+\frac{1}{2})\Gamma(m+\frac{1}{2})(u-m)!}{\pi(u+|m|)!} 2^{|m|+m} \times (2\alpha)^{(|m|+m)/2}(-2\beta)^{(|m|-m)/2}\rho^{u-|m|} \times C_{u-|m|}^{|m|+\frac{1}{2}}(\gamma/\rho) C_{u-m}^{m+\frac{1}{2}}(z). \quad (3.23)$$

Just as in I we can use the Clebsch–Gordan coefficients $C(\cdot ; \cdot \mid \cdot)$ to compute the general matrix element $\{v, n|\,\alpha, \beta, \gamma; \mathbf{e}\,|u, m\}$. The result is

$$\{v, n|\,\alpha, \beta, \gamma; \mathbf{e}\,|u, m\} = \sum_s \left[\frac{\pi(u-m)!(u+m)!}{(v-n)!(v+n)!} \times (v-u+s+n-m)!(v-u+s+m-n)!\right]^{\frac{1}{2}} \times C(u, 0; v-u+s, 0 \mid v, 0) \times C(u, m; v-u+s, n-m \mid v, n) \times \{v-u+s, n-m|\,\alpha, \beta, \gamma; \mathbf{e}\,|0, 0\}, \quad (3.24)$$

where s ranges over the finite set of nonnegative integer values for which the summand is defined. The computation of identities for Gegenbauer polynomials using these matrix elements is left to the reader.

4. MORE REDUCIBLE REPRESENTATIONS

In analogy with the procedure in Secs. 2 and 3, we shall briefly analyze the following two new classes of reducible representations of $\mathcal{E}_6$ on V: $R^-(u_0)$, $0 \leq \mathrm{Re}\, u_0 < 1$, $2u_0$ not an integer; and $\uparrow^-$.

A. $R^-(\mu_0)$, $0 \leq \mathrm{Re}\, u_0 < 1$, $2u_0$ not an integer

There is a countable basis $\{f_m^{(u)}\}$ for V such that $m = u,\ u-1,\ u-2, \cdots$, and $u = u_0,\ u_0 \pm 1,\ u_0 \pm 2, \cdots$.

There is a countable basis $\{f_m^{(u)}\}$ for V such that $m = u, u-1, \cdots, -u+1, -u$ and $u = 0, 1, 2, \cdots$.

For each representation the action of the infinitesimal operators on the basis vectors is given by

$$J^3 f_m^{(u)} = m f_m^{(u)}, \quad J^\pm f_m^{(u)} = (-u \pm m) f_{m\pm1}^{(u)}, \quad (4.1)$$

$$P^3 f_m^{(u)} = \frac{(u+m)(u-m)}{2u+1} f_m^{(u-1)},$$

$$P^+ f_m^{(u)} = \frac{-(u-m)(u-m-1)}{2u+1} f_{m+1}^{(u-1)}, \quad (4.2)$$

$$P^- f_m^{(u)} = \frac{(u+m)(u+m-1)}{2u+1} f_{m-1}^{(u-1)},$$

$$\mathbf{P}\cdot\mathbf{P} f_m^{(u)} \equiv 0, \quad \mathbf{P}\cdot\mathbf{J} f_m^{(u)} \equiv 0. \quad (4.3)$$

[If a vector $f_m^{(u)}$ on the right-hand side of one of the expressions (4.1)–(4.3) does not belong to the representation space, we set this vector equal to zero.]

The representations $R^-(u_0)$ and $\uparrow^-$ are degenerate cases of the irreducible representations $R_3(\omega, 0, u_0)$ and $\uparrow_4(\omega, 0)$ constructed in II, obtained formally by choosing a new basis $f_m'^{(u)} = \omega^{-u} f_m^{(u)}$ and going to the limit as $\omega \to 0$. Corresponding to each fixed value of u, the vectors $\{f_m^{(u)}\}$ form a basis for an irreducible representation of the subalgebra $sl(2)$ of $\mathcal{E}_6$. The finite-dimensional representations $D(2u)$ are induced by $\uparrow^-$, while the infinite-dimensional representations $\downarrow u$ are induced by $R^-(u_0)$.

According to our usual procedure, we search for models of these representations and compute their matrix elements. Unfortunately, $R^-(u_0)$ and $\uparrow^-$ have no models in two complex variables. However, the structure of the abstract recursion relations (4.1)–(4.3) is simple enough that we can compute the matrix elements of $R^-(u_0)$ and $\uparrow^-$ directly from the abstract relations. We then apply our results to a model in three complex variables, which does exist.

The matrix elements can be defined formally by

$$\mathbf{T}(\mathbf{w}, g) f_m^{(u)} = \exp(\alpha P^+ + \beta P^- + \gamma P^3) \exp(-b/dJ^+) \times \exp(-cdJ^-) \exp(-2 \ln dJ^3) f_m^{(u)} = \sum_{v,n} \{v, n|\, \mathbf{w}, g\,|u, m\} f_n^{(v)},$$

$$\mathbf{w} = (\alpha, \beta, \gamma), \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2).$$

The values assumed by the variables u, v, m, n depend on which of the representations $R^-(u_0)$ or $\uparrow^-$ we are studying. For the present we treat both representations simultaneously.

Since relations (4.1) and (2.1) are identical, it follows immediately that the matrix elements

$$\{v, n|\, \mathbf{0}, g\,|u, m\}$$

are given by Eq. (3.8). Furthermore, a simple induction argument based on (4.2) yields the results

$$\{v, n|\, 0, 0, \gamma; \mathbf{e}\,|u, m\} = \left(\frac{\gamma}{2}\right)^{u-v} \frac{\Gamma(v+\frac{3}{2})}{\Gamma(u+\frac{3}{2})} \frac{\Gamma(u+m+1)}{\Gamma(v+m+1)} \frac{(u-m)!}{(v-m)!} \frac{\delta_{m,n}}{(u-v)!} \quad \text{if } u - v \geq 0,$$
$$= 0 \quad \text{if } v - u > 0, \quad (4.4)$$

$$\{v, n|\, \alpha, 0, 0; \mathbf{e}\,|u, m\} = \left(\frac{-\alpha}{2}\right)^{u-v} \frac{\Gamma(v+\frac{3}{2})}{\Gamma(u+\frac{3}{2})} \frac{(u-m)!}{(2v-u-m)!} \frac{\delta_{m,n-u+v}}{(u-v)!} \quad \text{if } u - v \geq 0,$$
$$= 0 \quad \text{if } v - u > 0, \quad (4.5)$$

$$\{v, n|\, 0, \beta, 0; \mathbf{e}\,|u, m\} = \left(\frac{\beta}{2}\right)^{u-v} \frac{\Gamma(v+\frac{3}{2})}{\Gamma(u+\frac{3}{2})} \frac{\Gamma(u+m+1)}{\Gamma(2v-u+m+1)} \frac{\delta_{m,n-u+v}}{(u-v)!} \quad \text{if } u - v \geq 0,$$
$$= 0 \quad \text{if } v - u > 0. \quad (4.6)$$

The expression for the general matrix element $\{v, n|\, \mathbf{w}; \mathbf{e}\,|u, m\}$ of the representation $R^-(u_0)$ is rather complicated and we need not take the time to derive it. Similarly, we do not derive an expression for the most general matrix element of $\uparrow^-$, although this is not so complicated.[4]

The Model B operators (3.11) can be used to construct models of $R^-(u_0)$ and $\uparrow^-$ in three complex variables. In fact, relations (4.1)–(4.3) will be satisfied, provided that we choose the basis vectors as follows:

$$f_m^{(u)}[r, z, t] = \left(\frac{r}{2}\right)^u \frac{(u-m)!\,\Gamma(m+\frac{1}{2})}{\sqrt{2}\,\Gamma(u+\frac{3}{2})} C_{u-m}^{m+\frac{1}{2}}(z)(2t)^m, \quad (4.7)$$

where the possible values of the variables u, m are determined by the representation space to which the basis vectors belong. To see the equivalence between our models and certain recursion relations for Gegenbauer polynomials, we substitute (4.7) into (4.2):

$$\left[(1-z^2)\frac{d}{dz} + kz\right] C_k^\lambda(z) = (k + 2\lambda - 1) C_{k-1}^\lambda(z),$$

$$\left[z\frac{d}{dz} - k\right] C_k^\lambda(z) = 2\lambda C_{k-2}^{\lambda+1}(z),$$

$$2(\lambda - 1)\left[-z(1-z^2)\frac{d}{dz} - z^2 k + (k + 2\lambda - 1)\right] C_k^\lambda(z) = (k + 2\lambda - 1)(k + 2\lambda - 2) C_k^{\lambda-1}(z). \quad (4.8)$$

As was shown in Sec. 3, the differential operators (3.11) define a local representation of T_6 by operators $\mathbf{T}(\mathbf{w}, g)$, Eqs. (3.14), and (3.15). Furthermore, it is easy to see that the functions (4.7) form an analytic basis for such a representation. Thus the matrix elements defined by

$$\mathbf{T}(\mathbf{w}, g) f_m^{(u)} = \sum_{v,n} \{v, n \mid \mathbf{w}, g \mid u, m\} f_n^{(v)} \qquad (4.9)$$

are identical with those computed earlier in this section. In addition, we have the identity

$$\begin{aligned}\{v, n \mid \mathbf{w} + g\mathbf{w}'; gg' \mid u, m\} &= \sum_{l=-\infty}^{\infty} \sum_{k=0}^{\infty} \{v, n \mid \mathbf{w}, g \mid u + l, u + l - k\} \\ &\times \{u + l, u + l - k \mid \mathbf{w}', g' \mid u, m\}, \qquad (4.10)\end{aligned}$$

valid for g, g' in a suitably small neighborhood of $\mathbf{e}$. The following special cases of (4.9) are of interest: If $g = \mathbf{e}$, $\alpha = \beta = 0$, this identity becomes

$$[1 + \gamma^2 + 2\gamma z]^{k/2} C_k^\lambda((z + \gamma)[1 + \gamma^2 + 2\gamma z]^{-1/2}) = \sum_{l=0}^{k} \gamma^l \binom{2\lambda + k - 1}{l} C_{k-l}^\lambda(z); \qquad (4.11)$$

if $g = \mathbf{e}$ and $\beta = \gamma = 0$, one obtains

$$(1 - \alpha)^{k/2} C_k^\lambda(z(1 - \alpha)^{-1/2}) = \sum_{l=0}^{k/2} \alpha^l \binom{\lambda + l - 1}{l} C_{k-2l}^{\lambda+l}(z); \qquad (4.12)$$

if $g = \mathbf{e}$, and $\alpha = \gamma = 0$, there follows

$$[1 + \beta(1 - z^2)]^{k/2}(1 + \beta)^{\lambda-1/2} C_k^\lambda(z[1 + \beta(1 - z^2)]^{-1/2}) = \frac{\Gamma(2\lambda + k)}{\Gamma(\lambda)} \sum_{l=0}^{\infty} \left(\frac{\beta}{4}\right)^l \frac{\Gamma(\lambda - l)}{l!\,\Gamma(2\lambda + k - 2l)} C_k^{\lambda-l}(z), \quad |\beta| < 1. \qquad (4.13)$$

5. A CLASS OF IRREDUCIBLE REPRESENTATIONS

We now turn our attention to a new class of *irreducible* representations of $\mathcal{T}_6$ listed in Ref. 7:

$R_3'(\zeta, u_0)$, $\zeta \neq 0$, $0 \leq \operatorname{Re} u_0 < 1$, $2u_0$ not an integer.

There is a countable basis $\{f_m^{(u)}\}$ for the representation space V such that $m = u, u - 1, u - 2, \cdots$, and $u = u_0, u_0 \pm 1, u_0 \pm 2, \cdots$. The action of the infinitesimal operators on the basis vectors is given by

$$J^3 f_m^{(u)} = m f_m^{(u)}, \quad J^\pm f_m^{(u)} = (-u \pm m) f_{m\pm1}^{(u)}, \qquad (5.1)$$

$$P^3 f_m^{(u)} = \frac{-\zeta}{(2u + 1)(u + 1)} f_m^{(u+1)} + \frac{\zeta m}{u(u + 1)} f_m^{(u)} + \frac{\zeta(u + m)(u - m)}{(2u + 1)u} f_m^{(u-1)}, \qquad (5.2)$$

$$P^+ f_m^{(u)} = \frac{-\zeta}{(2u + 1)(u + 1)} f_{m+1}^{(u+1)} - \frac{\zeta(u - m)}{u(u + 1)} f_{m+1}^{(u)} - \frac{\zeta(u + m)(u - m - 1)}{(2u + 1)u} f_{m+1}^{(u-1)}, \qquad (5.3)$$

$$P^- f_m^{(u)} = \frac{\zeta}{(2u + 1)(u + 1)} f_{m-1}^{(u+1)} - \frac{\zeta(u + m)}{u(u + 1)} f_{m-1}^{(u)} + \frac{\zeta(u + m)(u + m - 1)}{(2u + 1)u} f_{m-1}^{(u-1)}, \qquad (5.4)$$

$$\mathbf{P} \cdot \mathbf{P} f_m^{(u)} = 0, \quad \mathbf{P} \cdot \mathbf{J} f_m^{(u)} = -\zeta f_m^{(u)}. \qquad (5.5)$$

[The representations $R_3'(\zeta, u_0)$ can be obtained formally from the representations $R_3(\omega, q, u_0)$ by setting $q = -\zeta/\omega$ and passing to the limit as $\omega \to 0$.]

$R_3'(\zeta, u_0)$ has no models in two complex variables. However, in three variables the type F operators[7] provide a model:

$$J^3 = t\frac{\partial}{\partial t}, \quad J^\pm = t^{\pm1}\left(z\frac{\partial}{\partial z} \pm t\frac{\partial}{\partial t} \mp \frac{z}{2}\right),$$
$$P^3 = 2\zeta z^{-1}, \quad P^\pm = \pm 2\zeta t^{\pm1} z^{-1}, \quad \zeta \in \mathbb{C}. \qquad (5.6)$$

As is easy to verify, these operators satisfy the commutation relations of $\mathcal{T}_6$. Furthermore,

$$\mathbf{P} \cdot \mathbf{P} \equiv 0, \quad \mathbf{P} \cdot \mathbf{J} = -\zeta.$$

Corresponding to this model, the basis vectors are determined up to a multiplicative constant by relations (5.1)–(5.4) and may be given by

$$f_m^{(u)}(z, t) = \frac{(-1)^{-u}}{\Gamma(-2u)} M_{m,-u-\frac{1}{2}}(z) t^m = \frac{1}{\Gamma(-2u)} M_{-m,-u-\frac{1}{2}}(-z) t^m, \qquad (5.7)$$

where the functions

$$M_{\chi,\mu}(z) = e^{-z/2} z^{\mu+1/2} {}_1F_1(\tfrac{1}{2} + \mu - \chi; 1 + 2\mu; z)$$

are Whittaker functions.[9] In fact, expressions (5.1) are equivalent to the recursion relations

$$\left(z\frac{d}{dz} \pm m \mp \frac{z}{2}\right) M_{m,\mu}(z) = (\mu + \tfrac{1}{2} \pm m) M_{m+1,\mu}(z), \qquad (5.8)$$

while expressions (5.2)–(5.4) are equivalent to the relations

$$z^{-1} M_{m,\mu}(z) = M_{m,\mu-1}(z) + \frac{m}{2(\mu + \frac{1}{2})(\mu - \frac{1}{2})} M_{m,\mu}(z) + \frac{(m - \mu - \frac{1}{2})(m + \mu + \frac{1}{2})}{4\mu(\mu + \frac{1}{2})} M_{m,\mu+1}(z), \qquad (5.9)$$

[9] W. Magnus, F. Oberhettinger, and R. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer-Verlag, Berlin, 1966), 3rd ed.

$$z^{-1}M_{m,\mu}(z) = M_{m+1,\mu-1}(z) + \frac{(m+\mu+\frac{1}{2})}{2(\mu+\frac{1}{2})(\mu-\frac{1}{2})} M_{m+1,\mu}(z) + \frac{(m+\mu+\frac{1}{2})(m+\mu+\frac{3}{2})}{4\mu(\mu+\frac{1}{2})} M_{m+1,\mu+1}(z), \tag{5.10}$$

$$z^{-1}M_{m,\mu}(z) = -M_{m-1,\mu-1}(z) + \frac{(m-\mu-\frac{1}{2})}{2(\mu+\frac{1}{2})(\mu-\frac{1}{2})} M_{m-1,\mu}(z) - \frac{(m-\mu-\frac{1}{2})(m-\mu-\frac{3}{2})}{4\mu(\mu+\frac{1}{2})} M_{m-1,\mu+1}(z). \tag{5.11}$$

We now prove some auxiliary lemmas which will enable us to extend the representation $R_3'(\zeta, u_0)$ of $\mathcal{T}_6$ to a local representation of T_6. In the following, all operators and basis vectors are assumed to satisfy relations (5.1)–(5.5).

Lemma 1:

$$(P^3)^k f_u^{(u)} = (2\zeta)^k k!\,\Gamma(2u+1) \times \sum_{n=0}^{k} \frac{(-1)^n(2u+2n+1)}{n!(k-n)!\Gamma(2u+n+k+2)} f_u^{(u+n)}, \quad k = 0, 1, 2, \cdots.$$

Proof: Expression (5.2) and induction on k.

Corollary 1:

$$e^{+z/2}(z)^{-u-k} = \sum_{n=0}^{k} \binom{k}{n} \frac{\Gamma(2u+2n+2)}{\Gamma(2u+n+k+2)} M_{u,-u-n-1/2}(z) \quad k = 0, 1, 2, \cdots.$$

By definition,

$$M_{u,-u-k-1/2}(z) = e^{z/2}z^{-u-k}\,{}_1F_1(-k; -2u-2k; -z).$$

From this equation and Corollary 1, it is an easy computation to obtain the identity

$$z^r M_{u,-u-k-\frac{1}{2}}(z) = \sum_{n=0}^{k+s} \binom{k+s}{n} \frac{\Gamma(2u-2s+2n-2r+2)}{\Gamma(-s+2u-2r+n+k+2)} \times {}_3F_2(-k, -s-k+n, +s-2u-n-k+2r-1; -2u-2k, -s-k; 1) M_{u-s-r,-u+s+r-n-\frac{1}{2}}(z), \quad k, s, \pm r = 0, 1, 2, \cdots, \tag{5.12}$$

expressing the function $z^r M_{u,-u-k-\frac{1}{2}}(z)$ as a linear combination of Whittaker functions $M_{u+s,-u-s-n-\frac{1}{2}}(z)$. [Compare this expansion with Eq. (3.2) of II.]

Let k be a nonnegative integer.

Lemma 2:

$$(P^3)^k f_m^{(u)} = \sum_{l=0}^{u-m+k} (2\zeta)^k \binom{u-m+k}{l} \times \frac{\Gamma(2m+2l+2)}{\Gamma(u+m+k+l+2)} \frac{\Gamma(-2m-2l)}{\Gamma(-2u)} \times {}_3F_2(m-u, -k+m-u+l, -u-m-k+l-1; -2u, -k+m-u; 1)(-1)^{u-m+l} f_m^{(m+l)}.$$

Lemma 3:

$$(P^+)^k f_m^{(u)} = \sum_{l=\max(u-m-2k,0)}^{u-m} (2\zeta)^k \binom{u-m}{l} \times \frac{\Gamma(2m+2k+2l+2)}{\Gamma(u+m+2k+l+2)} \times \frac{\Gamma(-2m-2k-2l)}{\Gamma(-u-m-l)} \frac{(2k)!}{(m-u+2k+l)!} \times (-1)^{u-m+k+l} f_{m+k}^{(m+k+l)}.$$

Lemma 4:

$$(P^-)^k f_m^{(u)} = \sum_{l=u-m}^{u-m+2k} \frac{\Gamma(2m-2k+2l+2)}{\Gamma(u+m+l+2)} \times \frac{\Gamma(-2m+2k-2l)}{\Gamma(-u-m)} \frac{\Gamma(-2m+2k-l)}{\Gamma(-u-m+2k-l)} \times \frac{(2k)!\,(-1)^{u-m-l}}{(u-m+2k-l)!(l-u+m)!} (2\zeta)^k f_{m-k}^{(m-k+l)}.$$

Proof: For our model these results can be obtained easily from expression (5.12). Since the lemmas are valid for the model, they must be true for the abstract representation $R_3'(u_0, \zeta)$.

According to Schafke[10] (Chap. 8), the functions (5.7) form an analytic basis for $R_3'(u_0, \zeta)$. Thus, this Lie-algebra representation can be extended to a local group representation of T_6. The matrix elements $\{v, n\,|\mathbf{w}, g|\,u, m\}$ can be defined by formulas analogous to (4.9) and satisfy the addition theorems (4.10). In particular, the matrix elements are completely determined by Lie algebra relations (5.1)–(5.4). We now compute the most important of these matrix elements.

Since Eqs. (5.1) and (2.1) are formally identical, the matrix elements $\{v, n|\,\mathbf{0}, g\,|u, m\}$ are given by (3.8).

[10] F. W. Schafke, *Einfuhrung in die Theorie der Speziellen Funktion der Mathematischen Physik* (Springer-Verlag, Berlin, 1963).

The elements of the form $\{v, n \mid \mathbf{w}, \mathbf{e} \mid u, m\}$ can be computed directly from Lemmas 2–4. In particular,

$$\begin{aligned}\{v, n| 0, 0, \gamma; \mathbf{e} |u, m\} &= \delta_{m,n} \sum_k \frac{(2\zeta\gamma)^k}{k!}\binom{u-m+k}{v-m} \\ &\quad\times \frac{\Gamma(2v+2)\Gamma(-2v)}{\Gamma(u+v+k+2)\Gamma(-2u)} \\ &\quad\times {}_3F_2(m-u, -k+v-u, -k+v-u \\ &\qquad -2m-1; -2u, -k+m-u; 1)(-1)^{v-u}. \end{aligned} \tag{5.13}$$

If $m = u$, this simplifies to

$$\begin{aligned}\{v, n| 0, 0, \gamma; \mathbf{e} |u, u\} &= \delta_{u,n} \frac{(2v+1)\Gamma(2u+1)(-1)^{v-u}}{(v-u)!(2\zeta\gamma)^{u+1}} I_{2v+1}(\sqrt{8\gamma\zeta}) \quad \text{if } v-u \geq 0, \\ &= 0 \quad \text{if } u-v > 0. \end{aligned} \tag{5.14}$$

Here $I_\lambda(z)$ is a modified Bessel function.[9] (5.13) and (5.14) are entire functions of $\zeta\gamma$:

$$\begin{aligned}\{v, n| \alpha, 0, 0; \mathbf{e} |u, m\} &= \frac{(2\alpha\zeta)^{n-m}}{(n-m)!}\binom{u-m}{v-n} \\ &\quad\times \frac{\Gamma(2v+2)\Gamma(-2v)(2n-2m)!\,(-1)^{v-u}}{\Gamma(u+v+n-m+2) \times \Gamma(-u-v+n-m)(v-u+n-m)!} \\ &\qquad \text{if } n-m \geq |u-v|, \\ &= 0 \quad \text{otherwise;} \end{aligned} \tag{5.15}$$

$$\begin{aligned}\{v, n| 0, \beta, 0; \mathbf{e} |u, m\} &= \frac{(-2\beta\zeta)^{m-n}}{(m-n)!} \frac{\Gamma(2v+2)}{\Gamma(u+v+m-n+2)} \\ &\quad\times \frac{\Gamma(-2v)\Gamma(-v-n)}{\Gamma(-u-m)\Gamma(-u-v+m-n)} \\ &\quad\times \frac{(2m-2n)!\,(-1)^{v-u}}{(u-v+m-n)!\,(v-u+m-n)!} \\ &\qquad \text{if } m-n \geq |u-v|, \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{5.16}$$

The operators $\mathbf{T}(\mathbf{w}, g)$ defining the multiplier representation induced by the Lie derivatives (5.6) take the form

$$[\mathbf{T}(\alpha, \beta, \gamma; \mathbf{e})f](z, t) = \exp\left[\frac{2\zeta}{z}(\alpha t - \beta t^{-1} + \gamma)\right] \cdot f(z, t), \tag{5.17}$$

$$\begin{aligned}[\mathbf{T}(\mathbf{0}, g)f](z, t) &= \exp\left[\frac{z}{2}\left(\frac{bt}{d+bt} - \frac{c}{at+c}\right)\right] \\ &\quad\times f\left(\frac{zt}{(at+c)(d+bt)}, \frac{at+c}{d+bt}\right), \\ &\quad |c/at| < 1, \quad |bt/d| < 1, \\ g &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2). \end{aligned} \tag{5.18}$$

By construction, the basis functions of our model must satisfy the identity

$$[\mathbf{T}(\mathbf{w}, g)f_m^{(u)}](z, t) = \sum_{v,n} \{v, n| \mathbf{w}, g |u, m\} f_n^{(v)}(z, t). \tag{5.19}$$

We examine some special cases of this identity. If $\mathbf{w} = \mathbf{0}$, (5.19) simplifies to

$$\begin{aligned}&\exp\left[\frac{z}{2}\left(\frac{bt}{d+bt} - \frac{c}{at+c}\right)\right] \\ &\quad\times \left(1+\frac{c}{at}\right)^m \left(1+\frac{bt}{d}\right)^{-m} (1+bc)^{-m} \\ &\quad\times M_{m,-m-k-\frac{1}{2}}\left(\frac{zt}{(at+c)(d+bt)}\right) \\ &= \sum_{l=0}^{\infty} \frac{d^l a^k b^{k-l}}{l!} k!\, \frac{F(-l, -2m-k; k-l+1; bc/ad)}{\Gamma(k-l+1)} \\ &\quad\times M_{m+k-l,-m-k-\frac{1}{2}}(z) t^{k-l}, \\ &\quad ad - bc = 1, \quad |c/at| < 1, \quad |bt/d| < 1. \end{aligned} \tag{5.20}$$

If $b = 0$, (5.20) reduces to

$$\begin{aligned}&e^{-(zc/2)/(1+c)} M_{m,-m-k-1/2}\left(\frac{z}{1+c}\right)(1+c)^m \\ &\quad = \sum_{l=0}^{\infty} \binom{2m+k}{l} M_{m-l,-m-k-1/2}(z) c^l, \\ &\quad |c| < 1, \quad k = 0, 1, 2, \cdots. \end{aligned} \tag{5.21}$$

In particular, if $k = 0$, this last expression yields the generating function

$$z^{-m} \exp\left(\frac{z(1-c)}{2(1+c)}\right)(1+c)^{2m} = \sum_{l=0}^{\infty} \binom{2m}{l} M_{m-l,-m-\frac{1}{2}}(z) c^l.$$

If $c = 0$, Eq. (5.20) reduces to

$$\begin{aligned}&\exp\left[\frac{zb}{2(1+b)}\right](1+b)^{-m} M_{m,-m-k-\frac{1}{2}}\left(\frac{z}{1+b}\right) \\ &\quad = \sum_{l=0}^{k} \binom{k}{l} M_{m+l,-m-k-\frac{1}{2}}(z) b^l, \quad |b| < 1. \end{aligned} \tag{5.22}$$

Another interesting special case of (5.19) can be obtained by setting $g = \mathbf{e}$, $\alpha = \beta = 0$, and $u = m$:

$$\exp\left(\frac{a}{z} + \frac{z}{2}\right) = \sum_{l=0}^{\infty} \frac{\Gamma(2u + 2l + 2)}{l!} a^{-u-\frac{1}{2}} I_{2u+2l+1}(2a^{\frac{1}{2}}) \times z^u M_{u,-u-l-\frac{1}{2}}(z). \quad (5.23)$$

6. WEISNER'S METHOD

In this section we will be concerned exclusively with the differential operators (3.11) which provide a realization of $\mathcal{G}_6$ in three variables. So far, these operators have been used to construct identities for special functions which are simultaneous eigenfunctions of J^3, $C_{1,0}$, and $\mathbf{P}\cdot\mathbf{P}$. Furthermore, our identities have been valid only for group elements in a sufficiently small neighborhood of $\{\mathbf{0}, \mathbf{e}\}$. However, we can follow a method introduced by Weisner[6] and use the operators (3.11) to derive identities for special functions in which the above restrictions are lifted. We make the following observations. If $f(r, z, t)$ is a solution of the equation $\mathbf{P}\cdot\mathbf{P}f = -\omega^2 f$, i.e.,

$$\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{t^2}{r^2}\frac{\partial^2}{\partial t^2} - \frac{2t}{r^2}\frac{\partial}{\partial t} + \frac{(1 - z^2)}{r^2}\frac{\partial^2}{\partial z^2} - \frac{2zt}{r^2}\frac{\partial^2}{\partial t \partial z} - \frac{2z}{r^2}\frac{\partial}{\partial z} + \omega^2\right] f(r, z, t) = 0, \quad (6.1)$$

then the function $\mathbf{T}(\mathbf{w}, g)f$, formally defined by Eqs. (3.14) and (3.15), is also a solution of (6.1). This remark is true whenever the formal expression for $\mathbf{T}(\mathbf{w}, g)f$ can be interpreted as an analytic function in (r, z, t) and is a consequence of the fact that the differential operators (3.11) commute with $\mathbf{P}\cdot\mathbf{P}$. In addition, if f is a solution of the equation

$$Lf = (x_1 J^+ + x_2 J^- + x_3 J^3 + y_1 P^+ + y_2 P^- + y_3 P^3) f = \lambda f$$

for complex constants x_i, y_i, λ, then $\mathbf{T}(\mathbf{w}, g)f = f'$ is a solution of an equation of the same form $L'f' = \lambda f'$ where $L' = \mathbf{T}(\mathbf{w}, g)L\mathbf{T}^{-1}(\mathbf{w}, g)$, i.e.,

$$\begin{aligned}
x_1' &= a^2 x_1 - b^2 x_2 + ab x_3, \\
x_2' &= -c^2 x_1 + d^2 x_2 - c\,d x_3, \\
x_3' &= 2ac x_1 - 2b\,d x_2 + (1 + 2bc) x_3, \\
y_1' &= a^2 y_1 - b^2 y_2 + ab y_3 \\
&\quad + \alpha[-ac x_1 + 2b\,d x_2 - (1 + 2bc) x_3] \\
&\quad + \gamma[a^2 x_1 - b^2 x_2 + ab x_3], \\
y_2' &= -c^2 y_1 + d^2 y_2 - c\,d y_3 \\
&\quad + \beta[2ac x_1 - 2b\,d x_2 + (1 + 2bc) x_3] \\
&\quad + \gamma[c^2 x_1 - d^2 x_2 + c\,d x_3], \\
y_3' &= 2ac y_1 - 2b\,d y_2 + (1 + 2bc) y_3 \\
&\quad + \alpha[-2c^2 x_1 + 2d^2 x_2 - 2c\,d x_3] \\
&\quad + \beta[-2a^2 x_1 + 2b^2 x_2 - 2ab x_3].
\end{aligned} \quad (6.2)$$

As an application of these remarks, consider a simultaneous eigenfunction of the commuting operators $\mathbf{P}\cdot\mathbf{P}$, P^3, J^3:

$$\mathbf{P}\cdot\mathbf{P}f = -f, \; P^3 f = \lambda f, \; J^3 f = mf, \; \lambda, m \in \mathcal{C}. \quad (6.3)$$

A straightforward computation shows that the solutions of (6.3) are of the form

$$f(r, z, t) = [t(z^2 - 1)^{\frac{1}{2}}(\lambda^2 - 1)^{\frac{1}{2}}]^m \times e^{\lambda r z} I_{\pm m}(r(z^2 - 1)^{\frac{1}{2}}(\lambda^2 - 1)^{\frac{1}{2}}).$$

Choosing the I_m solution, we note the validity of the expansion

$$f(r, z, t) = r^{-\frac{1}{2}} \sum_{k=0}^{\infty} a_k(\lambda) I_{m+k+\frac{1}{2}}(r) C_k^{m+\frac{1}{2}}(z) t^m, \quad (6.4)$$

giving f as a sum of simultaneous eigenfunctions of the operators $\mathbf{P}\cdot\mathbf{P}$, $C_{1,0}$, J^3.[10] It remains only to compute $a_k(\lambda)$. Since f is symmetric in z and λ, $a_k(\lambda) = b_k C_k^{m+\frac{1}{2}}(\lambda)$. Furthermore, if $\lambda = 1$, then

$$f(r, z, t) = \left(\frac{rt}{2}\right)^m \frac{e^{rz}}{\Gamma(m + 1)},$$

which has the well-known expansion

$$\left(\frac{r}{2}\right)^m e^{rz} = \sum_{k=0}^{\infty} \left(\frac{2}{r}\right)^{\frac{1}{2}} \Gamma(m + \tfrac{1}{2})(m + k + \tfrac{1}{2}) \times I_{m+k+\frac{1}{2}}(r) C_k^{m+\frac{1}{2}}(z)$$

(see I, Corollary 7, or Ref. 9). This last expansion enables us to compute the coefficients $a_k(\lambda)$ with the result

$$[(z^2 - 1)(\lambda^2 - 1)]^{-m/2} e^{r\lambda z} I_m(r[(z^2 - 1)(\lambda^2 - 1)]^{\frac{1}{2}}) = \frac{2^{2m+1}}{(2\pi r)^{\frac{1}{2}}} [\Gamma(m + \tfrac{1}{2})]^2 \sum_{k=0}^{\infty} \frac{k!\,(m + k + \frac{1}{2})}{\Gamma(2m + k + 1)} \times I_{m+k+\frac{1}{2}}(r) C_k^{m+\frac{1}{2}}(z) C_k^{m+\frac{1}{2}}(\lambda) \quad (6.5)$$

convergent for all z, $\lambda \in \mathcal{C}$.

Similarly, it is easy to show that

$$j(r, z, t) = \left(\frac{t}{\lambda(z^2 - 1)^{\frac{1}{2}}}\right)^m e^{\lambda z r} I_m[\lambda r(z^2 - 1)^{\frac{1}{2}}]$$

is a solution of the equations

$$\mathbf{P}\cdot\mathbf{P}j = 0, \quad P^3 j = \lambda j, \quad J^3 j = mj. \quad (6.6)$$

There exists an expansion of the form

$$j(r, z, t) = \sum_{k=0}^{\infty} a_k(\lambda) r^{m+k} C_k^{m+\frac{1}{2}}(z) t^m, \quad (6.7)$$

expressing j as a sum of simultaneous eigenfunctions of $\mathbf{P}\cdot\mathbf{P}$, $C_{1,0}$, and J^3. The constants $a_k(\lambda)$ can be evaluated by setting $z = 1$ on both sides of Eq. (6.7). The result is

$$\Gamma(m + 1)[r(z^2 - 1)^{\frac{1}{2}}]^{-m} e^{rz} I_m[r(z^2 - 1)^{\frac{1}{2}}] = \sum_{k=0}^{\infty} \frac{\Gamma(2m + 1)}{\Gamma(2m + k + 1)} C_k^{m+\frac{1}{2}}(z) r^k, \quad (6.8)$$

convergent for all r, $z \in \mathcal{C}$.[9]

As a final example we consider

$$h(r, z, t) = r^{-\frac{1}{2}} I_{u+\frac{1}{2}}(r) C_k^{u-k+\frac{1}{2}}(z) t^{u-k}, \quad u \in \not{C},$$

$2u$ not an integer, $k = 0, 1, 2, \cdots$. h is a solution of the simultaneous equations

$$\mathbf{P} \cdot \mathbf{P} h = -h, \; C_{1,0} h = u(u+1)h, \quad J^3 h = (u-k)h. \tag{6.9}$$

Note that the function

$$h' = \mathbf{T}(0, \tfrac{1}{2}, 0; \mathbf{e})h = \left(r^2 + \frac{(1-z^2)r}{t}\right)^{(-u+k-\frac{1}{2})/2} \times I_{u+\frac{1}{2}}\left[\left(r^2 + \frac{(1-z^2)r}{t}\right)^{\frac{1}{2}}\right] C_k^{u-k+\frac{1}{2}} \times \left[rz\left(r^2 + \frac{(1-z^2)r}{t}\right)^{-\frac{1}{2}}\right](1+rt)^{u-k}, \quad 0 < |t| < |r^{-1}|, \tag{6.10}$$

can be expanded in a Laurent series in t. Thus, the following expansion is valid:

$$h'(r, z, t) = \sum_{n=-\infty}^{\infty} \sum_{s=\max([-2n],0)}^{\infty} a_{n,s} s!\, \Gamma(n + \tfrac{1}{2}) \times r^{-\frac{1}{2}} I_{n+s+\frac{1}{2}}(r) C_s^{n+\frac{1}{2}}(z)(2t)^n. \tag{6.11}$$

We now determine the constants $a_{n,s}$. According to expression (6.2), $h' = \mathbf{T}(0, \frac{1}{2}, 0; \mathbf{e})h$ satisfies the equation $(J^3 + \frac{1}{2}P^-)h' = (u-k)h!$. This implies the recursion relation

$$2(u-k-n)a_{n,s} = \frac{-a_{n+1,s-2}}{(2n+2s-1)} + \frac{(2n+s+1)(2n+s+2)}{(2n+2s+3)} \times a_{n+1,s} \tag{6.12}$$

for the coefficients $a_{n,s}$. On the other hand, if $z = 1$, (6.11) reduces to a power series in t:

$$h'(r, 1, t) = \binom{2u-k}{k} r^{-u+k-\frac{1}{2}} I_{u+\frac{1}{2}}(r)(1+rt)^{u-k}, \quad |rt| < 1. \tag{6.13}$$

By comparing (6.13) with the well-known expansion

$$\left(\frac{r}{2}\right)^{-u+k} I_{u+\frac{1}{2}}(r) = \sum_{l=0}^{\infty} \frac{\Gamma(k+l+\frac{1}{2})\Gamma(-u+k+l)(k+2l+\frac{1}{2})}{l!\,\Gamma(-u+k)\Gamma(u+l+\frac{3}{2})} \times I_{k+2l+\frac{1}{2}}(r), \quad k = 0, 1, 2, \cdots,$$

[see I, Eq. (5.10), or Ref. 9, p. 129], we find

$$a_{n,s} = \frac{(2)^{-u+k+2n}}{(\pi)^{\frac{1}{2}}} \binom{2u-k}{k} \binom{-u+n+k+l-1}{l} \times \frac{\Gamma(u-k+1)\Gamma(n+k+l+\frac{1}{2})}{\Gamma(u-k-n+1)\Gamma(u+l+\frac{3}{2})} \times \frac{(n+k+2l+\frac{1}{2})}{(2n+k+2l)!} \quad \text{if } s = k+2l, \; k, l = 0, 1, 2, \cdots,$$
$$= 0 \quad \text{if } s \neq k+2l, \quad k, l = 0, 1, 2, \cdots. \tag{6.14}$$

Thus, we have computed $a_{n,s}$ for $n \geq 0$. However, formulas (6.14) make sense for all integers n such that $2n + s \geq 0$, and they satisfy the recursion relations (6.12) even when n is negative. Therefore, formulas (6.14), defined for all integers n and nonnegative integers s such that $2n + s \geq 0$, are the solution to our problem:

$$\frac{k!\,\Gamma(u-k+\frac{1}{2})}{\Gamma(2u-k+1)}\left(r^2 + \frac{(1-z^2)r}{t}\right)^{(-u+k-\frac{1}{2})/2} I_{u+\frac{1}{2}}\left[\left(r^2 + \frac{(1-z^2)r}{t}\right)^{\frac{1}{2}}\right] C_k^{u-k+\frac{1}{2}}\left[rz\left(r^2 + \frac{(1-z^2)r}{t}\right)^{\frac{1}{2}}\right](1+rt)^{u-k}$$
$$= \sum_{n=-\infty}^{\infty} \sum_{l=\max([-n-k/2],0)}^{\infty} 8^{-u+k+n} \binom{-u+n+k+l-1}{s} \frac{\Gamma(n+k+l+\frac{1}{2})\Gamma(n+\frac{1}{2})(k+2l)!\,(n+k+2l+\frac{1}{2})r^{-\frac{1}{2}}}{(2n+k+2l)!\,\Gamma(u+l+\frac{3}{2})\Gamma(u-k-n+1)}$$
$$\times I_{n+k+2l+\frac{1}{2}}(r) C_{k+2l}^{n+\frac{1}{2}}(z)t^n, \quad 0 < |t| < |r^{-1}|, u \in \not{C}, \quad 2u \text{ not an integer}, \quad k = 0, 1, 2, \cdots.$$

The above examples should suffice to indicate the scope of Weisner's approach—though many other results could be obtained using the same method.