Preface

This book is concerned with the relationship between symmetries of a linear second-order partial differential equation of mathematical physics, the coordinate systems in which the equation admits solutions via separation of variables, and the properties of the special functions that arise in this manner. It is an introduction intended for anyone with experience in partial differential equations, special functions, or Lie group theory, such as group theorists, applied mathematicians, theoretical physicists and chemists, and electrical engineers. We will exhibit some modern group-theoretic twists in the ancient method of separation of variables that can be used to provide a foundation for much of special function theory. In particular, we will show explicitly that all special functions that arise via separation of variables in the equations of mathematical physics can be studied using group theory. These include the functions of Lamé, Ince, Mathieu, and others, as well as those of hypergeometric type.

This is a very critical time in the history of group-theoretic methods in special function theory. The basic relations between Lie groups, special functions, and the method of separation of variables have recently been clarified. One can now construct a group-theoretic machine that, when applied to a given differential equation of mathematical physics, describes in a rational manner the possible coordinate systems in which the equation admits solutions via separation of variables and the various expansion theorems relating the separable (special function) solutions in distinct coordinate systems. Indeed for the most important linear equations, the separated solutions are characterized as common eigenfunctions of sets of second-order commuting elements in the universal enveloping algebra of the Lie symmetry algebra corresponding to the equation. The problem of expanding one set of separable solutions in terms of another reduces to a problem in the representation theory of the Lie symmetry algebra.

Although this method is simple, elegant, and very useful, it has as yet been applied to relatively few differential equations. (At the time of this writing, the wave equation $(\partial_{tt}-\Delta_3)\Psi=0$ is still under intensive study.) Moreover, few theorems have yet been proved that delineate the full scope of the method. It is the author's hope that the present work, which is aimed at a general audience rather than at specialists, will convince the reader that group-theoretic methods are singularly appropriate for the study of separation of variables and special functions. It is also hoped that this work will encourage others to enter the field and solve the many interesting problems that remain.

The ideas relating Lie groups, special functions, and separation of variables spring from a number of rather diverse historical sources. The first deep work on the relationship of group representation theory and special functions is commonly attributed to E. Cartan (27). However, the first detailed use of the relationship for computational purposes is probably found in the papers of Wigner. Wigner's work on this subject began in the 1930s and is given an elementary exposition in his 1955 Princeton lecture notes. These notes were later expanded and updated in a book by Talman (124).

A second major contributor to the computational theory is Vilenkin, who wrote a series of papers commencing in 1956 and culminating in his book (128). This encyclopedic treatise was strongly influenced by the explicit constructions of irreducible representations of the classical groups due to Gel'fand and Naimark (e.g., (41)). Vilenkin (and Wigner) obtain special functions as matrix elements of operators defining irreducible group representations.

Another precursor of our theory is the factorization method. The method was discovered by Schrödinger and applied to solve the time-independent Schrödinger equation for a number of systems of physical interest (e.g., (117)). This useful tool for computing eigenvalues and recurrence relations for solutions of second-order ordinary differential equations was developed by several authors, including Infeld and Hull (52), who summarized the state of the theory as of 1951. An independent and somewhat different development was given by Inoui (53).

The author contributed to this theory by showing, in 1964 (80), that the factorization method was equivalent to the representation theory of four Lie algebras.

Another approach to the subject matter of this book is contained in three remarkable papers by Weisner (133–135), the first appearing in 1955. Weisner showed the group-theoretic significance of families of generating functions for hypergeometric, Hermite, and Bessel functions. In these papers are also found examples of separable coordinate systems characterized in terms of Lie algebra symmetry operators. Weisner's theory is extended and related to the factorization method in the author's monograph (82). This monograph is primarily devoted to the representation theory of local Lie groups rather than the theory of global Lie groups, which is treated in the works of Talman and Vilenkin.

We should also mention Truesdell's monograph on the F equation (126), which demonstrated how generating functions and integral representations for special functions can be derived directly from a knowledge of the differential recurrence relations obeyed by the special functions. By 1968 it was recognized that Truesdell's technique fits comfortably into the group-theoretic approach to special functions (82).

A major theme in the present work is that separable coordinate systems for second-order linear partial differential equations can be characterized in terms of sets of second-order symmetry operators for the equations. This idea is very natural from a quantum-mechanical point of view. Moreover, since the work of Lie, it has been known to be correct for certain simple coordinates, such as spherical, cylindrical, and Cartesian (i.e., subgroup) coordinates. For a few important Schrödinger equations, such as the equation for the hydrogen atom, operator characterizations of a few nonsubgroup coordinates were well known (9, 30). However, the explicit statement of the relationship between symmetry and separation of variables appeared for the first time in the 1965 paper (138) by Winternitz and Fris. These authors gave group-theoretic characterizations of the separable coordinate systems corresponding to the eigenvalue equations for the Laplace-Beltrami operators on two-dimensional spaces with constant curvature. This work was extended by Winternitz and collaborators in (74, 106, 139, 140). Finally, the author in collaboration with C. P. Boyer and E. G. Kalnins has classified group theoretically the separable coordinate systems for a number of important partial differential equations and investigated the relationship between the classification and special function theory. One interesting feature of this work, primarily due to Kalnins, has been the discovery of many new separable systems that are not contained in such standard references as (97). A second feature has been the development of a group-theoretic method that makes it possible to derive identities for nonhypergeometric special functions, such as Mathieu, Lamé, spheroidal, Ince, and anharmonic oscillator functions, as well as for the more familiar hypergeometric functions.

Prerequisites for understanding this book include some acquaintance with Lie groups and algebras (i.e., homomorphism and isomorphism of groups and algebras) such as can be found in (43) and (85). However, the examples treated here are very explicit and can be understood with only a minimal knowledge of Lie theory. Secondly, it is assumed that the reader has some experience in the solution of partial differential equations by separation of variables, in, say, rectangular, polar, and spherical coordinates.

Due to limitations of space, time, and the author's competence, it has been found necessary to omit certain topics. The most important among these is the theory of spherical functions on groups. This topic, a generalization of the theory of spherical harmonics, has an extensive literature (e.g., [47, 130]). Moreover, spherical functions were recently used to derive an addition theorem for Jacobi polynomials (68, 119). However, spherical functions are always associated with subgroup coordinates, and even for the most elementary equations considered in this book, they fail to encompass all of the special functions that arise via separation of variables.

Boundary value problems have also been omitted, even though symmetry methods are important for their solution (see [16]). This last reference, as well as (105) and (38) contain discussions of symmetry techniques for finding solutions of nonlinear partial differential equations, a subject that has been omitted here because its ultimate forum is not yet clear.

I should like to thank Paul Winternitz for helpful discussions leading to the basic concepts relating symmetry and separation of variables. Finally, I wish to thank Charles Boyer and Ernie Kalnins, without whose research collaboration this book could not have been written.

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