Math 8669: Combinatorial Theory

HW 3: Due Monday April 30, 2018

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Remark: Please do at least six of the following nine problems.

1) Deduce and prove a formula for the $f^{k,1^{n-k}}$, the number of Standard Young Tableaux of shape $\lambda = [k, 1^{n-k}]$.

2) (i) What is the image of $RSK(\pi)$ where $\pi(1) = 4, \pi(2) = 3, \pi(3) = 2, \pi(4) = 6, \pi(5) = 5, \pi(6) = 1$?
   i.e. $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 6 & 5 & 1 \end{pmatrix}$
   (ii) What is the image of

$$RSK^{-1}\left(\begin{array}{ccc} 1 & 2 & 5 \\ 2 & 4 & 6 \\ 3 & & \end{array}, \begin{array}{ccc} 1 & 4 & 5 \\ 2 & 6 & 6 \\ 3 & & \end{array}\right)?$$

3) Prove the Newton-Girard Identities that state for all $k \geq 1$ that

$$kh_k = \sum_{i=1}^{k} h_{k-i}p_i, \quad \text{and} \quad ke_k = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i}p_i.$$

   Hint: Recall how we proved $\sum_{i=0}^{k} (-1)^i h_{k-i} e_i = 0$ for all $k \geq 1$; it may also be useful to consider logarithmic derivatives.

4) (i) Expand $h_{3,1}$ in terms of the e_λ's
   (ii) Expand $e_{2,2}$ in terms of the p_λ's.

5) (i) Expand $s_{2,1}$ as a polynomial in variables $\{x_1, x_2, x_3\}$.

Hint: You may use a computer algebra package for this computation.

(ii) Expand $s_{3,2,1}$ as a polynomial in variables $\{x_1, x_2, x_3, x_4\}$.

Hint: You may use a computer algebra package for this computation.

(iii) Conjecture and prove an expansion for $s_{k,k-1,\ldots,3,2,1}$ in terms of $\{x_1, x_2, \ldots, x_{k+1}\}$.

6) (a)-(c) Compute (using combinatorial formulas rather than a computer algebra package) the Schur expansions of

$$p_{3,2}, \qquad h_4 \cdot s_{2,2,2}, \qquad \text{and} \qquad s_{3,2} \cdot s_{4,1}.$$

7) (a) How many Standard Young Tableaux are there of shape $(4, 2, 1, 1)$?

(b) Draw all SYT of shape $(2, 2, 1, 1)$.

(c) Find the number of permutations in S_{12} with a longest increasing sequence of length 6 and longest decreasing sequence of length 4.

8) In this problem, you will give an inductive proof of the **Hook Length Formula**, i.e. the number of Standard Young Tableaux, of shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$, is given by

$$f^\lambda = \frac{n!}{\prod_{c \in \lambda} h_c},$$

where h_c is the number of boxes in the hook emanating from cell c.

Let $\ell_i = \lambda_i + k - i$ (assuming that $\lambda_1 \geq \lambda_2 \geq \ldots$)

Let $\Delta(\ell_1, \ldots, \ell_k) = \prod_{i<j}(\ell_i - \ell_j)$.

Let $F(\ell_1, \ldots, \ell_k) = \frac{n!\Delta(\ell_1,\ldots,\ell_k)}{\ell_1!\ell_2!\cdots\ell_k!}$.

(a) Show that $\frac{n!}{\prod_{c \in \lambda} h_c} = F(\ell_1, \ldots, \ell_k)$.

(b) Show that

$$\sum_{i=1}^{k} x_i \Delta(x_1, \ldots, x_i + t, \ldots, x_k) = (x_1 + x_2 + \cdots + x_k + \binom{k}{2} t)\Delta(x_1, \ldots, x_k).$$

(c) Show that $n \cdot \Delta(\ell_1, \ldots, \ell_k) = \sum_{i=1}^{k} \ell_i \Delta(\ell_1, \ldots, \ell_i - 1, \ldots, \ell_k)$.

(d) Show that $f^\lambda = \sum_{i=1}^{k} f^{(\lambda_1, \ldots, \lambda_i - 1, \ldots, \lambda_k)}$, where we define $f^{(\lambda_1, \ldots, \lambda_i - 1, \ldots, \lambda_k)}$ to be zero if $\lambda_i = \lambda_{i+1}$.

(e) Conclude that $f^\lambda = \frac{n!}{\prod_{c \in \lambda} h_c}$.

9) (a) Recall that the symmetry group of the cube is S_4. Let V be the the representation obtained by S_4 acting on the six faces of the cube. How does V decompose into irreducibles?

(b) Consider the Spect module $V = S^{3,2}$ (an irreducible representation of $S_{\{1,2,3,4,5\}}$) and the trivial representation W for $S_{\{6,7,8\}} \cong S_3$. What is the decomposition of $V \otimes W \uparrow_{S_5 \times S_3}^{S_8}$ into irreducible representations?