Math 8680: Cluster Algebras and Quiver Representations

Homework 1 (Due Wednesday February 25, 2015)

I encourage collaboration on the homework, as long as each person understands the solutions, writes them up in their own words, and indicates on the homework page their collaborators. You may use computer algebra packages for calculations but should also briefly describe your calculations in words in this case.

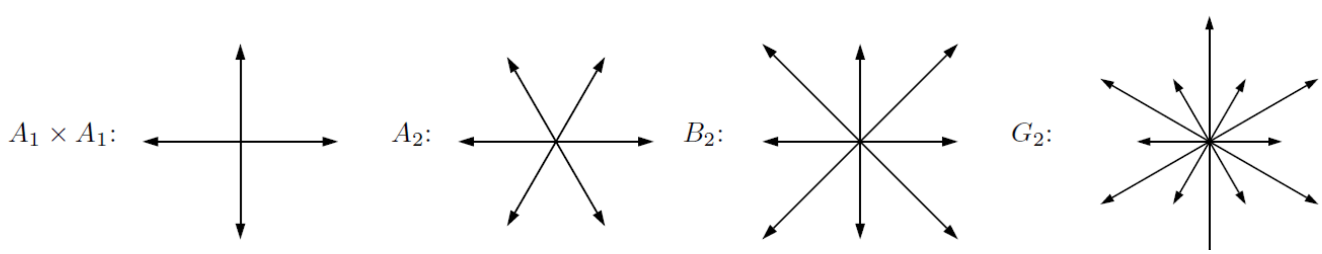
1) Consider the cluster algebra of geometric type defined by the initial labeled seed given by

$$\mathbf{x} = \{x_1, x_2, u_1, u_2, u_3\} \text{ and } B = \begin{bmatrix} 0 & 2 \\ -1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

a) Compute all cluster variables generating this cluster algebra.

b) Try to find a geometric model for this cluster algebra.

2) Consider the following families of vectors in two-dimensional space.



a) Verify that each of these collections satisfy the axioms of a root system.

b) Prove that these four root systems are the only root systems of rank 2, up to scaling and rotation.

Hint: Argue why you may assume that the unit vector e_1 is included in your root system.

c) Show that $\{e_j - e_i : i, j \in \{1, 2, \ldots, n+1\}, i \neq j\}$ is a rank n root system. (Called A_n)

d) Show that $\{\pm e_i : i \in \{1, 2, \ldots, n\}\} \cup \{\pm e_i \pm e_j : 1 \leq i < j \leq n\}$ is a rank n root system. (Called B_n)

e) Show that $\{\pm 2e_i : i \in \{1, 2, \ldots, n\}\} \cup \{\pm e_i \pm e_j : 1 \leq i < j \leq n\}$ is a rank n root system. (Called C_n)

f) Show that $\{\pm e_i \pm e_j : 1 \leq i < j \leq n\}$ is a rank n root system. (Called D_n)

Hint: You will use similar techiques for exercises (c)-(f) so feel free to reuse your work.

3) In this problem, you will prove some of the results needed for the *finite type classification*.

a) Prove that an n-by-n matrix B is skew-symmetrizable if and only if following two conditions on B hold:

(i) For all pairs $1 \leq i, j \leq n$, either b_{ij} and b_{ji} are both zero, or the product $b_{ij}b_{ji}$ is negative. This condition is called *sign-skew-symmetric*.

(ii) For all subsequences $[i_1, i_2, \ldots, i_k]$ of $\{1, 2, \ldots, n\}$, we have

$$b_{i_1 i_2} b_{i_2 i_3} \cdots b_{i_k i_1} = (-1)^k b_{i_2 i_1} b_{i_3 i_2} \cdots b_{i_1 i_k}.$$

We say that a skew-symmetrizable matrix B is 2-*finite* if B satisfies the bound $|b'_{ij} b'_{ji}| \leq 3$ for all i, j and all $B' = [b_{ij}]$ mutation-equivalent to B. Note that B is 2-finite if and only if $\mu_k(B)$ is 2-finite (for any k).

b) Prove that if B is not 2-finite, then the corresponding cluster algebra $\mathcal{A}(\mathbf{x}, B)$ is not of finite type.

c) Prove that if B is skew-symmetrizable and 2-finite, then every triangle in the diagram $\Gamma(B)$ of B is (i) oriented cyclically, and (ii) has weights $\{1, 1, 1\}$ or $\{1, 2, 2\}$.

d) Using (c) or the definition above, show that the following matrices are not 2-finite:

$$A_2^{(1)} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \; B_3^{(1)} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \\ 1 & -1 & 1 & 0 \end{bmatrix}, \; C_3^{(1)} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \text{ and } D_5^{(1)} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

4) Consider the cluster algebra $\mathcal{A}(2,2)$ with exchange matrix $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$. Let z be the Laurent polynomial $z = \frac{x_1^2 + x_2^2 + 1}{x_1 x_2}$.

a) Show that for all $n \in \mathbb{Z}$, with respect to the cluster $\{x_n, x_{n+1}\}$ that $z = \frac{x_n^2 + x_{n+1}^2 + 1}{x_n x_{n+1}}$.

This is called a **conserved quantity**.

b) Show that the cluster algebra exchange relation linearizes as $x_{n+1} = z x_n - x_{n-1}$ in this case.

5) a) Let $\{f_n\}$ be a sequence of rational functions defined by the following initial conditions and recurrence: $f_1 = x_1,\ f_2 = x_2,\ f_3 = x_3$, and for $n \geq 3$, $f_n f_{n-3} = f_{n-1} f_{n-2} + 1$. Use the Caterpillar Lemma or model this sequence by a cluster algebra to show that f_n is a Laurent polynomial in $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$ for all $n \geq 1$.

b) By a similar method, show that Somos-5, defined as $f_1 = x_1,\ f_2 = x_2,\ f_3 = x_3,\ f_4 = x_4,\ f_5 = x_5$, and for $n \geq 5$, $f_n f_{n-5} = f_{n-1} f_{n-4} + f_{n-2} f_{n-3}$ gives rise to Laurent polynomials in $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_4^{\pm 1}, x_5^{\pm 1}]$.