Math 8680: Cluster Algebras and Quiver Representations

Homework 2 (Due Wednesday April 8, 2015)

I encourage collaboration on the homework, as long as each person understands the solutions, writes them up in their own words, and indicates on the homework page their collaborators. You may use computer algebra packages for calculations but should also briefly describe your calculations in words in this case.

1) (a) Consider the quiver Somos 4 quiver Q, potential W, and corresponding bipartite tiling T as in Lecture 12 (page 2). List the perfect matchings of T and compute the Kasteleyn matrix and characteristic polynomial. What is the corresponding Newton polygon?

   (b) Apply the Goncharov-Kenyon procedure to this polygon. What bipartite tiling do you get?

   (c) and (d) Repeat the last two questions for the square-octagonal bipartite tiling on Lecture 12 (page 4).

2) (a) Prove the Lemma from Lecture 14, i.e. given a Kasteleyn weighting on the edges of a bipartite planar graph G and a $2k$-cycle L enclosing exactly ℓ vertices in its interior, prove that the alternating product of Kasteleyn weights on the edges of L is $(-1)^{k+\ell+1}$.

   (b) Let G be a bipartite graph on a torus where edges are each labeled with a distinct Boltzmann weight. Suppose we add signs to obtain a Kasteleyn weighting on edges. Let K be the corresponding Kasteleyn matrix where we include factors of $z_1^{\pm}$ and $z_2^{\pm}$ corresponding to edges crossing the homology of the torus.

   Recall from class that the terms in the expansion of $\det K$ will each be perfect matchings of G.

   Show that if M_1 and M_2 both correspond to the terms containing the same monomial $z_1^a z_2^b$, then the signs on these terms agree.

3) (a) Let Δ be a right triangle with side lengths 1, 2, and $\sqrt{5}$, as in Lecture 15. Consider the bipartite tiling T_Δ and corresponding quiver with potential (Q_Δ, W_Δ) obtained by the Goncharov-Kenyon construction.

   Prove that the superpotential algebra $A := \mathbb{C}Q_\Delta / I_{W_\Delta}$ is isomorphic to $\mathbb{C}[x, y, z] \rtimes \mathbb{Z}_2$.

   Hint and Update from Lecture 15: Using the notation from Lecture, let $(x, 1) = A + E$, $(y, 1) = D + C$, $(z, 1) = F + B$, and $(1, \epsilon) = e_1 - e_2$. Complete the isomorphism by describing (x, ϵ), (y, ϵ), and (z, ϵ), and showing that the correct relations are satisfied.

   (b) and (c) Repeat this problem (i.e. show that that superpotential algebra A is isomprhic to a twisted group ring $\mathbb{C}[x, y, z] \rtimes G$ for a finite abelian group $G \subset SL_3(\mathbb{C})$) for Δ a right triangle with side lengths $(2, 2, \sqrt{8})$ and $(1, 3, \sqrt{10})$.