Math 8680 : Spring 2015 1/21/15

Lecture 1

Welcome to "Topics in Combinatorics: Cluster Algebras, Tilings, and Physics"

Graduate Level topics course in Algebraic Combinatorics. I assume familiarity with groups, rings, modules (8202) but not cluster algebras or physics.

What are cluster algebras: Cluster Algebras are a unifying structure for understanding phenomena in a variety of topics in mathematics and physics.

History: They were first defined by Fomin & Zelevinsky in 2000 during their study of
- total positivity
- Lusztig's dual canonical bases for algebraic groups
- Coordinate rings of Grassmannians, $SL(N)$, ...

In these numerous examples, their goal was to write down an algebra where generators could be grouped together in overlapping subsets called clusters. Each cluster was related to others by binomial exchange relations.

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Prototypical Example:

Consider the Grassmannian (coordinate ring) $Gr(2, n)$.

e.g. $Gr(2,5) = \{$ 2-dim planes in 5 dims $\}$

$= \left\{ \text{Matrices } \underbrace{\begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \end{bmatrix}}_{\text{full rank}} \right\} \Big/ \left\{ \overset{\text{non-singular}}{\begin{bmatrix} A & B \\ C & D \end{bmatrix}} \text{ acting on left} \right\}$

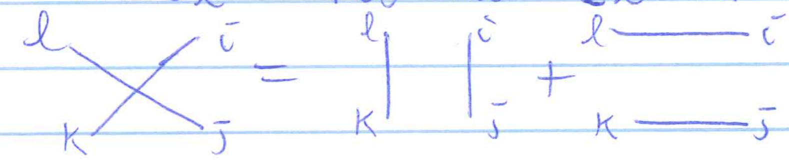
corresponds to change of basis for 2-plane

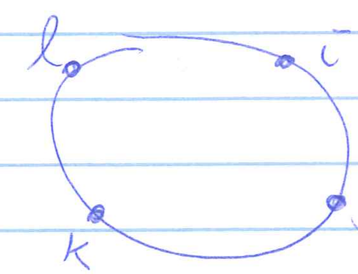
coordinates of $\mathbb{C}[Gr(2,n)]$ (coord ring) given by Plücker coords

$P_{ij} :=$ det of 2×2 minor given by ith & jth cols

$\{P_{ij}\}$'s not independent however: if $i<j<k<l$,

$$P_{ik} P_{jl} = P_{ij} P_{kl} + P_{il} P_{jk}$$





Exercise (if you haven't seen this before)

For $Gr(2,5)$ or general $Gr(2,n)$ compute/show the Plücker relation.

Let us use Plücker coordinates to write $\mathbb{C}[Gr(2,5)]$ as a cluster algebra.

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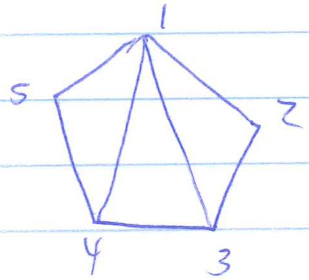
There are $10 = \binom{5}{2}$ Plücker coords, corresponding to the 10 chords of a pentagon

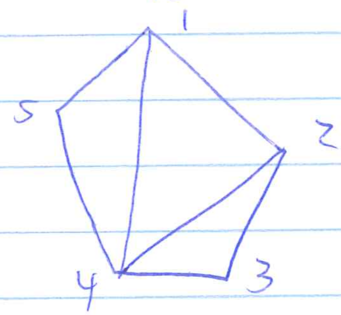
Plücker cords alg. ind. $\Longleftrightarrow$ the corresponding chords do not cross

Initial Cluster:



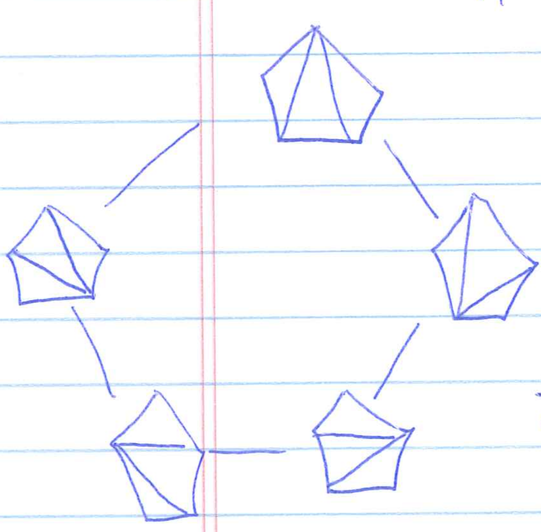
$P_{12}, P_{23}, P_{34}, P_{45}, P_{15}, P_{13}, P_{14}$

We can exchange P_{13} w/ P_{24} $\left(P_{13}P_{24} = P_{12}P_{34} + P_{23}P_{14}\right)$



$P_{12}, \ldots, P_{15}, \quad P_{24}, P_{14}$

Next, exchange P_{14} w/ P_{25}, and so on (pentagon of pentagons)



10 generators $\{P_{ij}\}$ = cluster variables

5 of them are frozen (appear in every cluster)

2 out of the remaining 5 in each cluster.

Exchange 1 at a time.

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This cluster algebra is known as <u>type A_2</u>
(For reasons we will esplain later in the course)

The connections between clusters via algebraic exchange relations is the <u>Hallmark</u> of Cluster Algebra theory.

Since their introduction by FZ, the connections between cluster algebras and other Fields have blossomed tremendously.

Here is a partial list of topics (from A. Zelevinsky Oct '12)

- Total Positivity
- Representation Theory
- String Theory
- Statistical physics models
- Quiver representations
- Non-commutative geometry
- Teichmüller theory
- Hyperbolic geometry
- Discrete integrable systems
- Poisson geometry
- Tropical geometry
- Polyhedral combinatorics

If you were in my topics course in 2011,
you saw

- Quiver representations
- Teichmüller theory
- Hyperbolic geometry

[math.umn.edu/~musiker/8680-11]

This course will focus on

- String theory
- Statistical physics models
- Discrete Integrable systems

Although most likely will touch on other connections as well.

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Course Logistics

Webpage: math.umn.edu/~musiker/8680-15/

Email: musiker@math.umn.edu

There will be no exams but I expect registered students

- to attend
- approximately 3 Problem sets during the course of the semester

[Mainly important to have you further digest material and gives me the opportunity to address common confusions.]

Grading

- First problem set will be handed out/posted next week. Due mid-to-late February

- I also will have a list of research papers highlighting further directions on the course webpage.

There will be class time the last week or two for graduate students to give presentations on these topics.

- For registered students, a presentation can take the place of a problem set.

  For auditing students (or interested postdocs), I can encourage presentations for your own edifications.

Presentation sign-ups will start mid-February.

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Course Logistics

Books: No required book.

Recommend "Lecture Notes on Cluster Algebras" by Robert March (2013, Zurich Lectures in Advanced Mathematics)

(First Third of the course will be based on Marsh.)

If you took my course in 2011, this first third will be similar to then so my 2011 lecture notes quite relevant now too.

Next: Recommend "Dimer Models and Calabi-Yau Algebras" by Nathan Broomhead (2012, Memoir of the AMS)

(Second part of course based on Broomhead and this provides our segway to Physics)

ArXiv 0901.4662 is essentially this book
If you don't want to purchase

Not as relevant this year, but a good book: "Cluster Algebras and Poisson Geometry" by Gekhtman-Shapiro-Vainshtein

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There are no books specifically on the physics connections yet (at least, none written for mathematicians) but lots of survey and research articles. [see special issue in J Phys A linked from website]

These are included on the course webpage and more will be added as course progresses.

The above books also on reserve or available through the library's website.

Office Hours: Feel free to drop by anytime I'm in. My door will be open. Or email for an appointment. [Vincent 251]

Official OH: MW 2:30 – 3:20
but dropping by in morning MWF usually better

Next, I will discuss the connections between cluster algebras and string theory foreshadowing second part of course.

Before that, any questions?

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The AdS/CFT correspondence

(anti-de Sitter / conformal field theory)

is a duality between quantum gravity
which is a model of spacetime in terms of a vacuum solution (gravitational fields in spacetime in the absence of masses) to Einstein's equations

and quantum field theory (in terms of string or M-theory)

Example: toric Sasaki-Einstein 5-manifold
(e.g. $AdS_5 \times S^5$)

corresponds to an $N=1$ superconformal field theory in 4 dim
(e.g. $N=4$ supersymmetric Yang-Mills theory)

Moral: To understand a supersymmetric superconformal quantum field theory,

important to understand geometry of a CY_3 manifold.

As we will see in this course, dimer models instrumental in understanding geometry in IR (infrared, i.e. low energy).

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Summary (based on Slides of Yang-Hui He, Dec. '14)

Standard Model (of particle physics)
$SU(3) \times SU(2) \times U(1)$ Gauge Theory

(symmetries of charges, spin, color, flavor, etc.)

superstring theory = 10 dim = $\underset{\text{spacetime}}{4} + \underset{\text{curled up}}{6}$

Dirichlet p-branes (D-branes) are $(p+1)$-dim subvarieties in $\mathbb{R}^{1,9}$ (9 space + 1 time) on which open strings can end

Brane World (theory): we live on D3 brane $\perp$ 6 dim affine variety $\mathcal{M}$

world volume = $(3+1)$-dim Quiver Gauge Theory with Supersymmetry (SUSY) Yang-Mills and product gauge group

The transverse variety $\mathcal{M}$ is local (affine, singular) Calabi-Yau 3-fold (cone over Sasaki-Einstein 5-manifold)

D3 brane probes the geometry of $\mathcal{M}$.

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Quiver Gauge Theory:

$$S = \int d^4x \left[\int d^4\theta \, \Phi_i^\dagger e^V \Phi_i + \left(\frac{1}{4g^2} \int d^2\theta \, \mathrm{Tr}\, W_\alpha W^\alpha + \int d^2\theta \, W(\Phi) + h.c. \right) \right]$$

w/ W = superpotential

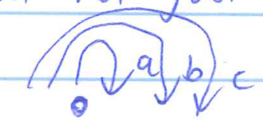
potential $V(\phi_i, \bar{\phi}_i) = \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 + \frac{g^2}{4} \left(\sum_i q_i |\phi_i|^2 \right)^2$

vacuum $\sim V(\phi_i, \bar{\phi}_i) = 0 \Rightarrow \begin{cases} \frac{\partial W}{\partial \phi_i} = 0 & F\text{-terms} \\ \sum_i q_i |\phi_i|^2 = 0 & D\text{-terms} \end{cases}$

$\mathcal{M}$ = vacuum moduli space
= solutions to F & D equations
= affine (singular) variety $\longrightarrow$ Rep'n Variety (of Quiver)

Most famous example

$\mathcal{N}=4$ supersymmetric (i.e. not just $\mathcal{N}=1$ SUSY, but higher) Yang Mills



$$W = \mathrm{Tr}(X[Y,Z]) = \mathrm{Tr}(XYZ - XZY)$$

if we vertex $\longleftrightarrow U(N)$, AdS/CFT corresp. gives N D3-branes transverse to flat $\mathbb{R}^6$.

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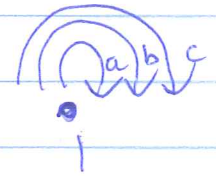
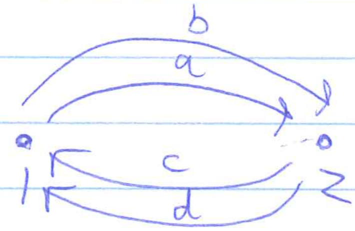
Consider the following two directed graphs (which we call <u>quivers</u> "think Robin Hood")

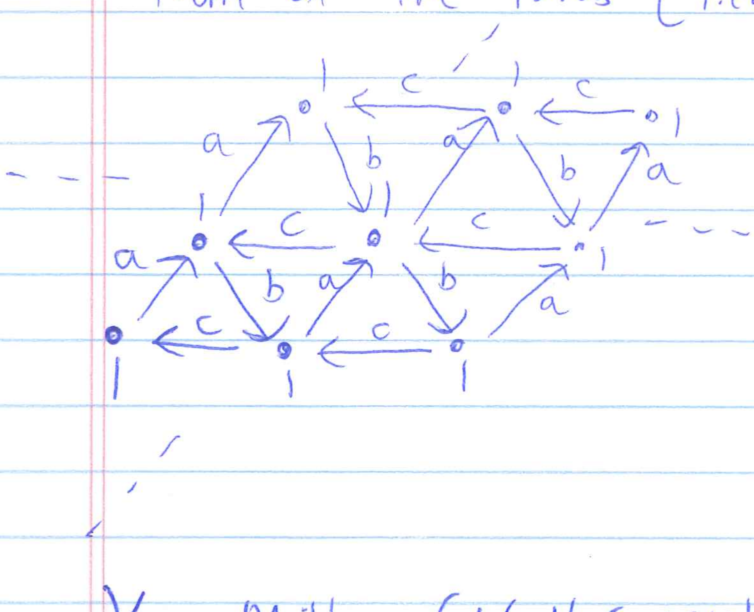
$Q_1 =$ 

$Q_2 =$ 

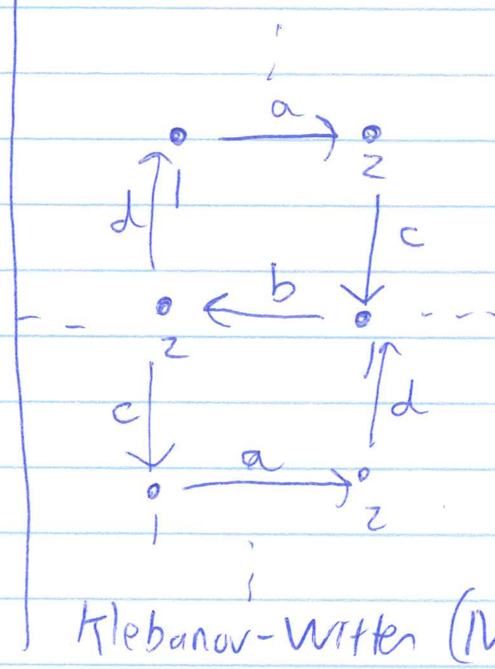
If you have seen cluster algebras before, you might be disturbed by the loops and 2-cycles but I hope by the end of the course you will think of this as

"Dr. Musiker: How I Learned to Stop Worrying and Love the Loops (and 2-cycles)"

I unfold Q_1 and Q_2 by drawing them on the torus (i.e. doubly periodically in the plane)



<u>Yang-Mills ($N=4$ SUSY)</u>



<u>Klebanov-Witten ($N=1$)</u>

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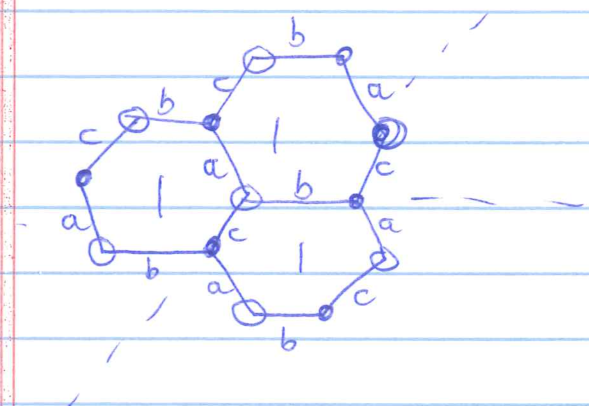
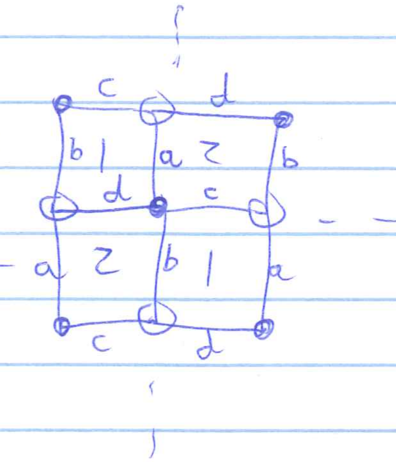
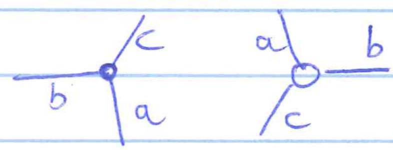
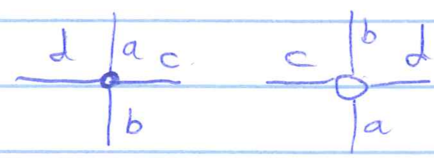
I next take the planar duals

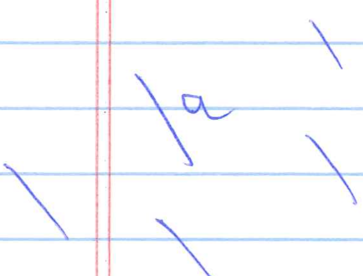
$T_1 =$ 

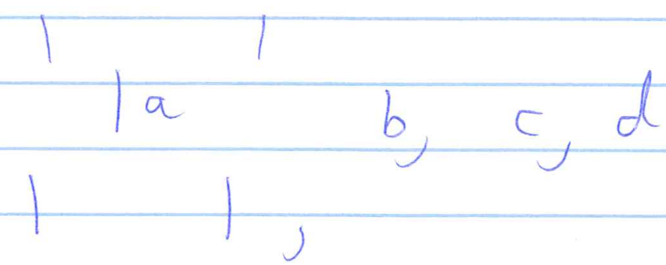
$T_2 =$ 



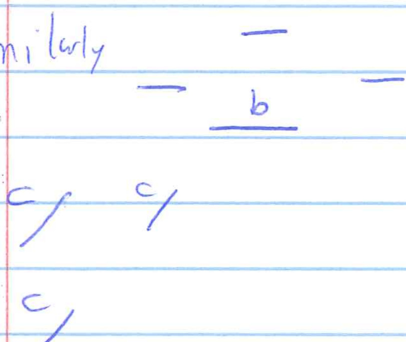


and periodic perfect matchings (dimer covers)





& similarly



Note: These are special cases, dimer covers often involve edges w/ more than 1 label.

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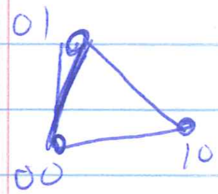
From techniques, we will discuss in this course,

three dimer covers of T_1

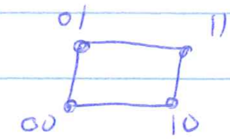
correspond to toric diagram



and toric variety with coord ring $\mathbb{C}^3$

four dimer covers of T_2

correspond to



and toric variety w/ coord ring.

$$\mathbb{C}[x, y, w, z]/(xy - zw)$$

singular conifold.

$\boxed{\text{AdS/CFT} = N \text{ D3-branes transverse to conifold singularity}}$

On the other hand, we can consider quiver representations of Q_1 & Q_2 (w/ potentials)

and get Jacobian algebras which are non-commutative resolutions of these coord. rings. (deformations)

Relations in these Jacobian algebra correspond to F-term relations which show up when physicists (string theorists) compute moduli space of vacua for cones over CY3-folds.

This course will study how combinatorics of dimers & tilings connects to both cluster algebras & problems in string theory.