2/25/14 <u>Lecture 11:</u> Quivers and algebras from dimer models

Today we begin Section 2.1 of

"Dimer Models and Calabi-Yau Algebras"
by Nathan Broomhead

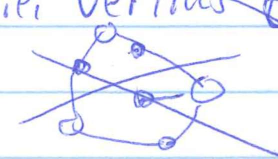
We begin with a <u>finite bipartite tiling</u> of a compact (oriented) Rieman surface Y.

By this, we mean a cell decomposition of Y by polygons ($2n$-gons since bipartite) such that the <u>vertices</u> (0-cells) and <u>edges</u> (1-cells) together form a bipartite graph.

We color the vertices black and white accordingly so that we have no ●—●, no ○—○, only ●—○'s.

The polygons give the 2-cells completing to our surface Y. Technically, the 2-cells are open disks bounded by the polygons. (Also called <u>faces</u>)

We also have no leaf edges (i.e. vertices of valence 1) in our cell decomposition

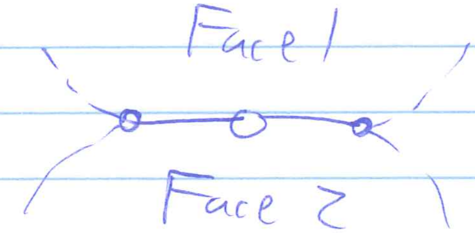


which if you were in Vic's class on <u>Graphs on surfaces</u> in the fall is diffeent.

2/25/15 ② Such a tiling is called a <u>dimer model</u>.

We do allow

— 2-valent vertices



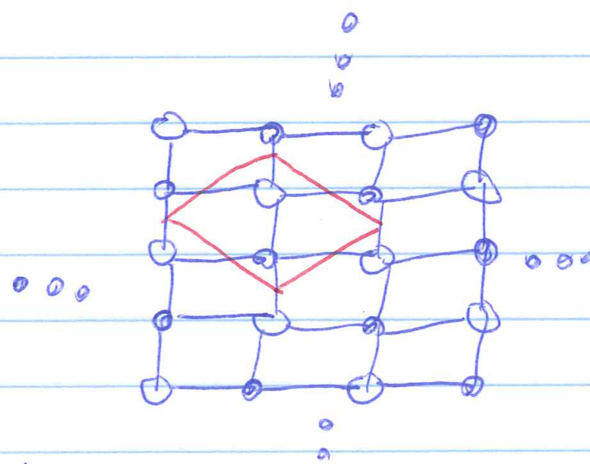
& — bigons 

For the time being.

<u>Examples</u>

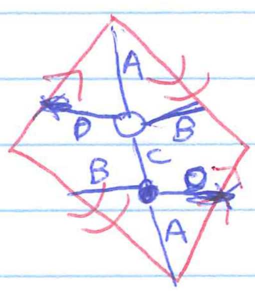
<u>E.g. 1</u> Checkerboard

on the 2-torus.

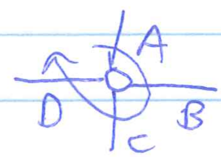


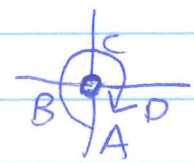
Here, illustrated as a doubly-periodic bipartite tiling of the plane, which is the universal cover of the 2-torus.

Alternatively, we can specify a fundamental domain [illustrated above in red]



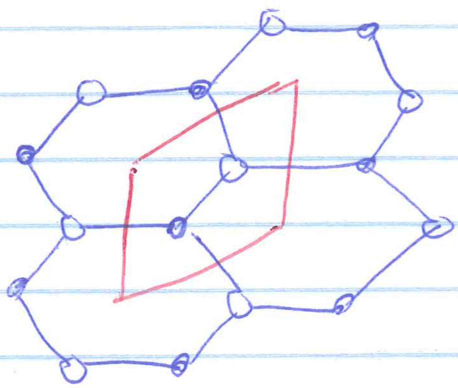
<u>locally</u>



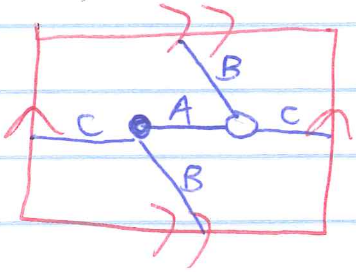


Note that edge A's midpoint is corner
edge B's midpoint is midpoint of
edge D's midpoint is midpoint of

2/25/15 3 Eg. 2 Hexagonal lattice



Alternatively



Remark: If you were in the Fall, you might recognize these two eg.'s as dessins de Enfants [From Bose-Gundry-He]

Eg. 1 $\longleftrightarrow$ $E = \{y^2 = x(x-1)(x+1)\}$, $B(x,y) = \frac{(x+1)^2}{4x}$ $J=1728$

$E = \{y^2 = x^3+1\}$, $B(x,y) = \frac{1}{2}(1+y)$ $J=0$

Alternatively

Eg. 2 $\longleftrightarrow$ $E = \{y^2 = x(x-1)(x-\lambda)\} \subset \mathbb{CP}^2$ w/ $\lambda = \frac{1+i\sqrt{3}}{2}$

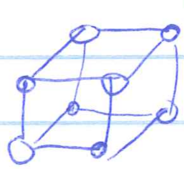
Belyi map: quotient by $g: Y \mapsto -\lambda Y$, $Z \mapsto -\lambda Z$, $X \mapsto X^2 - \lambda Z$

The connections between dessins and dimer models is an exciting area of research.

Talk to Yao-Rui Re (Minnesota Undergrad.) or Papers by Yang-Hui He (Oxford)

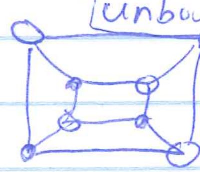
Eg. 3 On the 2-sphere, the cube gives a bipartite tiling (2-cell decomposition)

6 Faces

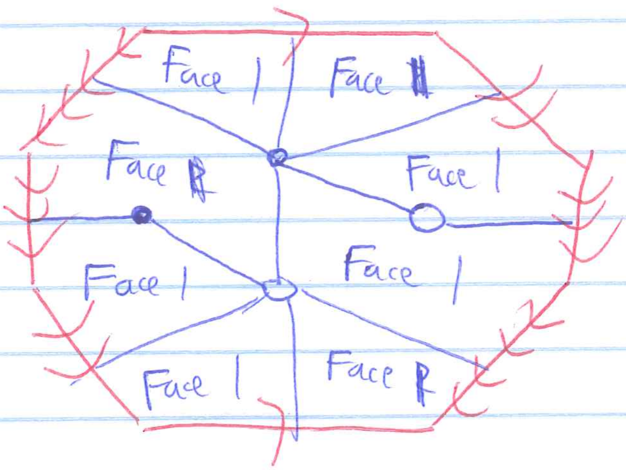


or stereographic projection

unbounded face

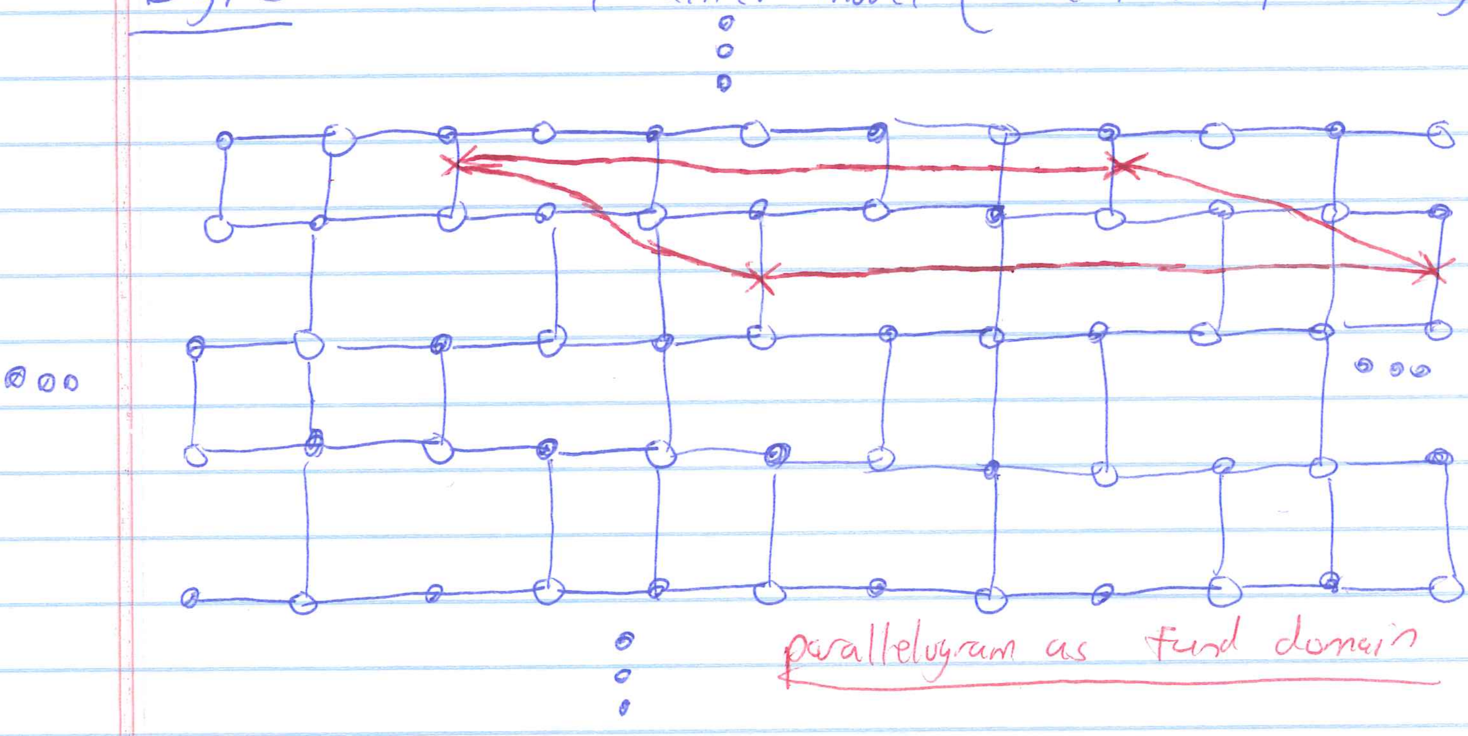


2/25/15 (4) **E.g. 4** On the genus 2 surface, we use for example an octagonal fundamental domain



and you can use 4g-gon for general genus g surface.

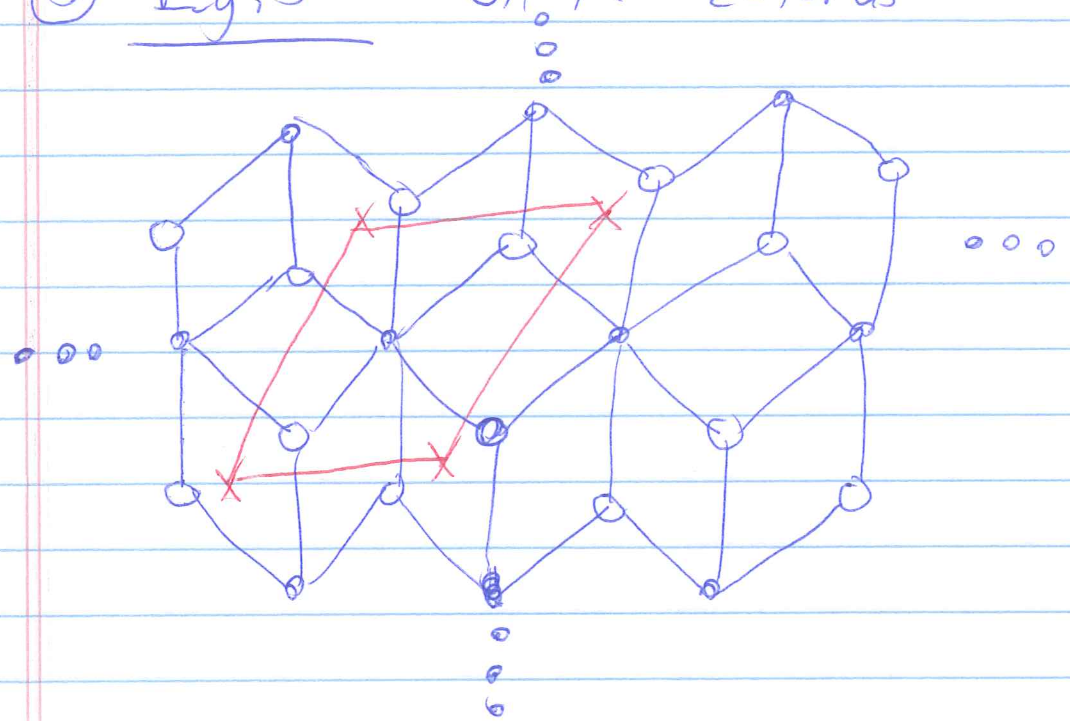
E.g. 5 sumos 4 dimer model (name to be explained later)



parallelogram as fund domain

Back on the 2-torus

2/25/15 (5) <u>Eg. 5</u> On the 2-torus



<u>Question:</u> How is Eg. 5 different than the other examples we've seen?

<u>Answer:</u> Odd # vertices in Fund. Domain

In fact one •, two ∘

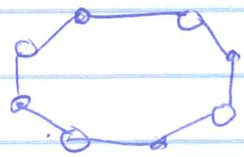
<u>Def'n</u>: A <u>dimer model</u> is called <u>balanced</u> if the number of • vertices = number of ∘ vertices.

We will usually consider only dimer models that are balanced, but allow unbalanced for the time being.

<u>Rem</u>: Bipartite graph ~~has~~ has no <u>perfect matchings</u>, i.e. <u>Full</u> collection of edges so every vertex touched exactly once if G is not balanced.

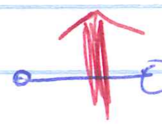
2/25/15 (6) To a dimer model (equivalently bipartite tiling), we can assign a <u>quiver</u>.

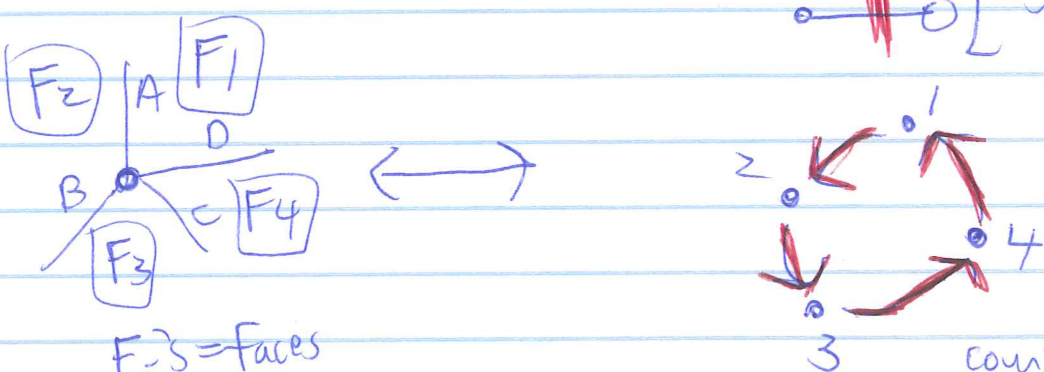
<u>First step:</u> Take dual graph

face  $\longleftrightarrow$ quiver vertex $\in Q_0$

vertex • or ○ $\longleftrightarrow$ cycle in quiver $\in Q_2$

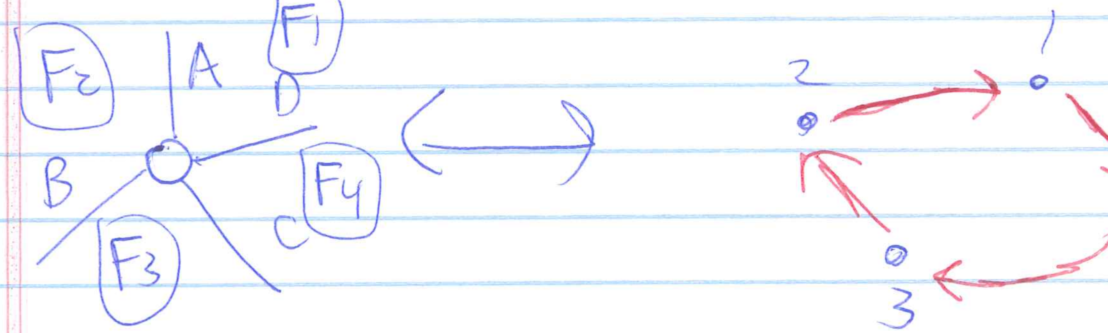
edge •—○ $\longleftrightarrow$ arrow in quiver $\in Q_1$

 [e.g. white on right]



<u>F_i's = faces</u>

<u>counter-clockwise</u>



<u>clockwise</u>

2/25/15 7

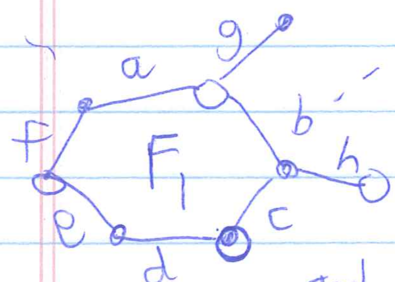
We get a homological chain complex for Riemann surface Y

$$\mathbb{Z}_{Q_2} \xrightarrow{\partial} \mathbb{Z}_{Q_1} \xrightarrow{\partial} \mathbb{Z}_{Q_0}$$ [Free abelian gps formally on Q_i]

and dual cochain complex in cohomology

$$\mathbb{Z}^{Q_0} \xrightarrow{d} \mathbb{Z}^{Q_1} \xrightarrow{d} \mathbb{Z}^{Q_2}$$ [$\mathbb{Z}$-linear functions on $\mathbb{Z}^{Q_i}$]

$d : \mathbb{Z}^{Q_1} \to \mathbb{Z}^{Q_2}$ coboundary map



$\longmapsto (F_1 \longrightarrow a+b+c+d+e+f)$

and in general, sum values of edges around a face to get its value.

Let $F \mapsto (-1)^F$ be the elt of $\mathbb{Z}^{Q_2}$ that takes value $+1$ on counterclockwise [i.e. black "faces"]

value -1 on clockwise [i.e. white "faces"]

The elt of $\mathbb{Z}_{Q_2}$ defined by

$\sum_{\text{face } F \in Q_2} (-1)^F \cdot F$ generates Ker $\partial : \mathbb{Z}_{Q_2} \to \mathbb{Z}_{Q_1}$. thus it is a <u>fund. class</u>, generator of <u>$H_2(Y) \cong \mathbb{Z}$</u>.

2/25/15 (8) The superpotential algebra

The path algebra of a quiver $Q = (Q_0, Q_1)$, which is often denoted as kQ or $\mathbb{C}Q$, is generated by all paths $a_1 a_2 \cdots a_k$ in Q_1 $\leftarrow \leftarrow \cdots \leftarrow$

plus $e_1, e_2, \dots, e_n$ defined as "lazy paths" corresponding to the $n = |Q_0|$ vertices of Q.

e_i starts and ends at vertex i.

We define multiplication in $\mathbb{C}Q$ as concatenation when possible, i.e.

$$\begin{cases} \underset{\leftarrow}{p} \circ \underset{\leftarrow}{q} = \underset{\leftarrow}{pq} & \text{if } t(p) = h(q) \\ 0 & \text{o.w.} \end{cases}$$

$t(e_i) = h(e_i) = i$ so $e_i^2 = e_i$ & $p e_i = \begin{cases} p & \text{if } t(p) = i \\ 0 & \text{o.w.} \end{cases}$

$e_i p = \begin{cases} p & \text{if } h(p) = i \\ 0 & \text{o.w.} \end{cases}$

Example $Q = \underset{1}{\circ} \leftarrow \underset{2}{\circ} \leftarrow \circ \cdots \circ \leftarrow \underset{n}{\circ}$

Claim: $\mathbb{C}Q \cong \{$upper triangular $n \times n$ matrices w/ entries in $\mathbb{C}\}$

$e_i \overset{\sim}{\longleftrightarrow} \begin{bmatrix} 0 & & & \\ & \ddots & & 0 \\ & & 1 & \\ 0 & & & \ddots & \\ & & & & 0 \end{bmatrix} = E_{ii}$, $\overbrace{\underset{i}{\circ} \leftarrow \circ \leftarrow \circ \cdots \leftarrow \underset{j}{\circ}}^{p} \overset{\sim}{\longmapsto} E_{ij}$

$i < j$

$E_{ij} E_{jk} = E_{ik}$, $E_{ij} \underset{e_j}{\underset{\parallel}{E_{jj}}} = E_{ij}$, $\underset{e_i}{\underset{\parallel}{E_{ii}}} E_{ij} = E_{ij}$

2/25/15 (9)

Beautiful theory regarding path algebras. Could study their indecomposable, Gabriel's Thm, they are nice examples of associative algebras which are Krull-Schmidt but ~~irreducible~~ indecomposable.

We instead consider variants = superpotential algebra, also sometimes known as Jacobian algebra of bipartite tiling $\overset{\text{duality}}{\longleftrightarrow} (Q_0, Q_1, Q_2)$.

Rem: Data (Q_0, Q_1) is indeed data of a quiver.

Let $\mathbb{C}Q$ be path algebra of (Q_0, Q_1).

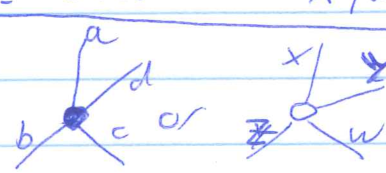
We use Q_2 to define an ideal of relations in $\mathbb{C}Q$.

Def: The superpotential (or potential) W of a bipartite tiling.

$$W := \sum_{F \in Q_2} (-1)^F \underbrace{\partial F}_{\text{elt of } \mathbb{Z}_{Q_1} \text{ give by boundary of face } F}$$

In other words, Q_2 are faces dual to vertices of the bipartite tiling. $\boxed{W = \ldots abcd - xywz \ldots}$ e.g.

In the tiling, vertices are



Picking an arbitrary starting point, we read arrows incident to • in counter-clockwise order, and to ∘ in clockwise order and then respectively add or subtract from W.

2/25/15 10 Rem: We would consider

$$W = \ldots + abcd - xywz + \ldots$$

& $$\widetilde{W} = \ldots + bcda - wzxy + \ldots$$

the same superpotential.

More rigorously, W lives in $\mathbb{C}Q_{cyc} := \mathbb{C}Q / [\mathbb{C}Q, \mathbb{C}Q]$

commutator $= \{pq - qp : p, q \in \mathbb{C}Q\}$

[In particular, notice that cyclically rotating a ~~term~~ term of W same as changing uv into vu.]

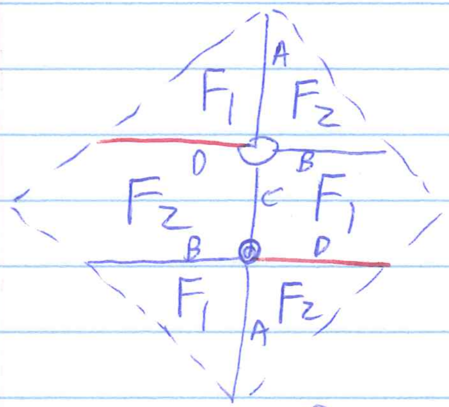
With W defined from Q_2, we define

$$I_W := \left(\frac{\partial}{\partial x_a} W : a \in Q_1 \right)$$

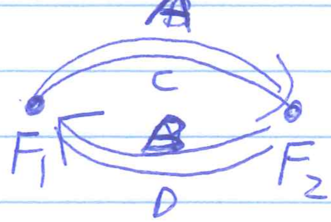
where $\frac{\partial}{\partial x_a}$ represents the "cyclic derivative" of W by arrow $a \in Q_1$,

meaning rotate terms in W until arrow a is in front (assuming a is in that term) and then take usual derivative in multivariate polynomial ring.

2/25/15 (11) Superpotential algebras for our examples



dual to



$Q_0 = \{F_1, F_2\}$

$Q_1 = \{A, B, C, D\}$

Q_2 corresponds to 

sometimes denoted

∂W

$\downarrow$

$$W = DCBA - ABCD$$
$$= ADCB - ABCD$$

$$I_W = \big(\underset{\partial_A}{DCB - BCD},\ \underset{\partial_B}{ADC - CDA},\ \underset{\partial_C}{BAD - DAB},\ \underset{\partial_D}{CBA - ABC}\big)$$

$\mathbb{C}$ () would be infinite-dimensional since would ~~be~~ generated by arbitrary large cycles.

But $\mathbb{C}$ () $/ I_W$ thought of as generated by

$e_1, e_2, A, B, C, D, AB, BC, CD,$ ~~DA~~ $DA,$
$BA, CB, DC,$ and AD

is easier to understand.

Claim: $\mathbb{C}$ () $/ I_W = \overbrace{Ae_1 \oplus Ae_2}^{A}$ where each $Ae_i \cong$ algebra over $\mathbb{C}[x,y,w,z]/(xy - wz)$

2/25/15

12

Notice that Ae_1 generated by $A, C, \underset{x_1}{\underset{\shortparallel}{AB}}, \underset{x_2}{\underset{\shortparallel}{AD}}, \underset{x_3}{\underset{\shortparallel}{CB}}, \underset{x_4}{\underset{\shortparallel}{CD}}$

Call these $\longrightarrow$ $x_1 \; x_2 \; x_3 \; x_4$

I_W tells us that $x_1 x_2 = ABAD \sim ADAB = x_2 x_1$

and similarly, $x_i x_j = x_j x_i$ for $1 \le i \le j \le 4$.

Thus it make sense that $\langle x_1, x_2, x_3, x_4 \rangle$ = polynomial ring in 4 vars mod relations

we also see $x_1 x_4 = ABCD \sim ADCB = x_2 x_3$

and in fact this is the only other relation.

In summary, $Ae_1 \cong \dfrac{\mathbb{C}[x_1, x_2, x_3, x_4]}{(x_1x_4 - x_2x_3)} \langle A, C \rangle$

Next week, we will see combinatorial point of view/model inspired by this identitization.

In fact, can let $x_1 = AB - BA$ to get

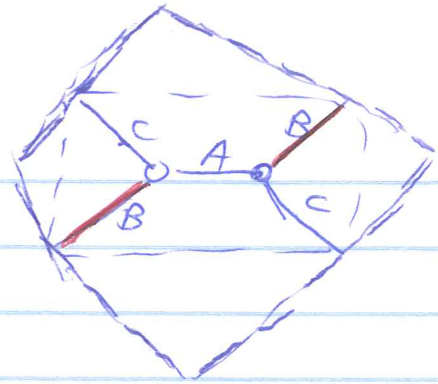
$x_2 = AD - DA$

$x_3 = CB - BC$

$x_4 = CD - DC$

$\mathbb{C}($$)/I_W \cong \dfrac{\mathbb{C}[x, y, w, z]}{(xy - wz)} \langle A, B, C, D \rangle$

2/25/15 13 E.g. 2 Hexagonal lattice



$W = ACB - ABC$

$I_W = (CB - BC,\ AC - CA,\ BA - AB)$

1 Face in the Fund. Domain so $Q = (Q_0, Q_1)$ =

$\mathbb{C}Q = \mathbb{C}\langle A, B, C\rangle$ free assoc. alg.
since any concatenations possible

but $\mathbb{C}Q / I_W = \mathbb{C}\langle A,B,C\rangle / [\langle A,B,C\rangle] \cong \underbrace{\mathbb{C}[A,B,C]}_{\text{ordinary poly ring.}}$