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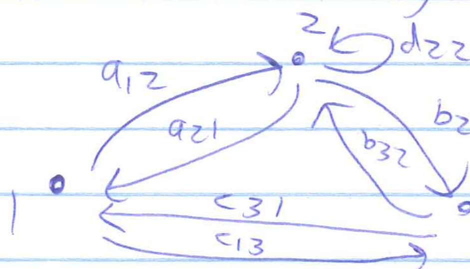
Lecture 12: Toric Actions and Perfect Matchings on Dimer Models

(Sections 2.2–2.3 of [Broomhead])

Consider the quiver

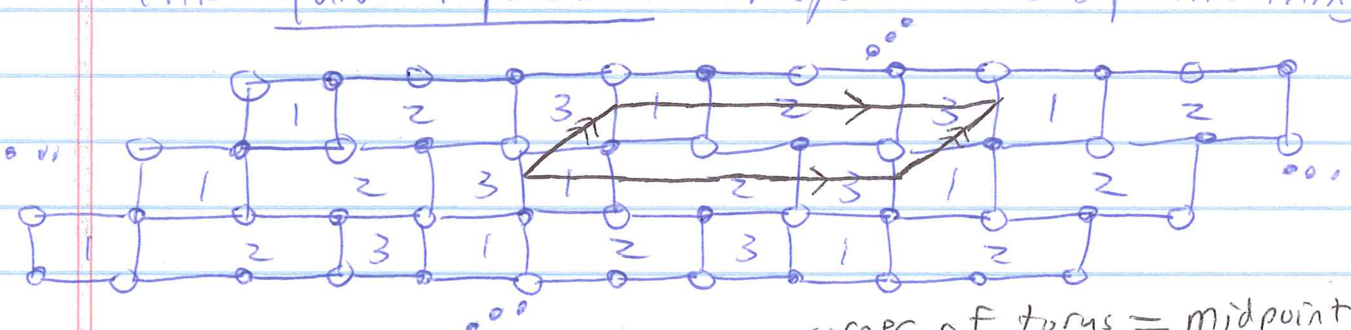


(Q_0, Q_1)

with potential $\boxed{W = a_{12}a_{21}c_{13}c_{31} + b_{23}b_{32}d_{22} - b_{32}b_{23}c_{31}c_{13} - a_{21}a_{12}d_{22}}$

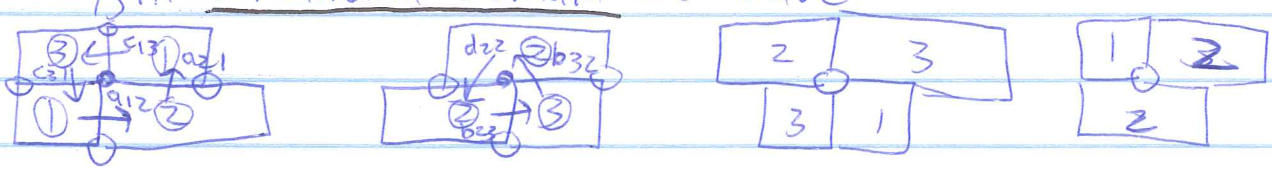
terms of W correspond to 2-cells of Q_2

This quiver + potential corresponds to the bipartite tiling



corner of torus = midpoint of $3\,|\,1$ edge

Locally, in fundamental domain we have



This dimer model called SPP, i.e. Suspended Pinch Point in the physics literature

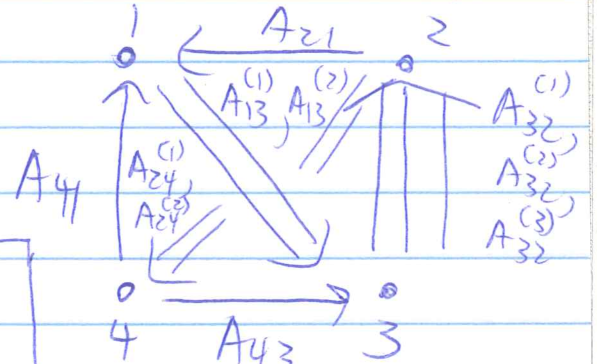
Claim: If we generalize the notion of mutation of quivers with potential to the case with 2-cycles and loops, then this (Q, W) is period 1.

3/2/15 2 Claim: SPP is period 1 (in the appropriate sense) and mutations correspond to the recurrence

$$x_n x_{n-3} = 2 x_{n-1} x_{n-2}.$$

Note: This is not a cluster algebra binomial exchange relation since $\exists\, x_i$ dividing both monomials.

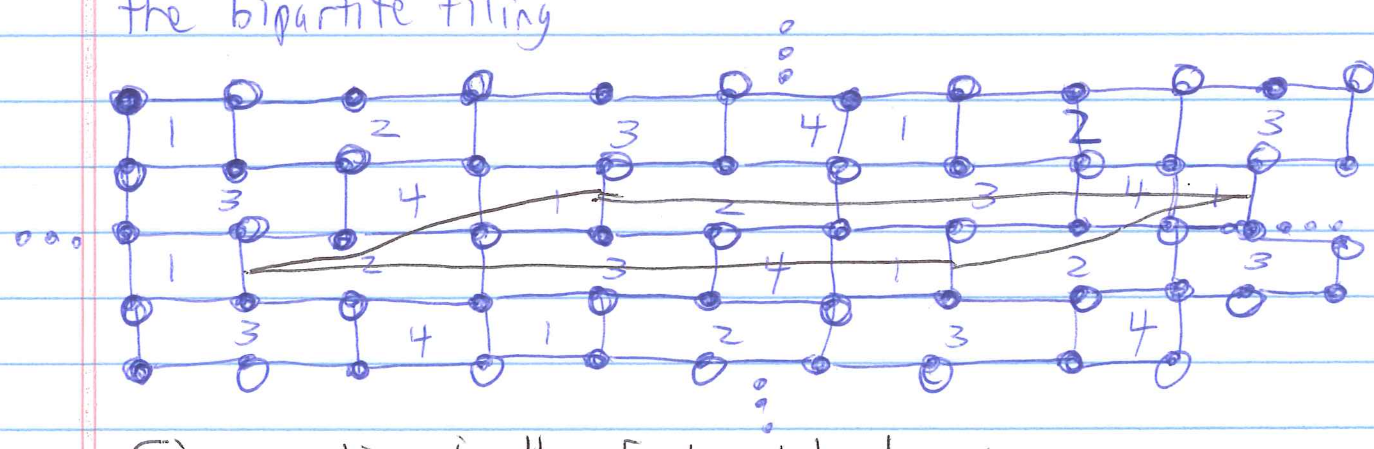
Somos-4 quiver $(Q_0, Q_1) =$



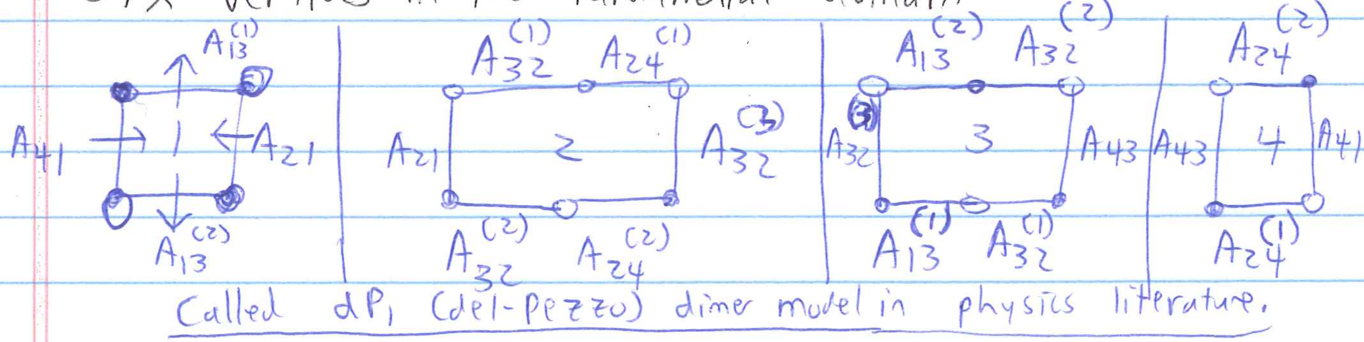
We use the potential

$$\boxed{\begin{aligned} W = {} & A_{13}^{(2)} A_{32}^{(2)} A_{21} + A_{13}^{(1)} A_{32}^{(3)} A_{24}^{(2)} A_{41} \\ & + A_{24}^{(1)} A_{43} A_{32}^{(1)} - A_{13}^{(2)} A_{32}^{(3)} A_{24}^{(1)} A_{41} \\ & - A_{24}^{(2)} A_{43} A_{32}^{(2)} - A_{13}^{(1)} A_{32}^{(1)} A_{21} \end{aligned}}$$

to encode Q_2. This triple (Q_0, Q_1, Q_2) corresponds to the bipartite tiling



Six vertices in the fundamental domain

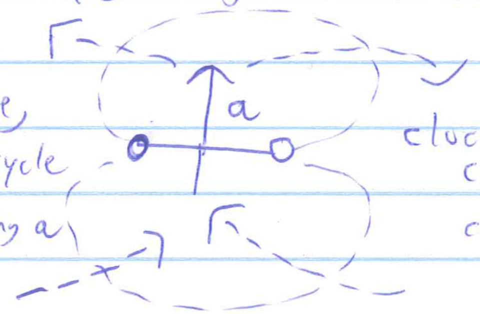


Called dP_1 (del-Pezzo) dimer model in physics literature.

3/2/15 ③ Notice in both of these examples, every arrow $a \in Q_1$ appears <u>twice</u> in the potential W, once w/ a pos. sign, once w/ a neg. sign.

This corresponds to the ability to glue cycles of Q_2 together in a periodic way

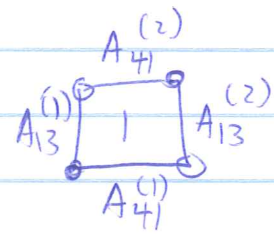
counter-clockwise, i.e. positive cycle in Q_2 containing a

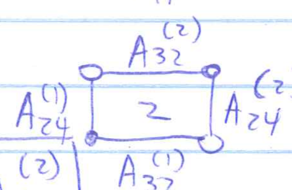


clockwise, i.e. negative cycle in Q_2 containing a

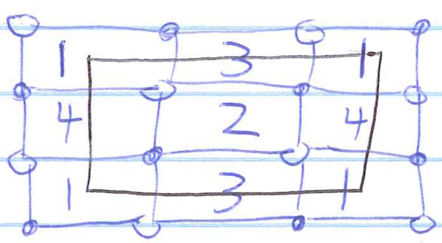
<u>Two more related examples</u>

$$(Q_0, Q_1) = \begin{tikzcd} 1 \arrow[r, shift left, "A_{13}^{(1)}"] \arrow[r, shift right, "A_{13}^{(2)}"'] & 3 \arrow[d, shift left, "A_{32}^{(1)}"] \arrow[d, shift right, "A_{32}^{(2)}"'] \\ 4 \arrow[u, shift left, "A_{41}^{(1)}"] \arrow[u, shift right, "A_{41}^{(2)}"'] & 2 \arrow[l, shift left, "A_{24}^{(1)}"] \arrow[l, shift right, "A_{24}^{(2)}"'] \end{tikzcd}$$



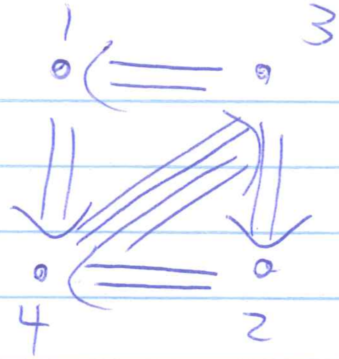


$$\boxed{\begin{aligned} W = {} & A_{13}^{(1)} A_{32}^{(2)} A_{24}^{(2)} A_{41}^{(1)} + A_{13}^{(2)} A_{32}^{(1)} A_{24}^{(1)} A_{41}^{(2)} \\ & - A_{13}^{(2)} A_{32}^{(2)} A_{24}^{(1)} A_{41}^{(1)} - A_{13}^{(1)} A_{32}^{(1)} A_{24}^{(2)} A_{41}^{(2)} \end{aligned}}$$



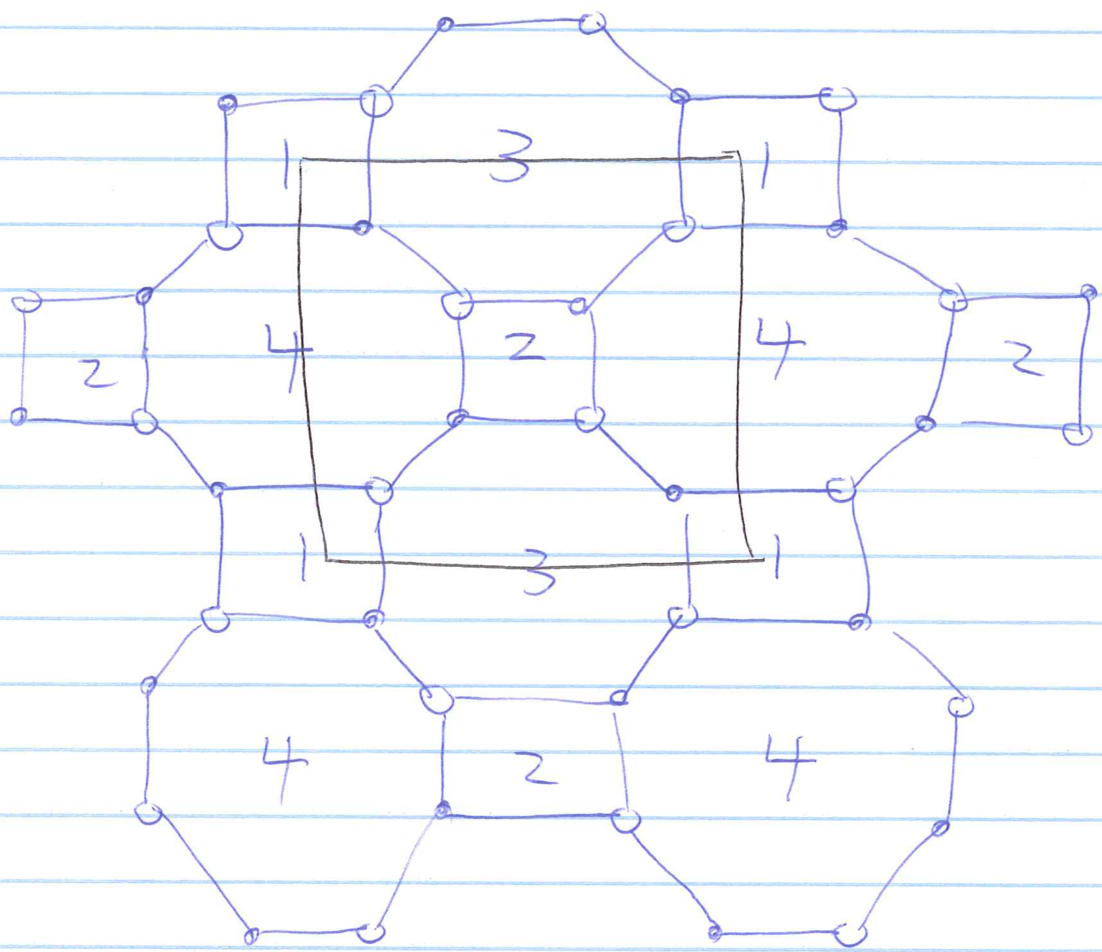
and

3/2/15 (4) $(Q_0, Q_1) =$



and we omit W/ Q_2

Corresponding bipartite tiling is (with rectangular fundamental domain)



Motivational Question: How are these two quiver + potentials (equiv. bipartite tilings) related?

We come back to this question later in the course.

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Let $A =$ superpotential algebra $\mathbb{C}Q/I_W$

<u>Def:</u> A <u>global symmetry</u> is a one-parameter subgroup

$\rho : \mathbb{C}^* \to \mathrm{Aut}(A)$ such that for each $t \in \mathbb{C}^*$,

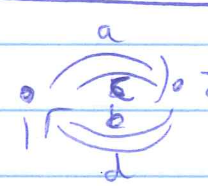
$\rho(t)$ defines the map $\rho(t) : X_a \mapsto t^{v_a} X_a$

for each $X_a \in A$ corresponding to $a \in Q_1$ and some $v \in \mathbb{Z}^{Q_1}$.

Note that ρ is a well-defined map to $\mathrm{Aut}(A)$ if it <u>acts homogeneously</u> on all terms in W.

Thus $d : \mathbb{Z}^{Q_1} \to \mathbb{Z}^{Q_2}$, the coboundary map satisfies $dv =$ constant function λ (for some $\lambda \in \mathbb{Z}$), called the <u>degree of ρ</u>.

<u>Example 1:</u> $Q =$



$, \quad W = abcd - adcb$

We can let ρ be defined as (of degree $\alpha + \beta + \gamma + \delta$)

$$\rho(t) : \begin{aligned} a &\mapsto t^{\alpha} a \\ b &\mapsto t^{\beta} b \\ c &\mapsto t^{\gamma} c \\ d &\mapsto t^{\delta} d \end{aligned}$$

[any assignment well-defined since both terms in potential W are the same, up to commutation.]

<u>Example 2:</u> (Somos-4 e.g.) Let $A_{ij} \mapsto t^{\alpha_{ij}} A_{ij}$

We would get relations $\alpha_{13}^{(2)} + \alpha_{32}^{(2)} + \alpha_{21} = \alpha_{13}^{(1)} + \alpha_{32}^{(2)} + \alpha_{24}^{(2)} + \alpha_{41} = \ldots$

So the space of global symmetries less than $\dim 10 = |Q_1|$.

3/2/15 6 Let $N = d^{-1}(\mathbb{Z}\cdot\vec{1}) \subset \mathbb{Z}^{Q_1}$

$$\begin{array}{c} \cap \\ \mathbb{Z}^{Q_2} \end{array}$$

and $N^+ := N \cap \mathbb{N}^{Q_1}$, i.e. allowing only nonneg integer values

N = one-parameter subgroup lattice of a complex torus $\mathbb{T} \leq \mathrm{Aut}(A)$ containing all global symmetries

Def: An R-symmetry is a global symmetry that acts with strictly positive weights (a.k.a. "charges") on all arrows.

I.e. R-symmetries = {"interior lattice points" of cone N^+}

Rem: Physics literature often defines R-charges to specifically have degree 2 but allows real weights rather than integer weights.

We will come back to global & R-symmetries momentarily.

Def: A perfect matching on a bipartite graph is a collection of distinguished edges such that each vertex is covered by exactly one edge.

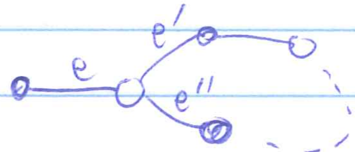
A dimer configuration is another name for perfect matching.

3/2/15 (7) Rem: A perfect matching can also be thought of as a 1-cochain $\pi \in \mathbb{Z}^{Q_1}$ with $\pi(e) \in \{0,1\}$ for all edges e and s.t. $d\pi = T$.

Thus we can think of perfect matchings as degree 1 elements of N^+.

Def: A dimer model is non-degenerate if every edge in the bipartite graph is contained in some perfect matching.

Example: If we allowed leaf edges, the dimer model would be degenerate (assuming it contains at least two edges)



Leaf edge e must be in every perfect matching $\Rightarrow$ e', e'' are not in any perfect matching.

Consequence: For any non-degenerate dimer model, the sum

$$\sum_{\substack{\text{perfect} \\ \text{Matchings} \\ M}} M$$

thought of as a sum of 1-cochains, i.e. as an element of N^+

is strictly positive on each arrow.

$\Rightarrow$ $\sum M$ defines an R-symmetry.

3/2/15 (8) **Lemma**: The cone N^+ is generated (over the integers) by the perfect matchings of the dimer model. Furthermore, the perfect matchings correspond to external elements of N^+.

PF: (Adaptation of Proof of Birkoff-von Neumann Theorem)

Let $P =$ cone generated by all 1-cochains corresponding to perfect matchings.

As argued above, $P \subset N^+$. ($\subset \mathbb{Z}^{Q_1}$)

On the other hand, let $v \in N^+$ with $\deg v > 0$. Pick black or white without loss of generality and let A be a subset of vertices of that color. Let $N(A) =$ neighbors of A, which have to be of the other color by bipartiteness.

Using duality A & $N(A)$ correspond to ~~cycles in~~ subsets of Q_2

Let $\langle \; , \; \rangle$ be the inner product between cochains $\mathbb{Z}^{Q_2}$ and chains $\mathbb{Z}_{Q_2}$

$$(\deg v) \cdot |A| = \sum_{\text{face } f \in A \subseteq Q_2} \langle d(v), f \rangle$$

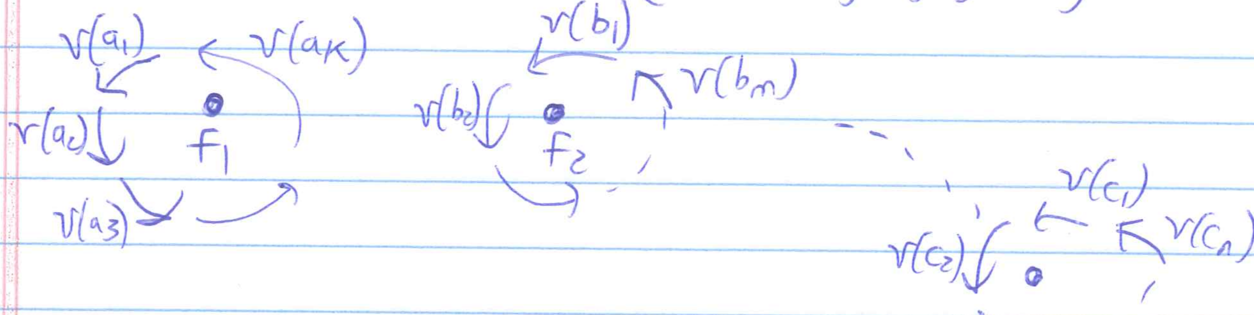
since $d(v)$ is constant $(\deg v)$ on each face of Q_2

$$= \sum_{\substack{f \in A \\ g \in N(A)}} \langle v, \partial f \cap \partial g \rangle$$

[see next page]

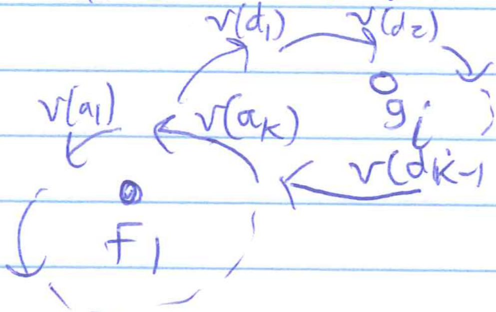
3/2/15 (9) $\sum_F \langle d(v), f)$ = sum of all these values

$(A \Leftrightarrow \{f_1, f_2, \dots, f_k\})$

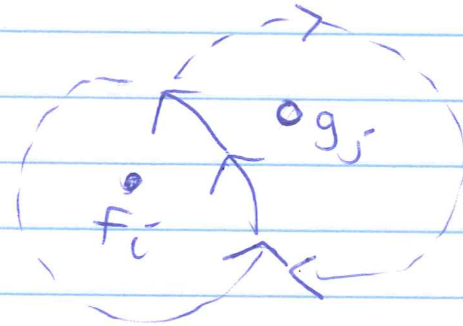


None of these cycles can share an edge, but can share and will with cycles around $g \in N(A)$

e.g.



or even



In fact every $v(a)$ for $a \in Q$, ~~adjacent~~ incident to a f_i will also be incident to a g_j, so we can look instead at

$$\sum_{\substack{f \in A \\ g \in N(A)}} \langle v, \partial f \cap \partial g\rangle \quad \text{as desired.}$$

$[\partial f \cap \partial g$ = class in $\mathbb{Z}_Q$ $\leftrightarrow$ sum of arrows in these boundaries$]$

Since $A \subseteq N(N(A))$, we also see

$$(\deg v) \cdot |N(A)| = \sum_{\substack{f \in N(A) \\ g \in N(N(A))}} \langle v, \partial f \cap \partial g\rangle \geq \sum_{\substack{f \in N(A) \\ g \in A}} \langle v, \partial f \cap \partial g\rangle = (\deg v)|A|.$$

We conclude $\underline{|N(A)| \geq |A|}$ for any subset A.

3/2/15 10 By Hall's Marriage Theorem, we conclude that this dimer model in fact has a perfect matching.

Note: We have shown a converse to the above: the existence of an R-symmetry $\Rightarrow$ there exists a perfect matching

Let π be this perfect matching (which is non-canonically determined by v)

By construction, our argument that $|N(A)| \geq |A|$ only used the edges a^* of the dimer model where $v(a) \neq 0$ thus we can assume that π includes only edges where $v(a) \geq 1$ $\Rightarrow$ $v - \pi$ is again in N^+

Thus we proceed inductively $\left(\begin{array}{c}\text{since } v-\pi \text{ is of degree} \\ \deg v - 1\end{array}\right)$

$\vec{0}$ is the only degree 0 element of N^+

$\Rightarrow$ $\boxed{v \text{ is a sum of } (\deg v) \text{ perfect matchings}}$

We are left to prove that the perfect matchings correspond to external elements.

Let $\pi_1, \ldots, \pi_n$ be elements of N^+ corresponding to perfect matchings. (Assume they are distinct.)

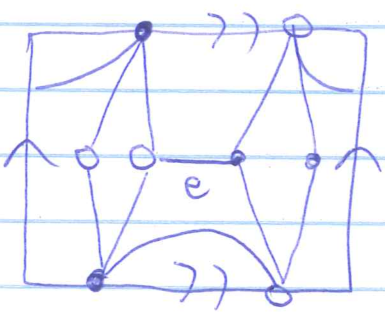
3/2/15 (11) Let $\sum_{s=1}^{n} k_s \Pi_s$ be a non-trivial convex combination (i.e. $k_s > 0$ for all s and $\sum_{s=1}^{n} k_s = 1$)

We need to show that $\sum_{s=1}^{n} k_s \Pi_s$ cannot be a perfect matching.

But each Π_s is a $\{0,1\}$-function, so $\sum_{s=1}^{n} k_s \Pi_s(\overset{\text{arrow } a}{a}) = 0$ or 1
only if $\Pi_1(a) = \Pi_2(a) = \dots = \Pi_n(a)$ $\forall$ arrows a
$\Rightarrow$ the Π_s's not distinct as needed.

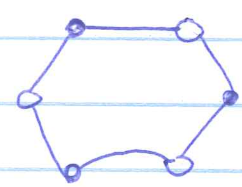
$\blacksquare$

Another example of a degenerate dimer model

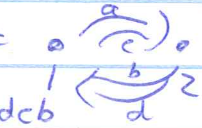


All perfect matchings contain edge e

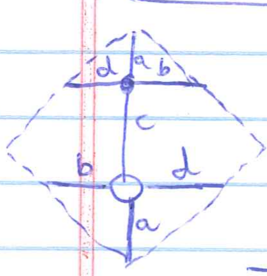
Taking this edge away, we see this bipartite graph is isomorphic to



Examples of N^+'s

e.g. 1) $(Q_0, Q_1) =$ 

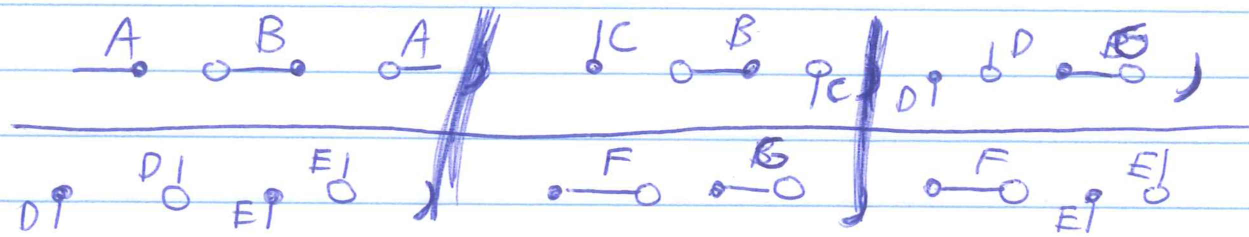
$W = abcd - adcb$



Four perfect matchings use edge a, b, c, or d

$\Rightarrow N^+ \cong \mathbb{Z}^4$, generated by $[1000]$, $[0100]$, $[0010]$, $[0001]$ as desired.

3/2/15 (12) <u>SPP e.g.</u> perfect matchings are



so in this case, N^+ has external generators

$[1100000]$, $[0110000]$, $[0001001]$,

$[0001100]$, $[0000011]$, $[0000110]$

Note that each of these is a global symmetry

e.g. $[1100000] \leftrightarrow$ Act on $A = \mathbb{C}Q/I_W$ by

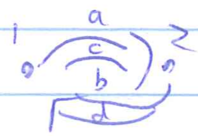
$c_{31} \mapsto t c_{31}$, $d_{22} \mapsto t d_{22}$, all other arrows fixed.

Notice that indeed all four terms of W contain c_{31} or d_{22} but not both

$$I_W = \begin{pmatrix} a_{21}c_{13}c_{31} - d_{22}a_{21},\ c_{13}c_{31}a_{12} - a_{12}d_{22}, \\ b_{32}d_{22} - c_{31}c_{13}b_{32},\ d_{22}b_{23} - b_{23}c_{31}c_{13}, \\ c_{31}a_{12}a_{21} - b_{32}b_{23}c_{31},\ a_{12}a_{21}c_{13} - c_{13}b_{32}b_{23}, \\ b_{23}b_{32} - a_{21}a_{12} \end{pmatrix}$$

Indeed each of these relations preserved by this action.

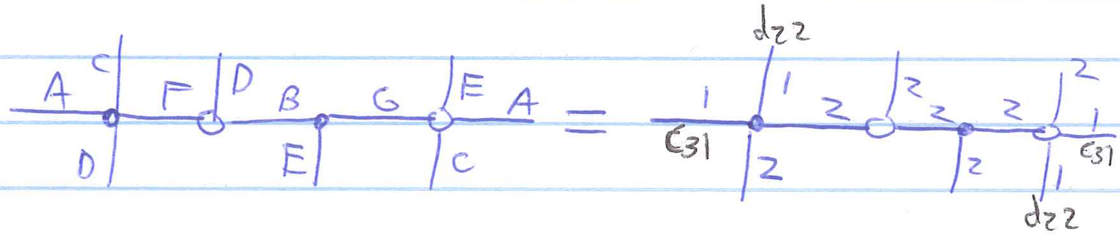
3/2/15 13 We get an R-symmetry by adding up values on all perfect matchings.

E.g. 

w/ $W = abcd - adcb$

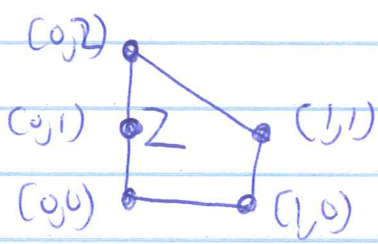
$[1,1,1,1]$ is the R-symmetry obtained

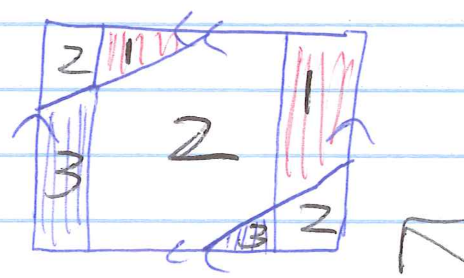
SPP: $[1\,2\,1\,2\,2\,2\,2]$ is the R-symmetry



Notice: sum incident to each vertex is constant, namely 6 in this case.

Toric Diagram [We will come back to its meaning later]





We will see later how R-symmetry allows us to redraw bipartite tiling as