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Lecture 13: Kasteleyn Theory

In summary, in an effort to understand global symmetries (which induce automorphisms of the superpotential algebra A), we study the cone N^+ corresponding to global symmetries with nonnegative weights.

Cone N^+ is generated by perfect matchings of the dimer model corresponding to A.

Today, we begin discuss techniques from algebraic combinatorics to enumerate perfect matchings of a dimer model (with weights).

We follow Sec 3 of Rick Kenyon's "Lectures on Dimers" arXiv: 0910.3129

Rem: Kenyon discuss dimer enumeration in the context of probability theory, but is essentially combinatorics. We give the probabilistic language here accordingly.

Computing the number of dimer coverings of a bipartite planar graph using the Kasteleyn-Temperley-Fisher technique:

Def: Let $G=(V,E)$ be a bipartite graph with a positive real function $w: E \to \mathbb{R}_{>0}$ on edges.

3/4/15 (2) We define a probability measure, called the Boltzmann measure, on dimer covers by

$\mu(G, w)$ has value on a given dimer cover, i.e. perfect matching M, of G

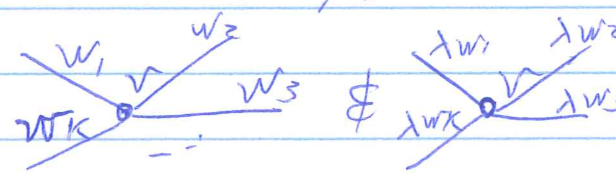
$$\boxed{\mu(M) = \frac{\prod_{e \in M} w(e)}{\sum_M \prod_{e \in M} w(e)}}$$

Note: $\prod_{e \in M} w(e)$ is also sometimes referred to as a Boltzmann weight of M.

Furthermore, the denominator is often abbreviated as Z, and called the partition function.

Gauge equivalence: Different weight functions w can easily give rise to the same measure μ.

For e.g., multiplying all $w(e)$'s, for edges e incident to a fixed vertex v, by a constant λ scales both the numerator and denominator by λ and leaves μ unchanged.



Def: If w and w' are related to one another by a sequence of scalings at vertices, as above, then we say weight functions w and w' are gauge-equivalent.

Prop: w and w' are gauge equivalent $\Leftrightarrow$ for every face $\{e_1, e_2, \ldots, e_{2k}\}$ in cyclic order, the alternating products of Boltzmann weights, i.e.

3/4/15 (3) $\dfrac{w(e_1)w(e_3)\cdots w(e_{2k-1})}{w(e_2)w(e_4)\cdots w(e_{2k})}$ and $\dfrac{w'(e_1)w'(e_3)\cdots w'(e_{2k-1})}{w'(e_2)w'(e_4)\cdots w'(e_{2k})}$

are equal. (In the case of a planar graph G.)

Pf: We use techniques from graph homology and cohomology. Recall we already have seen coboundary map $d: \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_1}$ defined by $g \mapsto dg$ where

$$dg(v_1 \to v_2) := g(v_2) - g(v_1)$$

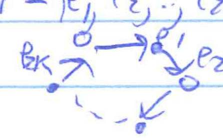
Today, we consider $d: \mathbb{R}^V \to \mathbb{R}^E$ for graph $G=(V,E)$ where we orient all edges of E as $\circ \to \bullet$.
we then define $f(\circ \xleftarrow{e} \bullet)$ as $-f(\circ \xrightarrow{e} \bullet)$ for $f \in \mathbb{R}^E$.

Warning: We are not considering G to be dual to Quiver (Q_0, Q_1) today, but simply borrowing same terminology to analyze G.

Def'n: We call $f \in \mathbb{R}^E$ satisfying $f(-e) = -f(e)$ (reverse orientation) a 1-form.

Claim: For a planar graph $G=(V,E)$, the set of 1-forms

F satisfying $df = 0$, [here, $d: \mathbb{Z}^E \to \mathbb{Z}^F$ by $f \mapsto df$ & $df(P) := f(e_1) - f(e_2) + f(e_3) - + \dots - f(e_{2k})$, a face $P = (e_1, e_2, \dots, e_{2k})$ in cyclic order]



called cocycles, is the same as the set of 1-forms satisfying $f \in \mathrm{Im}(d: \mathbb{R}^V \to \mathbb{R}^E)$, called coboundaries.

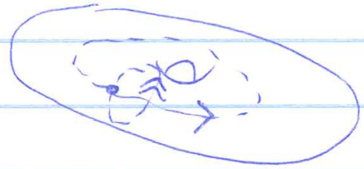
3/4/15 (4) Pf of Claim: This follows from standard holomogy theory

cocycles $= \operatorname{Ker} d: \mathbb{R}^E \to \mathbb{R}^F$
coboundaries $= \operatorname{Im} d: \mathbb{R}^V \to \mathbb{R}^E$

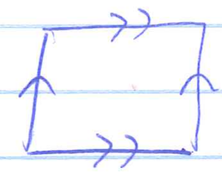
$H^1(G) := \operatorname{Ker}/\operatorname{Im}$ = cocycles/coboundaries
and when G is planar, after letting F's fill in G as 2-cells, $G \simeq$ disc or sphere S^2 (if we infinite face)

Either way, $H^1 = 0$.

Rem: If G is not planar but is embedded on a genus g surface instead, then $H^1(G) \cong \mathbb{R}^{2g}$
For example, for a bipartite tiling on a torus, we get $\operatorname{Ker} d / \operatorname{Im} d \cong \mathbb{R}^2$



i.e.



We come back to this case later.

With this claim in hand, and a choice of weight function $w: E \to \mathbb{R}$, consider the function $\underline{\log w}$ as a 1-form, i.e.

$\log w(\circ \xleftarrow{e} \bullet) := -\log(\circ \xrightarrow{e} \bullet)$.

By defn of gauge-equivalence, $w \sim w'$ if $\exists\, f \in \mathbb{R}^V$ s.t. $w'(e) = w(e) \cdot f(v_1) f(v_2)$ for edge $e = v_1 - v_2$.

By tweaking function f, we can also incorporate orientations $\circ \to \bullet$ and say $w'(e) = w(e) \cdot \frac{f(v_2)}{f(v_1)}$ for some $f \in \mathbb{R}^V$.

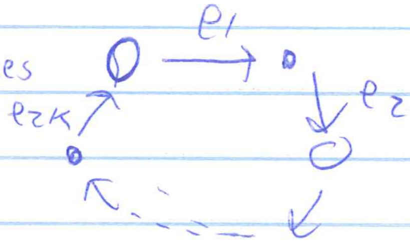
3/4/15 ⑤ Equivalently, $\log w' - \log w = df$ for some $f \in \mathbb{R}^V$.

Thus we need $\log w' - \log w$ is a coboundary
$\Leftrightarrow$ (in the planar graph case) that

$\log w' - \log w$ is a cocycle

$\Leftrightarrow d(\log w' - \log w) = 0 \Leftrightarrow d \log w' = d \log w$

$\Leftrightarrow \forall$ faces



$$\log w(e_1) - \log w(e_2) + - \cdots + \log w(e_{2k-1}) - \log w(e_{2k})$$
$$= \log w'(e_1) - \log w'(e_2) + - \cdots + \log w'(e_{2k-1}) - \log w'(e_{2k})$$

$\Leftrightarrow$ alternating products in terms of w and w' agree. ▨

<u>Rem:</u> If G is not planar, then w & w' are gauge-equivalent $\Leftrightarrow$

- the alternating product on all faces agree

& • the alternating product along any homology-cycle of the Riemann surface Y agree

<u>Moral:</u> We later study transformations on weights that preserve the alternating products around faces/cycles.

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Kasteleyn weighting

Def: A Kasteleyn weighting of a planar bipartite graph is a choice of sign for each edge (thought of as an undirected edge) such that

Each face F contains an $\begin{cases} \text{odd \# } -\text{'s \& odd \# } +\text{'s} & \text{if } F \text{ is a } 4k\text{-gon} \\ \text{even \# } -\text{'s \& even \# } +\text{'s} & \text{if } F \text{ is a } (4k+2)\text{-gon} \end{cases}$

(could also think of this Kasteleyn weighting as Boltzmann weights adorned w/ these choices of signs)

Aside:

One can also extend the notion of Kasteleyn weightings to be a choice of complex numbers z with $|z|=1$ for each edge, with the condition that

For each face F, the alternating product of weights is in $\begin{cases} \mathbb{R}_{<0} & \text{if } F \text{ is a } 4k\text{-gon} \\ \mathbb{R}_{>0} & \text{if } F \text{ is a } (4k+2)\text{-gon} \end{cases}$

Since the alternating products of Kasteleyn weights around faces only depends on the size of faces, it follows that for planar graphs, all Kasteleyn weightings are gauge-equivalent.

Consequently, any two Kasteleyn weightings differ by a sequence of transformations given by multiplying all edges incident to a vertex by the same constant, i.e. (4).

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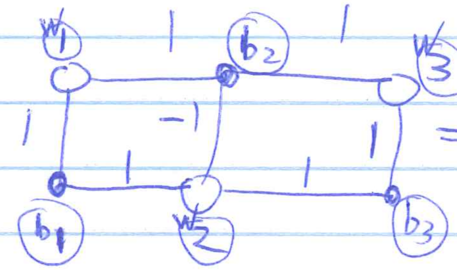
Given a <u>Kasteleyn weighting</u> w on a planar graph G, we define the Kasteleyn matrix $K(G)$, relative to this weighting, w to be the $|B| \times |W|$ matrix

$$\left[K_{bw}\right]_{\substack{b\in B\\ w\in W}} \quad \text{with} \quad \underset{\substack{\| \\ K(b,w)}}{K_{bw}} := \begin{cases} 0 & \text{if no edge } b - w \\ w(bw) & \text{if edge } b - w \text{ in } E \end{cases}$$

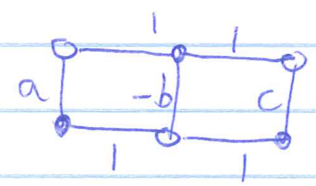
(Here $G=(V,E)$ with $V = B \sqcup W = \{\text{black vertices}\} \sqcup \{\text{white vertices}\}$.)

<u>Example:</u>



$$\Rightarrow K = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

We now add extra Boltzmann weights to describe K more clearly



$$\Rightarrow K = \begin{bmatrix} a & 1 & 0 \\ 1 & -b & 1 \\ 0 & 1 & c \end{bmatrix}$$

<u>Claim:</u> Gauge transformations (i.e. equivalences) correspond to multiplying on the left & right by a diagonal matrix (depending on whether you are scaling at a black vertex or a white vertex)

<u>Thm</u> $Z = |\det K|$ (if $|B| = |W|$).

<u>Example continued:</u> $\det K = -a - c - abc$

$\Rightarrow Z = a + c + abc$ ✓

For Boltzmann weights