3/9/15

Lecture 14: Kasteleyn Theory II: From Dimer Models to Toric Varieties

Recall from last week:

Given a planar graph G with Boltzmann ($\in \mathbb{R}_{>0}$) weights on edges, we adorn this weighting with signs to obtain a Kasteleyn weighting.

Kasteleyn Matrix K is $|B| \times |W|$ matrix with K_{bw} = Kasteleyn weighting on edge(s) between $\bullet$ and $\circ$
b w

Partition function $Z = \sum_{M \text{ a perfect matching of } G} \prod_{e \in M} w(e)$

$w(e)$: Boltzmann weights without signs

Thm: If $|B| = |W|$, then

$$Z = |\det K|.$$

Rem: If G embeds into genus $g \geq 1$ Riemann surface Y (i.e. G not planar), we still have

terms in Z, i.e. perfect matchings $\longleftrightarrow$ terms in $\det K$
(but signs on RHS might not all be the same)

3/9/15 (2) Proof: First note that if $|B| \neq |W|$, there are no perfect matchings and $Z = 0$.

We thus assume $|B| = |W|$ and K is square below.

Let $\tilde{G}$ be the complete bipartite graph on ~~[illegible]~~ $n = |B|$ black vertices and $n = |W|$ white vertices (with weights $w(b_i, w_j) \in \mathbb{R}_{\geq 0}$ on all edges)

We let $w(b_i, w_j) = 0$ in $\tilde{G}$ $\iff$ G has no edge (b_i, w_j).

With this convention $Z_{\tilde{G}} = Z_G$.

Note that $\det K = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) K(b_1, w_{\sigma(1)}) \cdots K(b_n, w_{\sigma(n)})$

and since σ is a permutation, it follows that σ corresponds to a perfect matching of $\tilde{G}$

However $K(b_i, w_j) = \pm w(b_i, w_j) \neq 0 \iff G$ has edge (b_i, w_j)

and so each nonzero term in the expansion of $\det K$ corresponds to a perfect matching of G.

Rem: This portion of proof did not use planarity.

It suffices to verify that the signs of all non-zero terms are the same.

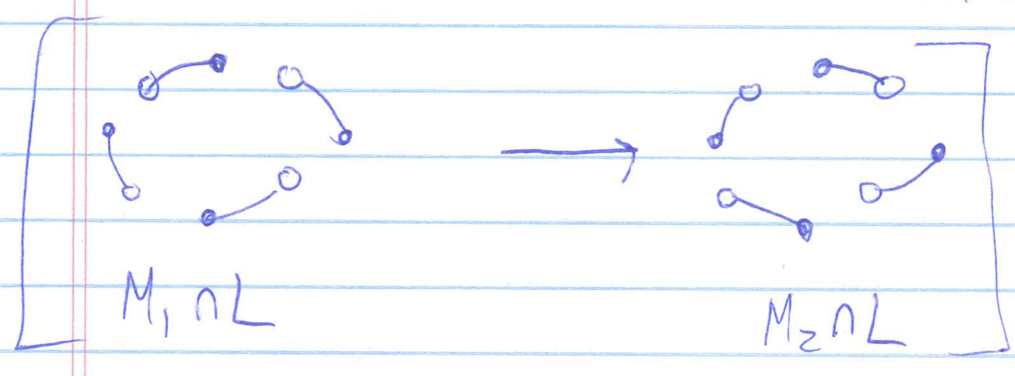
Draw two perfect matchings M_1 and M_2 as a superposition on G.

3/4/15 ③

Since $M_1 \cup M_2$ is degree at every vertex,
then $M_1 \cup M_2$ can be decomposed into connected components of loops including 2-loops, a.k.a. doubled edges.

Note that we can transform M_1 into M_2 by the following operation:

- For every loop L of size ≥ 4 (note that loops are of even size by bipartiteness)



$M_1 \cap L$ is one of two possible perfect matchings of a $(2k)$-cycle.

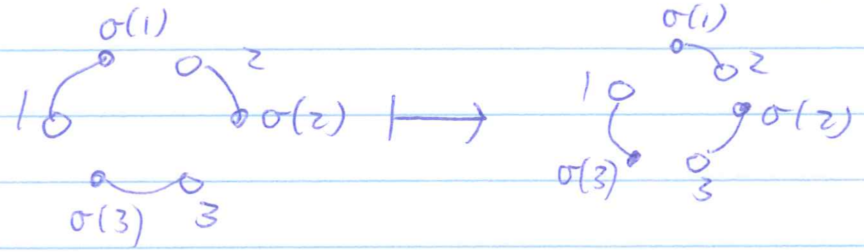
We define M_2 so that $M_2 \cap L$ is the other perf. matching.

Note that L can be a general cycle in graph G that encircles numerous faces (in the planar case) or even is homologous to a fundamental cycle of genus g Riemann surface Y (in general)

<u>Lemma</u>: Given a Kasteleyn weighting and a $(2k)$-cycle L enclosing ℓ vertices (in the planar case) then the alternating product of Kasteleyn weights
~~(where product of signs ± in the real case)~~
is $\underline{(-1)^{k+\ell+1}}$.

3/9/15 (✝) Each local change on a loop L of size $2k$ also changes σ by multiplying by a k-cycle

e.g.



so $\sigma'(1)=\sigma(3)$
$\sigma'(2)=\sigma(1)$
$\sigma'(3)=\sigma(2)$

Note also that l, the # interior vertices enclosed by L, must be even since enclosed vertices come from other loops (which are $2m$-gons) or doubled edges.

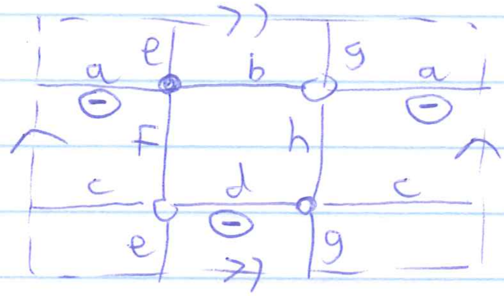
$\Rightarrow$ each local change by L alters alternating product of Kasteleyn weights by $(-1)^{k+1}$.

At the same time, sgn σ changes by $(-1)^{k+1}$ = sgn of a k-cycle.

Thus all perfect matchings of a planar graph yield the same sign for its product of Kasteleyn weights. ▨

We now consider all this theorem can be adapted to the case where G is on a Riemann surface Y of genus g (as opposed to planar).

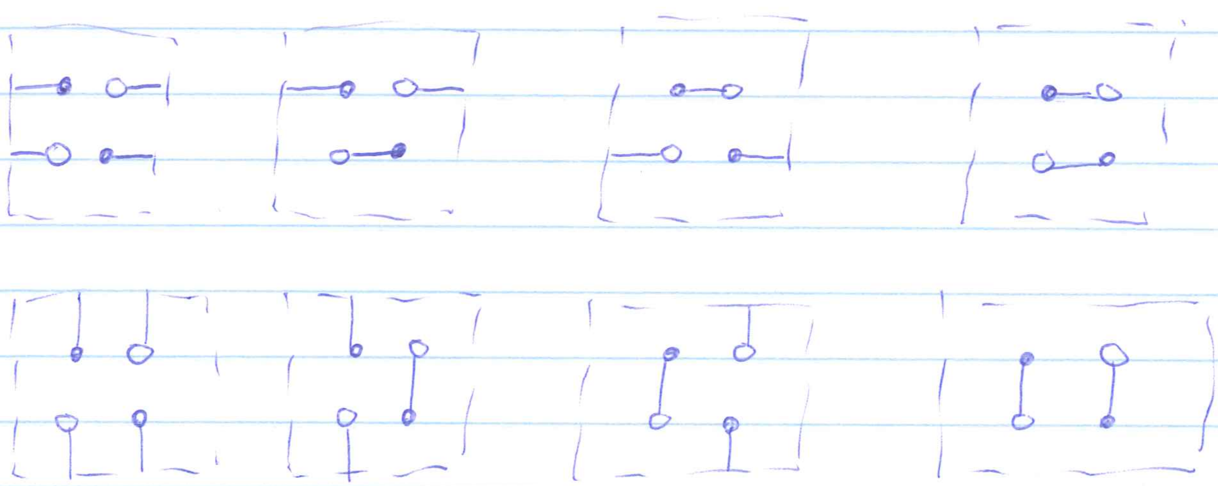
3/9/15 ⑤ Let us focus on $g=1$ case

E.g.  has

$$K = \begin{bmatrix} -a+b & e+f \\ g+h & c-d \end{bmatrix}$$

with $\det K = -ac+ad+bc-bd$
$-eg-eh-fg-fh$

In fact, there are <u>eight perfect matchings</u>

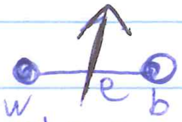


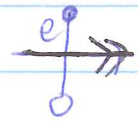
Since $g=1$, there are 2 fundamental cycles to torus; we wish to consider these cycles as data as we build K or Z.

3/9/15 (6)

Example continued: Let z_1 represent ↑ in torus

z_2 represent ⟶⟶ in torus

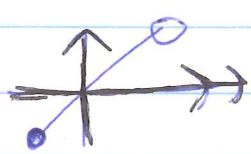
If we have , then in K_{bw} the contribution of edge e is scaled by z_1^{+1}.

Similarly,  leads to scaling contribution e by z_2^{+1}.

On the other hand,  (or equiv. ) scales contribution instead by z_1^{-1}

and  scales by z_2^{-1}

Lastly, we can have combinations of these contributions, e.g.

 would scale by $z_1^{+1} z_2^{-1}$.

With these rules, K becomes (for our e.g. cont.)

$$\begin{bmatrix} -az_1^{-1}+b & ez_2^{-1}+f \\ gz_2+h & cz_1 - d \end{bmatrix}$$

with det =

$-ac + adz_1^{-1} + bcz_1 - bd$
$-eg - ehz_2^{-1} - fgz_2 - fg$

Called the Characteristic polynomial

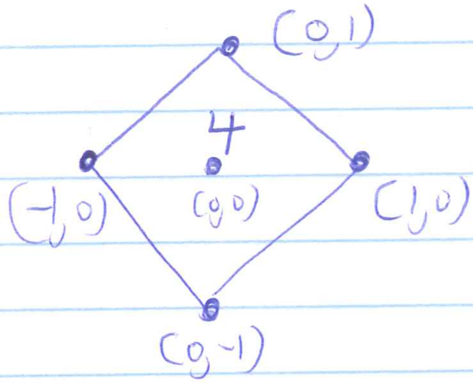
Rem: If we set a, b, c, d, e, f, g, h = 1, det K simplifies to

$\boxed{z_1^{-1} + z_1 - z_2^{-1} - z_2 - 4}$

Note that no terms cancelled.

3/9/15 ⑦ This Laurent polynomial in z_1 and z_2 determines a <u>Newton polygon Δ</u>, defined as convex hull of (i,j)'s $\in \mathbb{Z}^2$ appearing as exponents of <u>nonzero terms in $\det K$</u>.

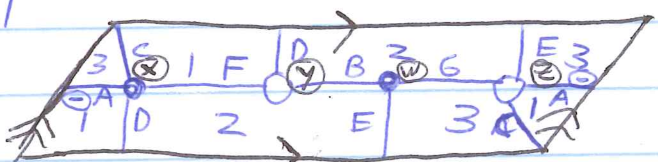
e.g.



We let $\|$coeff of $z_1^i z_2^j\|$ signify <u>multiplicity</u> of that point in Newton polygon Δ.

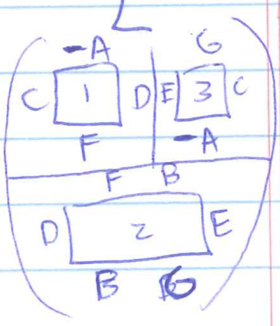
Note that vertices of Δ all have multiplicity 1.

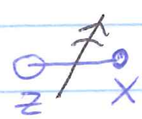
<u>Another example:</u> SPP

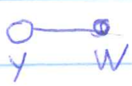


Let ⓧ, ⓨ, ⓦ, ⓩ label the four vertices.

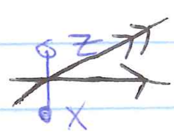
[With Kasteleyn signs ⊕ everywhere except ⊖ on A]



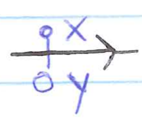
A 

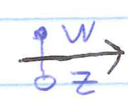
B 

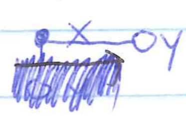
so $K(z_1, z_2) =$

C 

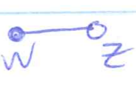
$$\begin{array}{c|cc} & y\circ & z\circ \\ \hline x\bullet & & \\ w\bullet & & \end{array}\quad \begin{bmatrix} Dz_1 + F & -Az_2^{-1} + Cz_1^{-1}z_2^{-1} \\ B & Ez_1 + G \end{bmatrix}$$

D 

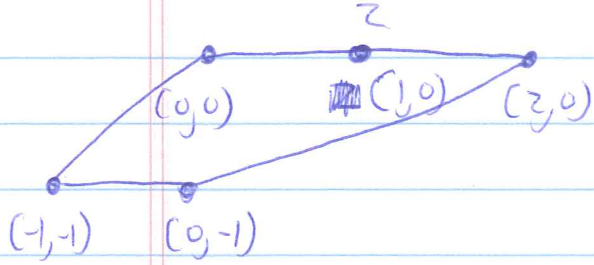
E 

F 

with $\det = DEz_1^2 + DGz_1 + EFz_1 + FG$
$+ ABz_2^{-1} - BCz_1^{-1}z_2^{-1}$

G 

3/9/15 (8) Newton polygon for SPP (with this choice of fund. domain)



Again, vertices of Δ have multiplicity 1.

We will see shortly that the Newton polygon corresponding to char poly $= \det K(z_1, z_2)$

is the toric diagram of a toric variety geometrically related to the superpotential algebra corresponding to $Q = (Q_0, Q_1)$ and W [terms $\leftrightarrow Q_2$].

Moreover, Newton polygon can be used to construct bipartite tiling directly.

Sec 2.2 of Goncharov-Kenyon

Given convex polygon N in $\mathbb{R}^2$ with vertices in $\mathbb{Z}^2$, let $\{e_1, e_2, \ldots, e_K\}$ = primitive integer vectors $\overrightarrow{(a,b)}$ [with $\gcd(a,b)=1$] corresponding to sides of N in counter-clockwise order.

By construction $\sum e_i = \vec{0}$.