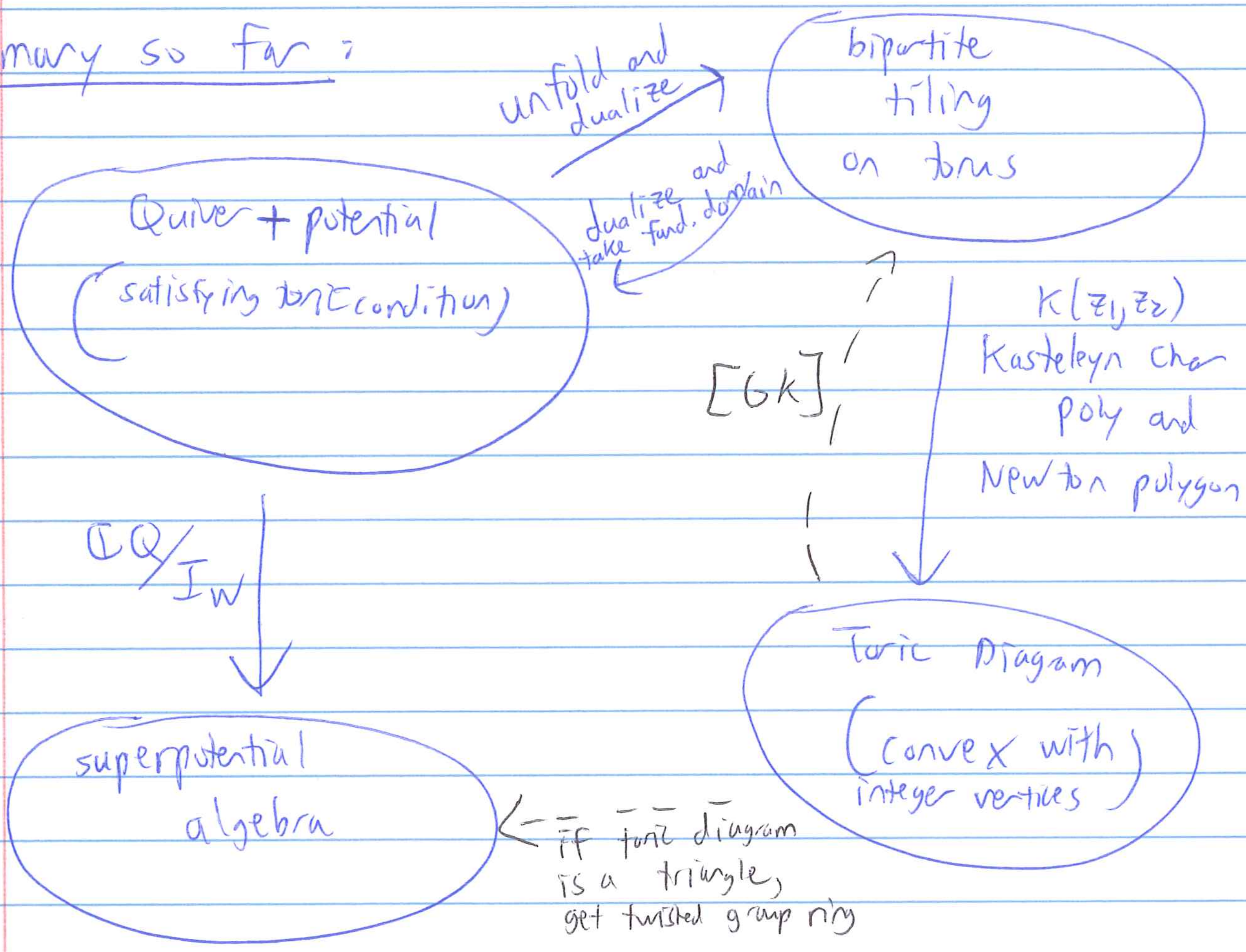
3/11/15 Lecture 15: From polygons to bipartite tilings I: basic examples and triangles

Summary so far:



Today: – Goncharov Kenyon construction
– orbifolds/Hexagonal Lattice/twisted group ring in special case when toric diagram is a triangle.

Sec 2.2 of Goncharov-Kenyon: Given convex polygon N in $\mathbb{R}^2$ with vertices in $\mathbb{Z}^2$, let $\{e_1, e_2, \ldots, e_k\}$ = primitive integer vectors ($\overrightarrow{(a,b)}$ with $\gcd(a,b)=1$) corresponding to sides of N in counter-clockwise order.

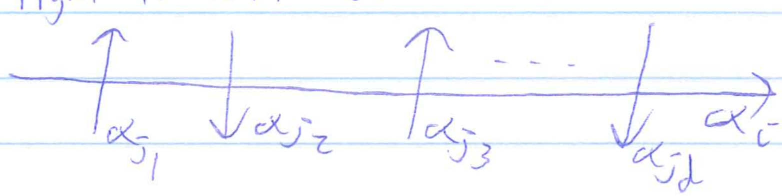
By construction, $\sum e_i = \vec{0}$.

3/11/15 (2) A Torus T can be represented as $\mathbb{R}^2/\mathbb{Z}^2$, so each $\vec{e_i}$ determines a homology class $[e_i] \in H_1(T, \mathbb{Z})$

In fact, there is a unique geodesic (line in $\mathbb{R}^2/\mathbb{Z}^2$) representing this class of direction given by $\vec{e_i}$.

Up to translations, place loops $(\alpha_1, \dots, \alpha_k)$ on T so oriented direction of α_i is $\vec{e_i}$ arranged so

- No triple intersections (i.e. generic config.)
- Total number of intersection points is minimal,
- Alternating strand condition: Following loop α_i we encounter loops $\alpha_{j_1}, \dots, \alpha_{j_d}$ in some order. We wish these to alternate crossing α_i from left-to-right to right-to-left or vice-versa

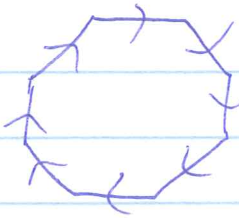
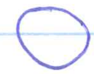


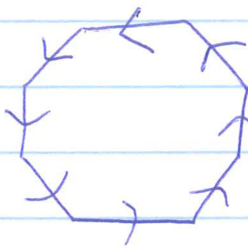
[want this condition for every α_i]

Yields an admissible surface graph on torus T whose 2-cells are polygons oriented in one of 3 ways

1) 2) 3)

clockwise   counter-clockwise   alternating

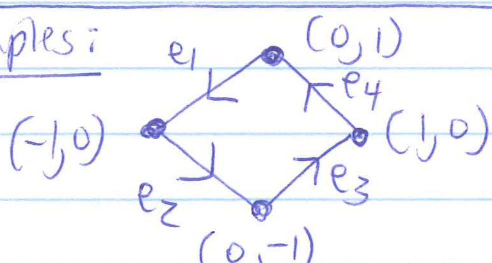
3/19/15 (B) By replacing  with white vertices ,

 with black vertices ,

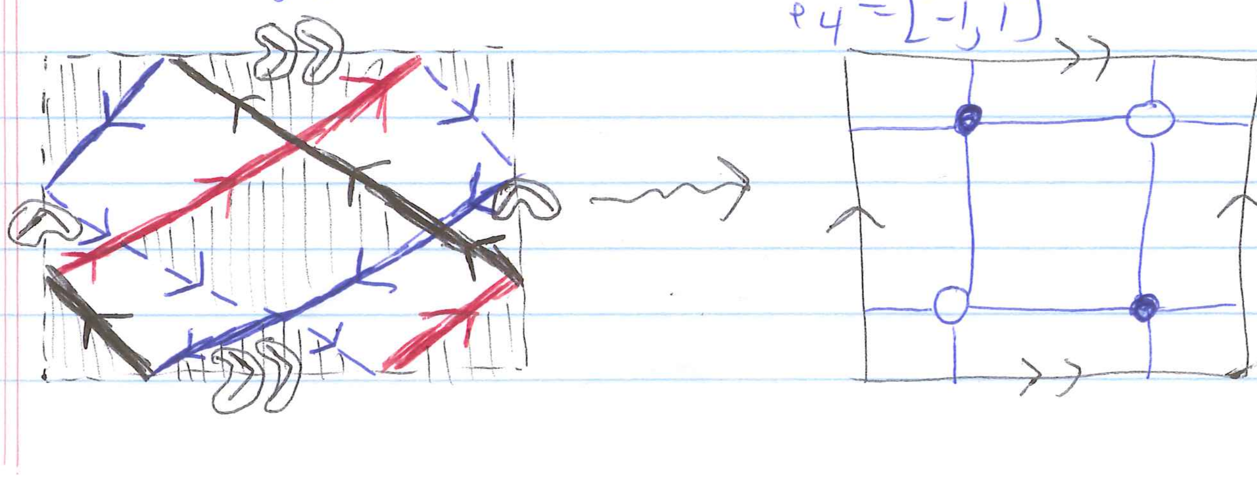
and contracting alternating regions,

we get a new bipartite graph
(using notions of <u>Triple Point Diagrams</u>
going back to unpublished work of Dylan Thurston)

<u>Examples:</u>

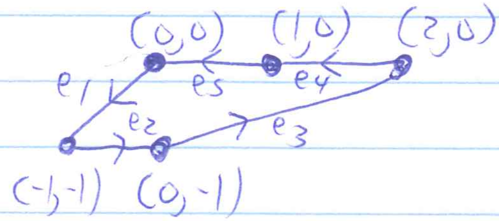


with $e_1 = [-1,-1]$
$e_2 = [1,-1]$
$e_3 = [1,1]$
$e_4 = [-1,1]$

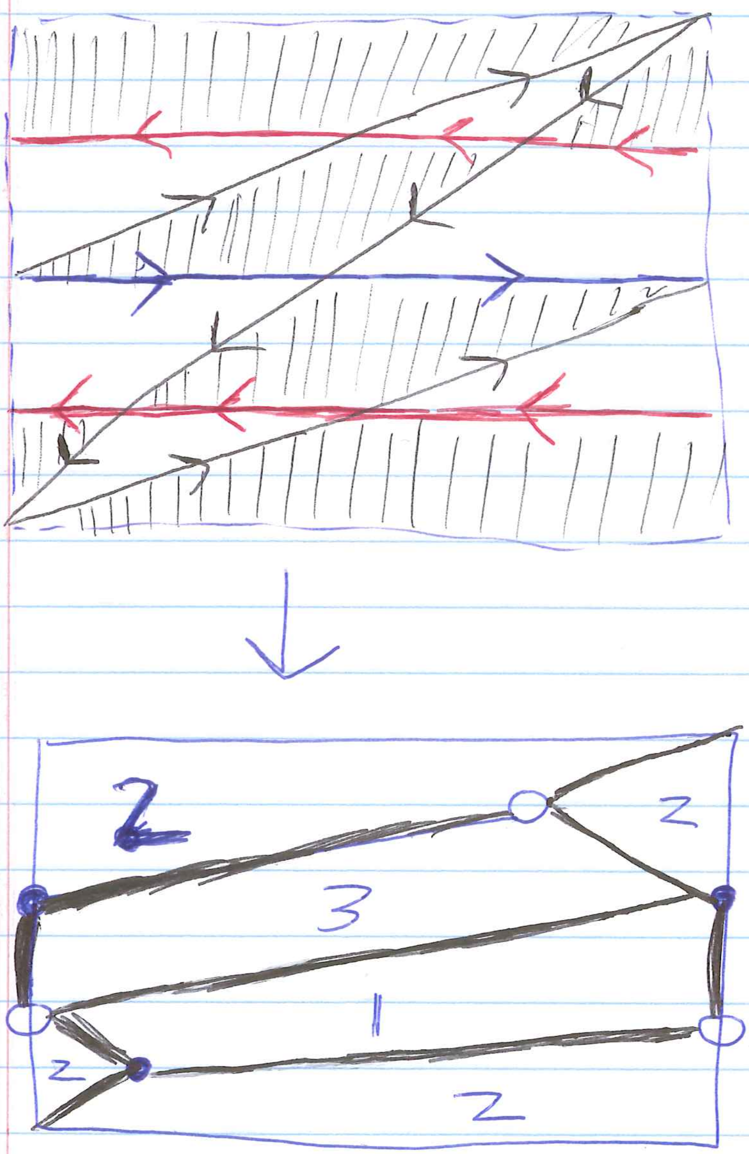


3/18/15 (4) Note that these clockwise and counter-clockwise regions are all quadrilaterals and correspond to 4-valent vertices.

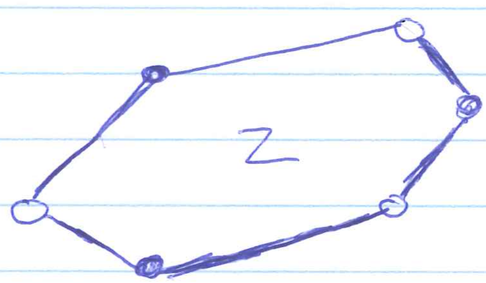
SPP example:



$e_1 = [-1, -1]$
$e_2 = [1, 0]$
$e_3 = [2, 1]$
$e_4 = [-1, 0]$
$e_5 = [-1, 0]$

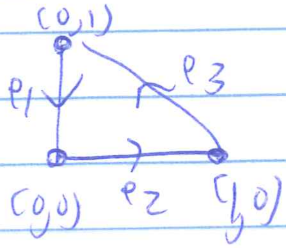


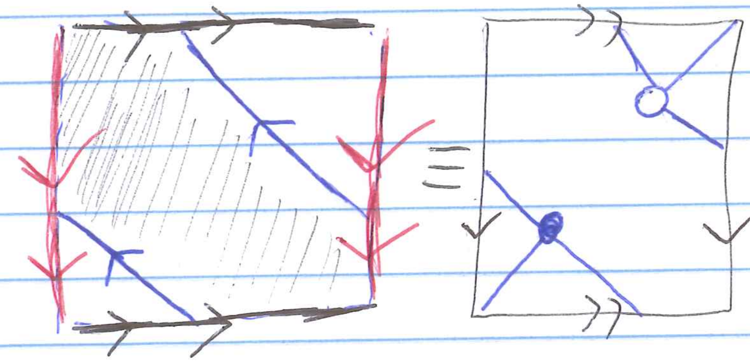
note that face 2 is a hexagon



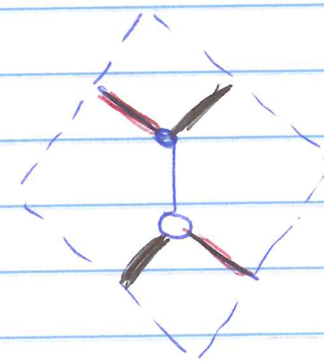
3/11/15 ⑤ We now focus on cases where Δ is a triangle

e.g. 1)

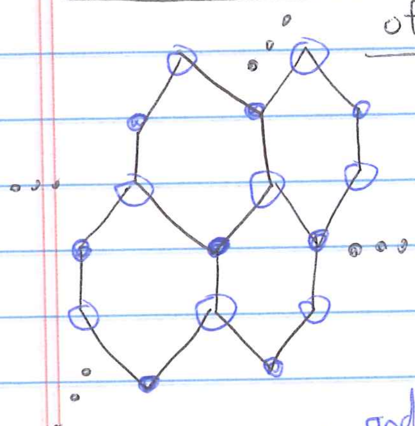




which shifts and straightens out to



Recovers the hexagon e.g. of a bipartite tiling



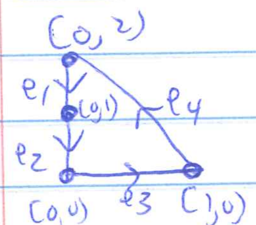
Recall that corresponding (Q, W) was

with $W = abc - acb$

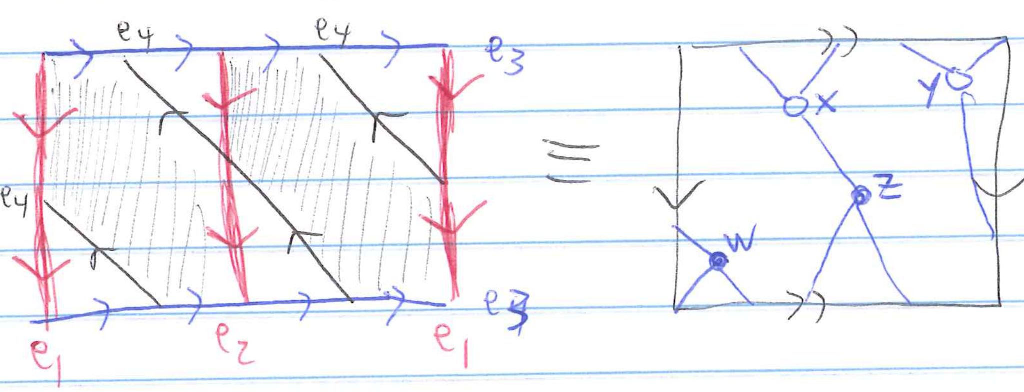
and superpotential algebra $A = \mathbb{C}\langle a,b,c\rangle / I_W$
with $I_W = (ab-ba, bc-cb, ac-ca)$

3 perfect matchings which yield $K(z_1, z_2) = \pm 1 \pm z_1 \pm z_2$ as well $\Rightarrow A \cong \boxed{\mathbb{C}[a,b,c]}$

e.g. 2)



$e_1 = e_2 = (0,-1)$, $e_3 = (1,0)$, $e_4 = (-1,2)$



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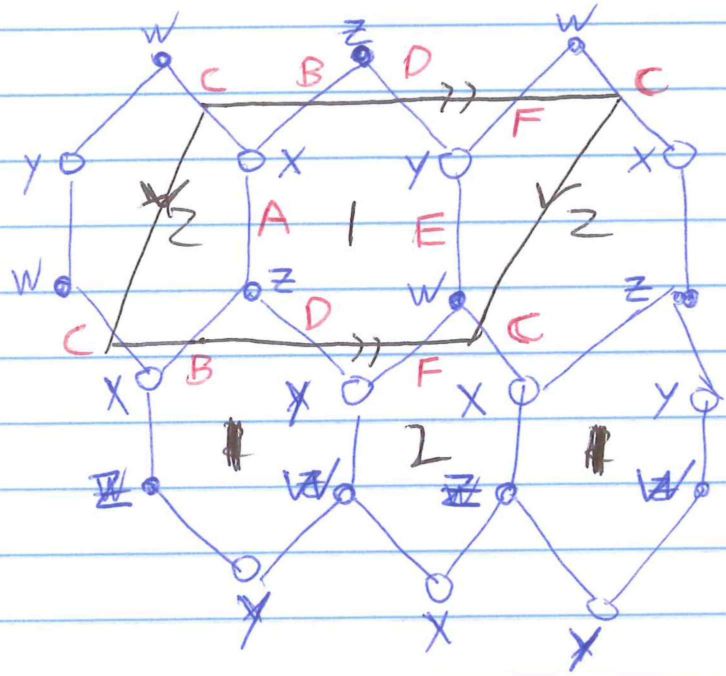
e.g. 2 continued (straightened out)



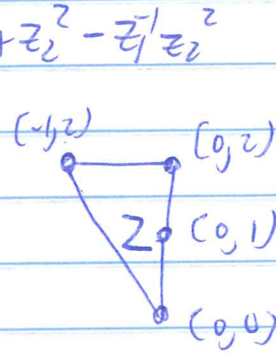
Point: bipartite tiling again involves only hexagons but fundamental domain is larger.

In particular, 2 faces rather than just 1

perfect matchings by Kasteleyn matrix
$$\begin{array}{c|cc} & z & w \\ \hline x & 1+z_2 & z_1^{-1}z_2 \\ y & z_2 & 1+z_2 \end{array}$$

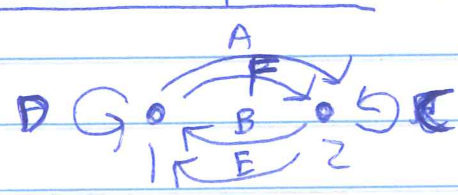
$\det K(z_1,z_2) = 1 + 2z_2 + z_2^2 - z_1^{-1}z_2^2$

with Newton polygon

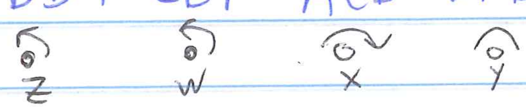


which agrees w/ $\triangle$ up to $SL_2(\mathbb{Z})$

quiver and potential



$W = ABD + CEF - ACB - DFE$



3/11/15 7 e.g. 2 cont superpotential algebra $A = \mathbb{C}Q/I_W$

Claim: $A \cong$ twisted group ring $\mathbb{C}[x,y,z] \rtimes \mathbb{Z}_2$.

Def: A twisted group ring $S \rtimes G$ (G acts on S) as an S-module defined with elements as pairs (s, g) $s \in S$, $g \in G$ s.t.

$$(s_1, g_1) \circ (s_2, g_2) := (s_1\, {}^{g_1}s_2,\ g_1 g_2)$$

where ${}^{g_1}s_2 :=$ result after g_1 left-acts on s_2.

In our e.g., let $\underset{\{1,\varepsilon\}}{\underset{\shortparallel}{\mathbb{Z}_2}}$ act on $\mathbb{C}[x,y,z]$ by

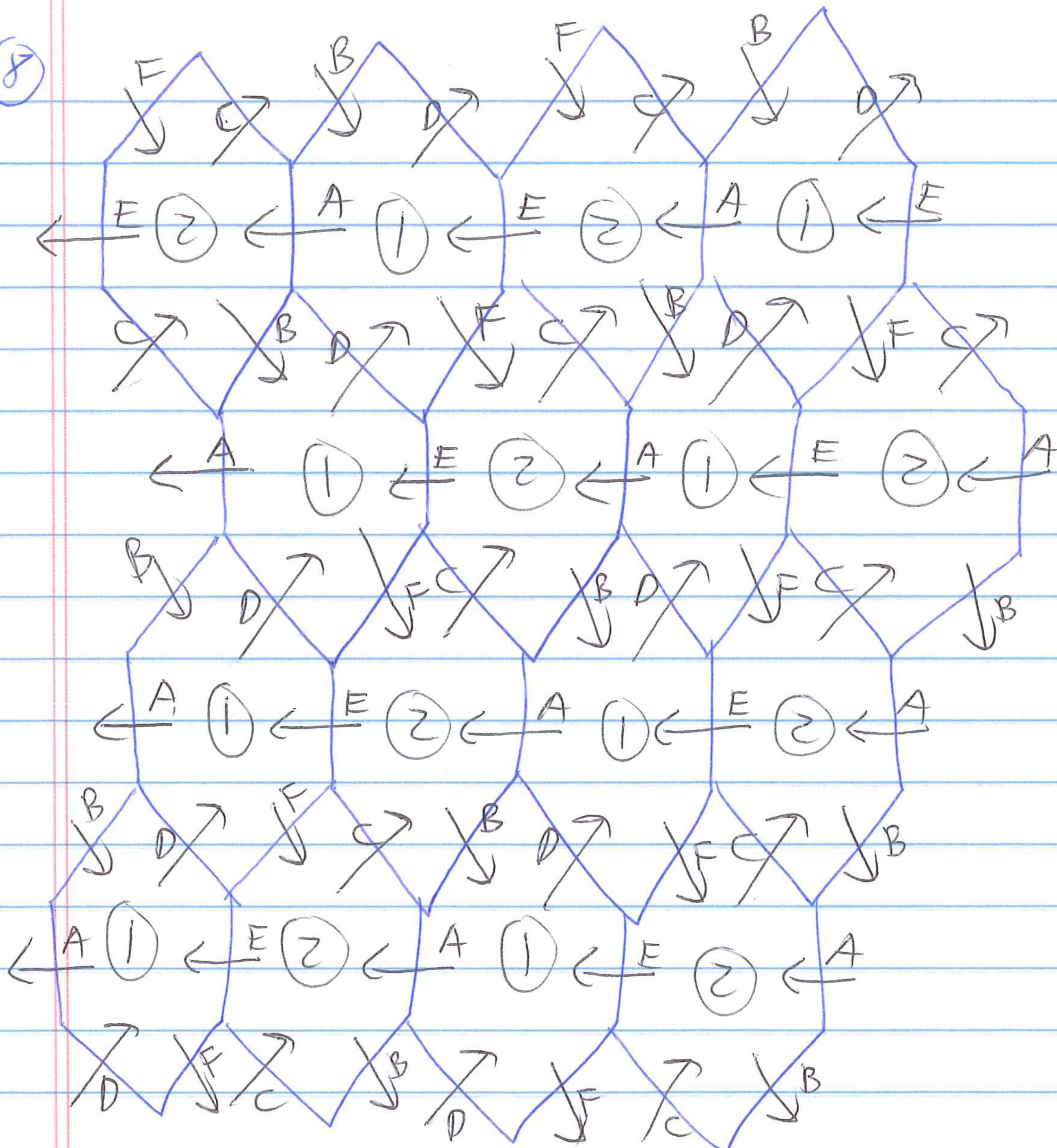
$\varepsilon x = -x$, $\varepsilon y = -y$, $\varepsilon z = z$.

i.e. ε represented by diagonal matrix $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ in $SL_3(\mathbb{C})$ of order 2.

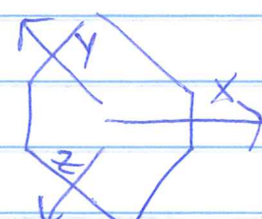
We explain full method for equating $\mathbb{C}Q/I_W$ to $\mathbb{C}[x,y,z] \rtimes G$ (G abelian $\leq SL_3(\mathbb{C})$) when $(Q, W) \longleftrightarrow$ triangular toric diagram later.

For now, draw unfolded quiver on hexagonal lattice

3/11/15 (8)



Three cardinal directions



Define in $\mathbb{C}Q$

$\tilde{X} \Leftrightarrow DF + CB \sim DF + BD \sim FC + BD \sim FC + CB$

on 1 on 2

$\tilde{Y} \Leftrightarrow DA + CE \sim \ldots$

on 1 on 2

$\tilde{Z} \Leftrightarrow AB + EF \sim \ldots$

on 1 on 2

[Equivalences in J_W]

3/11/15 (9)

Rem: multipl. by $\tilde{x}$ moves one unit in x-direction
and move face 1 to 2 and vice-versa

multipl. by $\tilde{y}$ " " y-direction
and move face 1 to 2 and vice-versa

BUT multipl. by $\tilde{z}$ " " z-direction
but face 1 $\circlearrowleft$, face 2 $\circlearrowleft$

corresponds to $\begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}$ diagonal action

Exercise: 1) Show $xy = yx$, $xz = zx$, $yz = zy$ in $\mathbb{C}Q/I_W$.

2) Letting $P(x,y,z) \in \mathbb{C}[x,y,z] \subset \mathbb{C}[x,y,z] \rtimes \mathbb{Z}_2$
by $\longmapsto (P(x,y,z), 1)$
come up with a way to define $(P(x,y,z), \varepsilon)$'s so that mult in Twisted Group Ring agrees with that in $\mathbb{C}Q/I_W$.

3) Use this to complete argument
$\mathbb{C}Q/I_W \cong \mathbb{C}[x,y,z] \rtimes \mathbb{Z}_2$
for this example of Q and W.