4/1/15 Lecture 17-18

We have the following schematic

Finite Abelian subgroup $G \subset SL_3(\mathbb{C})$

$\updownarrow$ McKay quiver $W = XZY - XYZ$

Quiver and potential unfolding to triangular lattice

$\downarrow$ superpotential algebra

twisted group ring $\mathbb{C}[x,y,z] \rtimes G$

Quiver and potential unfolding to triangular lattice $\xrightarrow{\text{unfold and dualize}}$ bipartite tiling on torus as fundamental domain of hexagonal lattice

bipartite tiling on torus as fundamental domain of hexagonal lattice $\downarrow$ Kasteleyn char poly and Newton polygon

Lattice Triangle $\uparrow$ Goncharov–Kenyon (Also known as "linear Hanany–Vegh")

Lattice Triangle

Two comments from last time:

Miles Reid notation $\frac{1}{3}(1,1,1)$ for $\mathbb{Z}_3(1,1,1)$

Geometric/Physics notation $\mathbb{C}^3/\mathbb{Z}_3$ & this is different than $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C} \sim \mathbb{Z}_3(2,1,0)$

Also $\mathbb{Z}_2(1,1,0)$ would be $\frac{1}{2}(1,1,0)$ & $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ in this notation

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2 We now focus on the two red arrows

1) Given a bipartite tiling as a fund. domain of hexagonal lattice, what is its associated toric diagram?

2) Applying Goncharov-Kenyon ([UY] refers to as linear Hanany-Vegh) to a lattice triangle, what possible bipartite tilings do we get?

To answer (1), we consider the more general question:
If $P(z,w) = \det K(z,w)$ for the Kasteleyn matrix corresponding to a bipartite tiling G, and we let $G_{n,m}$ be the dilation by taking a fundamental domain m times wider & m times higher

what is the new $P_{n,m}(z,w)$?

[See Theorem 3.3 of "Dimers and Amoebae" by Kenyon-Okounkov-Sheffield arXiv:0311005]

Thm $P_{n,m}(z,w) = \prod_{i=1}^{n} \prod_{j=1}^{m} P(\lambda^i z^{1/m}, \omega^j w^{1/m})$

where λ is a primitive nth root of unity and ω is a " mth " ".

Rem: By symmetries, the RHS is indeed a polynomial in z, w (rather than $z^{1/n}, w^{1/m}$).

[illegible]/15

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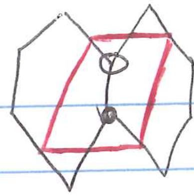
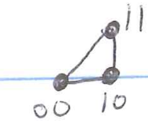
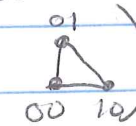
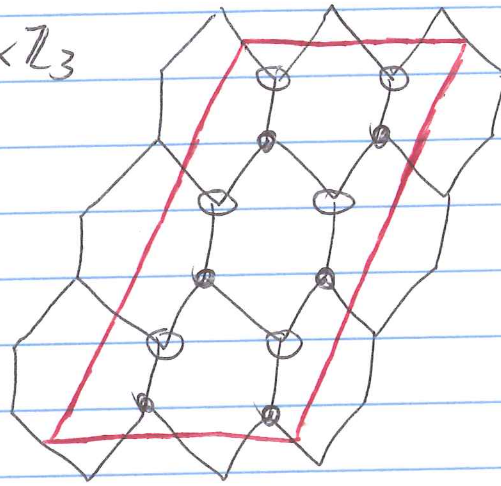
<u>Example</u>:  has $P(z,w) = 1 + z + zw$ 

(or up to $GL_2(\mathbb{Z})$, $1 + z + w$ )

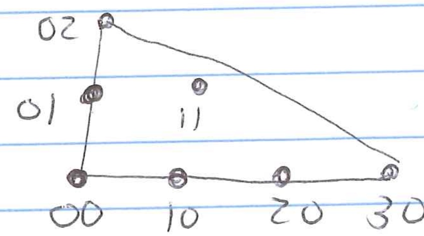
we can dilate by $\mathbb{Z}_2 \times \mathbb{Z}_3$



<u>up to $GL_2(\mathbb{Z})$</u>

$$P_{2,3}(z,w) = \left(1 + z^{1/2} + w^{1/3}\right)\left(1 - z^{1/2} + w^{1/3}\right)\left(1 + z^{1/2} + \rho w^{1/3}\right)\left(1 - z^{1/2} + \rho w^{1/3}\right)\left(1 + z^{1/2} + \rho^2 w^{1/3}\right)\left(1 - z^{1/2} + \rho^2 w^{1/3}\right)$$

$$= 1 - 3z + 3z^2 - z^3 + 2w + 6zw + w^2$$



If we ignore coeffs, and just care about the Newton polygon, we see it is dilated by n & m.

Here, for hexagonal lattice we get a <u>dilated triangle</u> as desired.

[illegible]/15

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Proof of theorem: Let $K_{n,m}(z,w)$ denote the Kasteleyn matrix for $G_{n,m}$, which we think of as a linear map from

$$\left\{\begin{matrix}\text{Vector space of}\\ \text{functions of \underline{white}}\\ \text{vertices of } G_{n,m}\end{matrix}\right\} \xrightarrow{K_{n,m}(z,w)} \left\{\begin{matrix}\text{Vector space of}\\ \text{functions of \underline{black}}\\ \text{vertices of } G_{n,m}\end{matrix}\right\}$$

$$V_W \qquad\qquad\qquad\qquad V_b$$

Let α = nth root of unity, β = mth root of unity,

$V_W^{\alpha,\beta}$ and $V_b^{\alpha,\beta}$ are subspaces of functions s.t. translation by one period in horizontal direction $\longleftrightarrow$ multi. by α
" " " " vertical " " $\longleftrightarrow$ multi. by β

Since $V_W = \bigoplus_{i=1, j=1}^{n\ m} V_W^{\alpha^i, \beta^j}$ & $V_b = \bigoplus_{i=1, j=1}^{n\ m} V_b^{\alpha^i,\beta^j}$ $\Rightarrow$

$K_{n,m}(z,w)$ is block diagonal

$$\begin{array}{cc} & V_b^{\alpha^i,\beta^j} \\ V_W^{\alpha^i,\beta^j} & \begin{bmatrix} & 0 & 0 & 0 \\ 0 & & 0 & 0 \\ 0 & 0 & & 0 \\ 0 & 0 & 0 & \end{bmatrix}\end{array}$$

$$\Rightarrow \det K_{n,m}(z,w) = \prod_{i=1, j=1}^{n\cdot m} \underbrace{\det K\big|_{V_W^{\alpha^i,\beta^j} \to V_b^{\alpha^i \beta^j}}}_{\substack{\| \\ \det K(\alpha^i z^{1/n},\ \beta^j w^{1/m})}}$$

[illegible]/15

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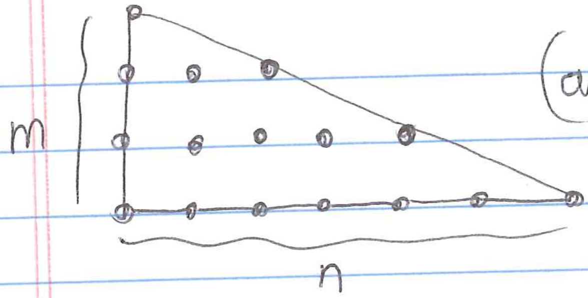
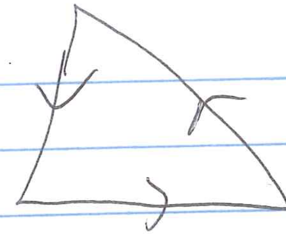
Up to $GL_2(\mathbb{Z})$, first assume our lattice triangle looks like



(area $\frac{mn}{2}$)



Hanany-Vegh algorithm technically would use primitive normals rather than primitives but essentially same as Goncharov Kenyon

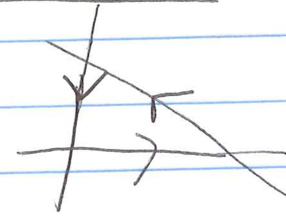
n horizontal primitives $(1,0)$

m vertical primitives $(0,-1)$

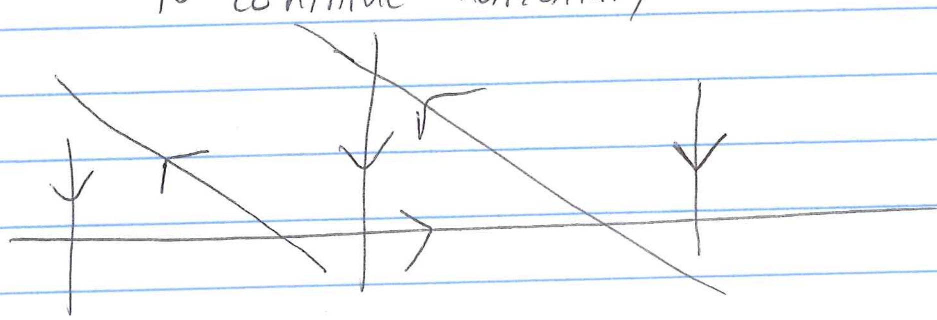
d diagonal primitives $\left(\frac{-n}{d}, \frac{m}{d}\right)$ where $d = \gcd(n,m)$

We wish to arrange these so we have alternating strand condition and no triple intersections.

Since only have 3 slopes to work with, locally must have configuration



Then to insure alternating strand condition, would need to continue horizontally as



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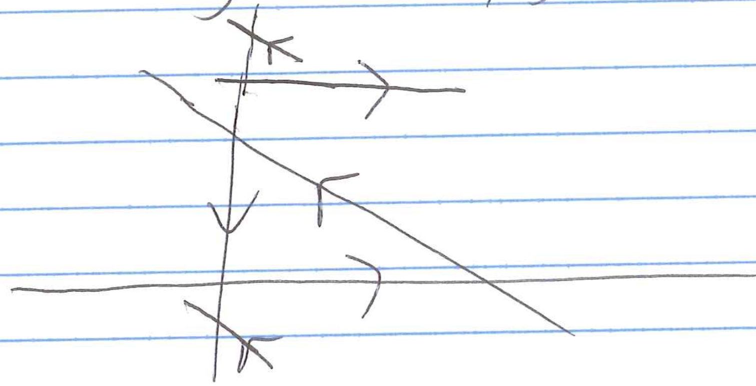
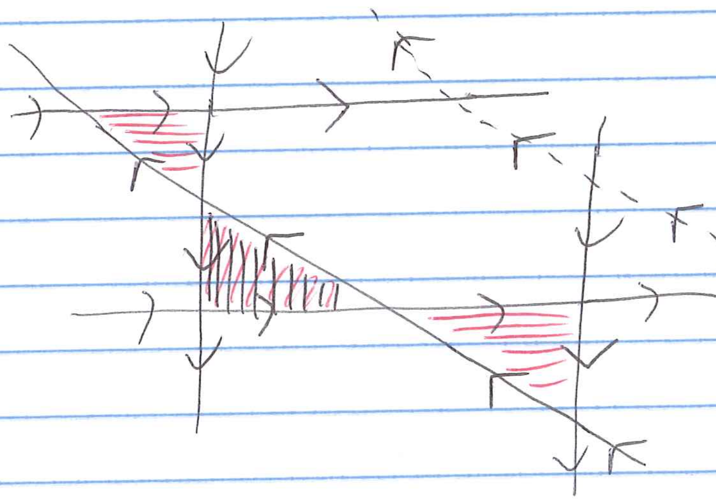
Furthermore, vertically, we would have



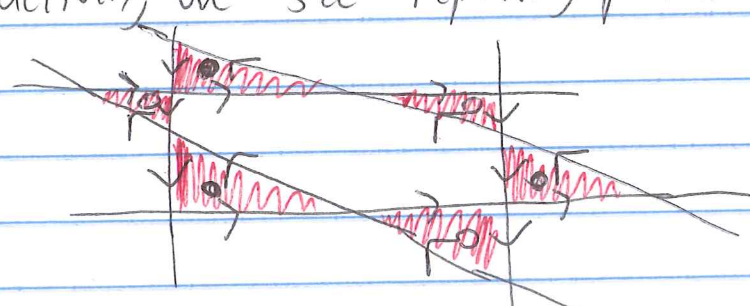
In other words, once we pick one slope and one line, the other lines crossing must alternate in slope

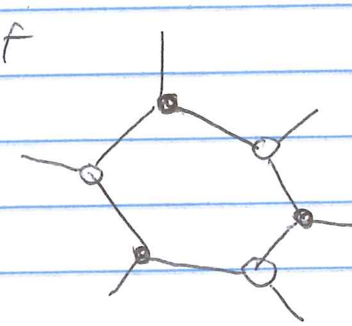
we hence locally have two adjacent triangles



and dotted diagonal line must be added to preserve alternating strand condition.

Inductively, we see repeating pattern of





5/[illegible]/15

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Counting # horizontal, vertical, and diagonal lines, we indeed get hexagonal lattice with fund domain n wide, m high

i.e. dual of McKay quiver corresponding to group $G = \mathbb{Z}_n \times \mathbb{Z}_m$ with

$$\chi_{nat}(\sigma) = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \chi_{nat}(\tau) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & w^{-1} \end{bmatrix}$$

Next time we will discuss how to view this in relation to orbifold

$\mathbb{C}^3 / \mathbb{Z}_n \times \mathbb{Z}_m$ and the associated

toric action

$$(x, y, z) \mapsto (\lambda x, \lambda^{-1} y, z)$$

$$(x, y, z) \mapsto (x, wy, w^{-1} z)$$

See Hanany–Kennaway

"Dimer models and toric diagrams"

arXiv: 0503149

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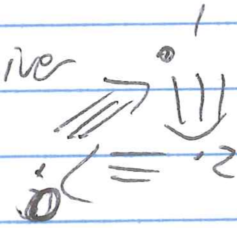
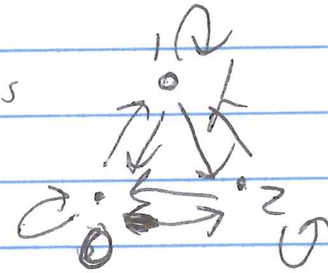
8

Going back to even $\mathbb{C}^3/\mathbb{Z}_3$ vs $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{C}$ cases we see dilation by integers is not the only possibility:

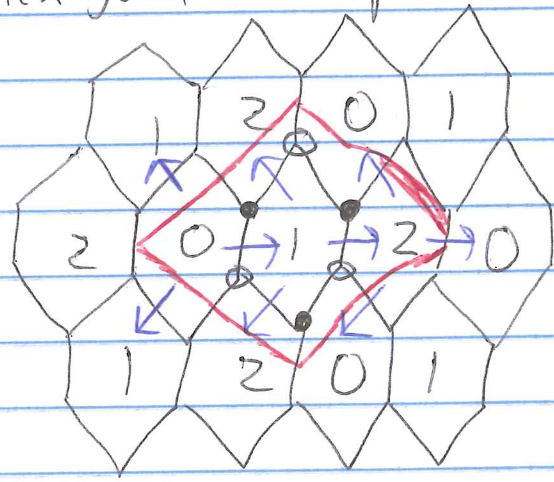
Recall: $\mathbb{Z}_3(1,1,1)$ has McKay quiver



and $\mathbb{Z}_3(2,1,0)$ has



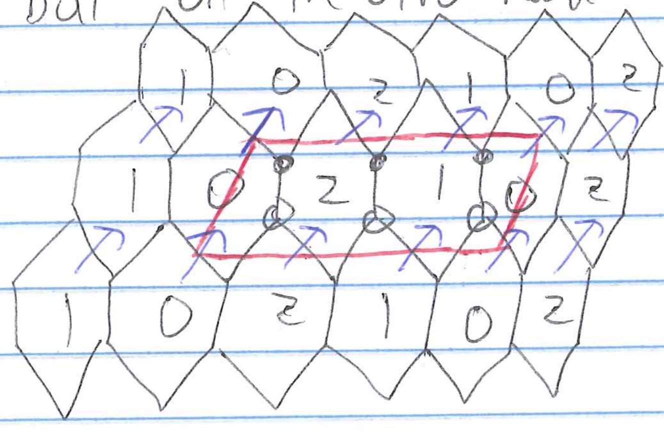
Using the potentials discussed last time, we get hexagonal bipartite tiling



For $\mathbb{Z}_3(1,1,1)$

[notice: tensoring by χ_a, χ_b, χ_c all increase index mod 3]

but on the other hand

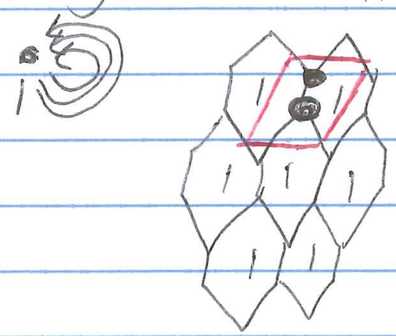


For $\mathbb{Z}_3(2,1,0)$

[tensoring by χ_c keeps index]

BOTH contain 6 vertices in fundamental domain

4/1/15 (9) In the second case, Fundamental parallelogram (of $\mathbb{C}^3$ case



)

has been dilated so 3-times as wide but with the same height.

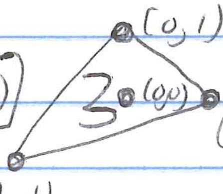
On the other hand, in the first case we have stretched both sides by $\sqrt{3}$

Consequence: does not fit into Kenyon-Okounkov-Sheffield Dilation Theorem.

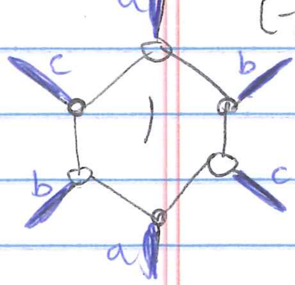
$= \boxed{3 + z + w + z^{-1}w^{-1}}$

In Fact, $K(z,w)$ leads to toric diagram

$\boxed{(1,0)+(0,1)+(-1,-1)=(0,0)}$



where 3 at (0,0) corresponds to the three perfect matchings



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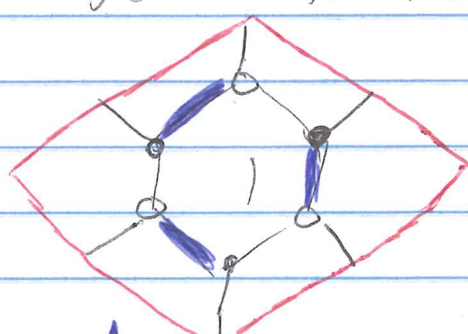

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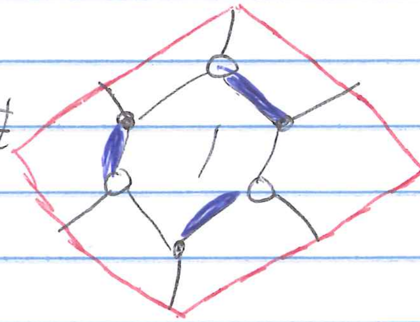
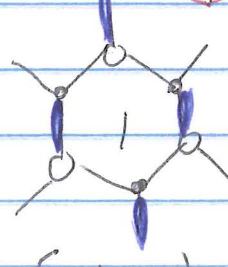
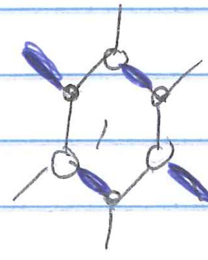


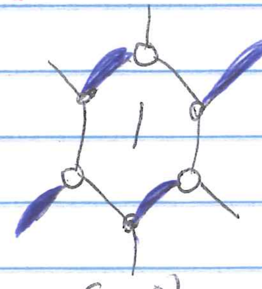
Other three perfect matchings:



(-1,-1)

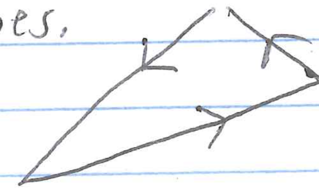


(0,1)



(1,0)

4/1/15 (10) If our lattice triangle is not a right triangle (up to $GL_2(\mathbb{Z})$), we can use same logic as before (pgs. 5~7) Still only three slopes.



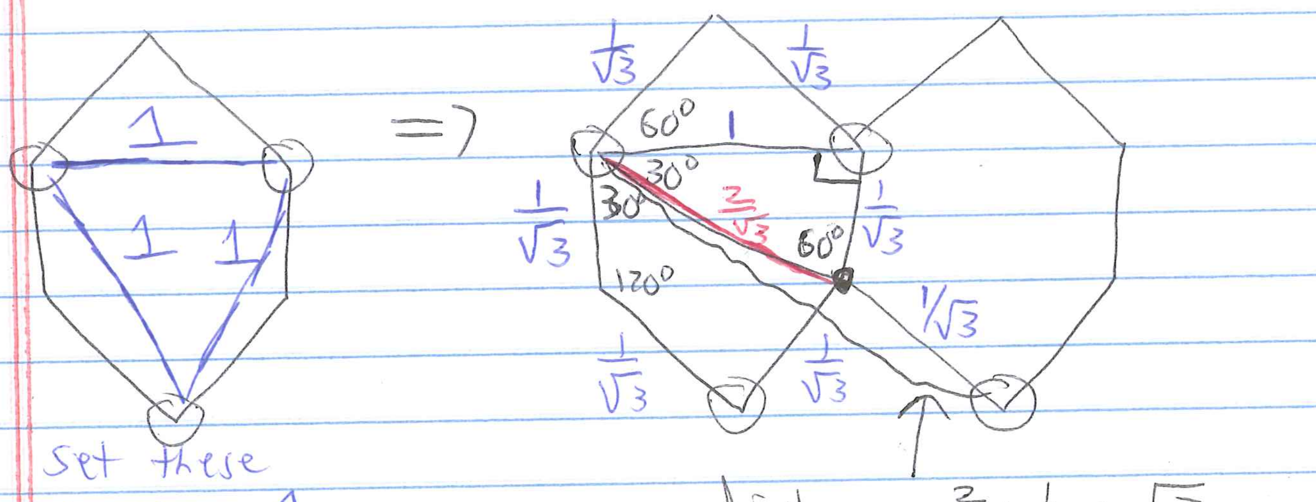
$\Rightarrow$ pattern of bipartite tiling is again a hexagonal lattice.

Assuming the area of the toric diagram is $\frac{n}{2}$,

we indeed see that every nth hexagon has the same label.

$\Rightarrow$ associated abelian group G is $\mathbb{Z}_n$ or $\mathbb{Z}_k \times \mathbb{Z}_m$ where $n = km$.

Remark: We see irrational dilation for $\mathbb{C}^3/\mathbb{Z}_3$ by geometry of regular hexagons.



Set these lengths as 1

distance $= \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \sqrt{3}$.