4/6/15 ① Lecture 19-20: F-terms, D-terms, Moduli space, and Master Space

Recall from Lecture 12, we defined global symmetries as one parameter-subgroups

$\rho: \mathbb{C}^* \to \mathrm{Aut}(A)$ [A = superpotential algebra $\mathbb{C}Q/\partial W$]

s.t. for each $t \in \mathbb{C}^*$, $\rho(t)$ defines the map

$\rho(t): X_a \longmapsto t^{v_a} X_a$ for each $X_a \in A$ corresponding to an arrow $a \in Q_1$. ($\underset{\text{vector}}{v = [\{v_a\}]_{a \in Q_1}} \in \mathbb{Z}^{Q_1}$)

ρ is only well-defined if it acts homogeneously on terms in W.

(Following Broomhead) We let N^+ = cone of v's with all entries nonnegative and proved that perfect matchings ~~or~~ of the bipartite tiling corresponding to (Q, W) were the generators of N^+ by

perf matching $M \longmapsto v_M$ = 0-1 vector with value 1 on $a \in Q_1 \iff e_a$ in M.

$\underset{\text{edge}}{e_a}$

e_a a

Today, we explain how N^+ is related to the toric diagram Δ corresponding to

- Kasteleyn char poly of N^+'s generators

using choice of fund. domain for our bipartite tiling.

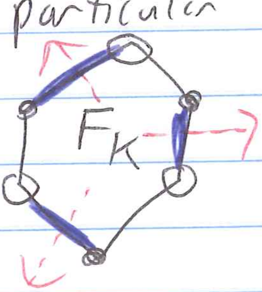
4/6/15 (2) Let $M = |Q_1| \times m$ (0-1)-matrix with rows indexed by edges $\{e_1, \ldots, e_{|Q_1|}\}$ of the bipartite tiling and columns indexed by perfect matchings $\{P_1, \ldots, P_m\}$.

We let entry $M_{ij} = \begin{cases} 1 & \text{if perfect matching } P_j \text{ contains } e_i \\ 0 & \text{otherwise} \end{cases}$

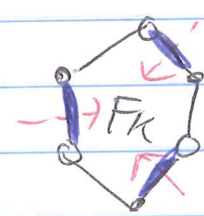
Matrix M will almost always have linear dependencies in the columns which we record in the $F \times m$ matrix Q_F

defined as $Q_F = (\ker M)^T$.

Additionally for any face F_K of our bipartite tiling, we define P_K^+ and P_K^- as the perfect matchings which each use exactly half of the edges around face F_K and look the same outside of this face. (we in particular let P_K^+ look like



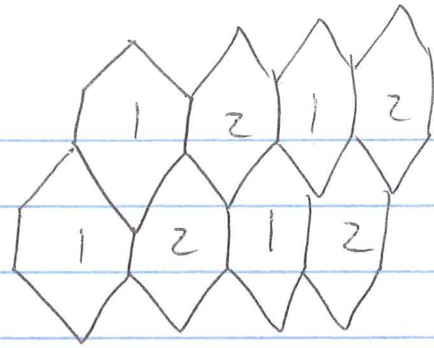
and P_K^- look like

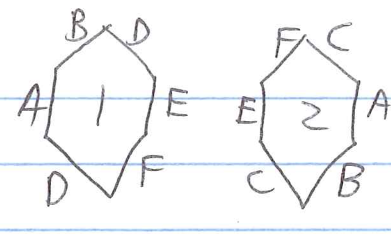


and extend arbitrarily to a perfect matching on the rest of the torus)

Rem: There are examples where constructing such P_K^+ and P_K^- are not possible. However the following construction sometimes still works after defining variants of P_K^+ & P_K^-.

4/6/15 ③ e.g.





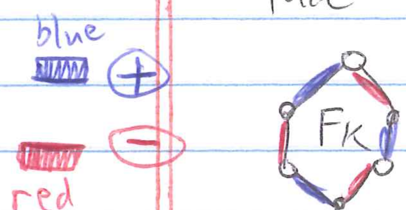
P_1^+ should contain edges A, D, and F.

However, 1) in this e.g. the fund. domain only contains two black vertices & two white vertices

2) we would need to include both copies of edge D incident to face 1 which result in two edges (D & F) incident to the bottom vertex.

We will come back to examples like this

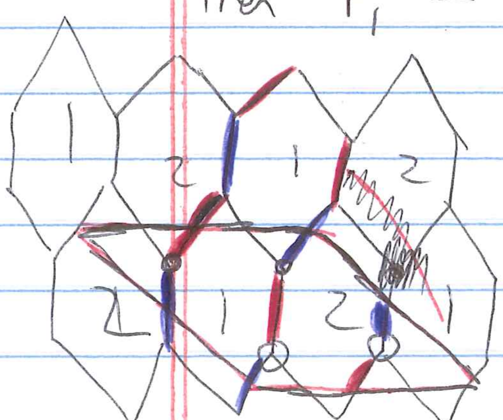
Consider the linear combination $P_K^+ - P_K^-$ for every face F_K $(K = 1, \ldots, |Q_0|)$ which results in



and all edges are cancelled outside of F_K

In degenerate e.g.'s like above, we abuse notation, and let "P_1^+" = A + D + F, "P_1^-" = B + D + E

then "P_1^+" − "P_1^-" = A + F − B − E (the two D's cancel)



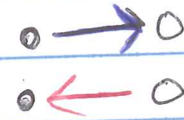
which separates the strip of faces 1 from the strip of faces 2.

Furthermore, agrees with alternating sum $\underline{A - B + D - E + F - D}$ around F_1.

4/6/15 (4) Ignoring this degeneracy issue for the moment, we get $\mathbb{Z}$-linear combinations of perfect matchings corresponding to alternating sum around each face F_K

we obtain

defining a flow

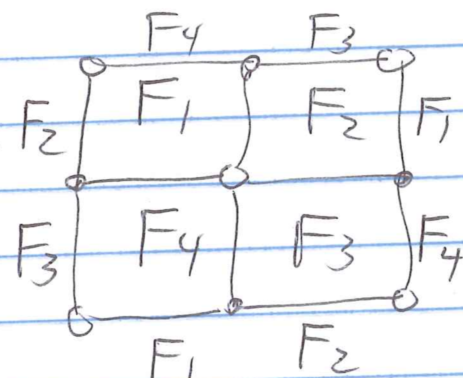


for pos

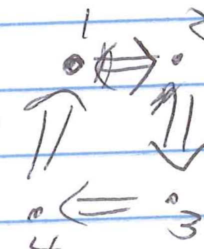
for neg

$(|Q_0|-1)$ of these are linearly independent since we can always define the outside of F_K to be the contour around every other face.

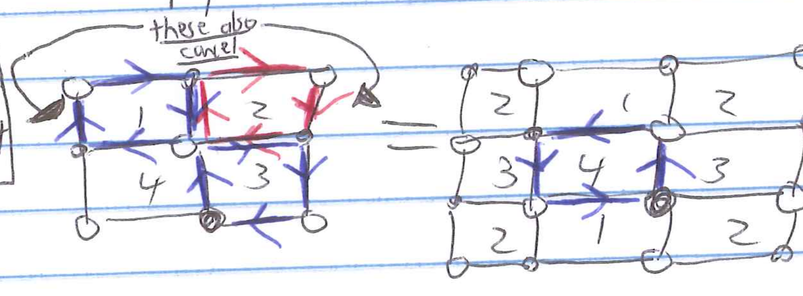
e.g. 4-checkerboard



corresponding to



$F_1 + F_2 + F_3 = F_4$



Let $Q_D = (|Q_0|-1) \times m$ matrix by picking all but one face and writing alternating sum around it as $\mathbb{Z}$-linear combo of perfect matchings.

Rem: As we will see later, even if $P_K^+ - P_K^-$ construction does not work, can always isolate a single face (alternating sum of its edges) as a $\mathbb{Z}$-linear combin. of P_i's.

4/6/15 ⑤ Build matrix $Q = \begin{bmatrix} Q_D \\ Q_F \end{bmatrix}$ (m = # perfect matchings)

Claim: For a bipartite tiling on a torus

$(\ker Q)^T = 3 \times m$ matrix

such that in row-echelon form, each column sums to 1.

$\Rightarrow$ cols of $(\ker Q)^T$ are coplanar.

projecting to the plane,

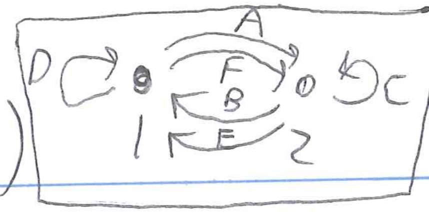
Claim: columns are vertices of toric diagram Δ with multiplicities.

Physics literature discusses a proof in "Moduli Spaces of Gauge Theories from Dimer Models: Proof of the Correspondence" by Franco and Vegh (arXiv:0601063)

After some examples, I will follow the proof in

[Broomhead, sec 2.3] using diffeent terminology.

4/6/15 (6) <u>Example 1</u> $(\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{C})$



5 perfect matchings : $\underbrace{AE}_{p_1}, \underbrace{AF}_{p_2}, \underbrace{BE}_{p_3}, \underbrace{BF}_{p_4}, \underbrace{CD}_{p_5}$

$$M = \begin{array}{c|ccccc} & p_1 & p_2 & p_3 & p_4 & p_5 \\ \hline A & 1 & 1 & 0 & 0 & 0 \\ B & 0 & 0 & 1 & 1 & 0 \\ C & 0 & 0 & 0 & 0 & 1 \\ D & 0 & 0 & 0 & 0 & 1 \\ E & 1 & 0 & 1 & 0 & 0 \\ F & 0 & 1 & 0 & 1 & 0 \end{array}$$

$$Q_F = (\ker M)^T = \begin{bmatrix} 1 & -1 & -1 & 1 & 0 \end{bmatrix} \quad (p_1\ p_2\ p_3\ p_4\ p_5)$$

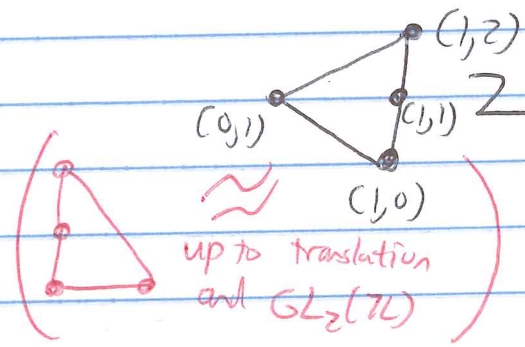
As discussed earlier, Face 1 $\to \begin{bmatrix} 0 & 1 & -1 & 0 & 0 \end{bmatrix}$ $(p_1\ p_2\ p_3\ p_4\ p_5)$
AF−BE

in fact Face 2 $\to \begin{bmatrix} 0 & -1 & 1 & 0 & 0 \end{bmatrix}$

(and clear linear dependence)

w.l.o.g. $Q_D = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 \end{bmatrix}$ $(p_1\ p_2\ p_3\ p_4\ p_5)$

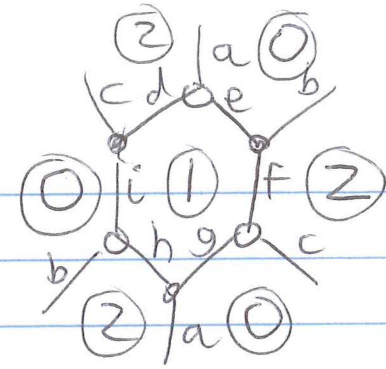
$$Q = \begin{bmatrix} 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$



$$(\ker Q)^T = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{matrix} (1)+(2) \\ (2)+(3) \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 \end{bmatrix}$$

✓

4/6/15 ⑦ Example 2 $\mathbb{C}^3/\mathbb{Z}_3$



afi, bdg, ceh, abc, dfh, egi

six perfect matchings

$$M = \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{array}\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Q_F = \ker M = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix}$$

$$F_0 = g - c + i - b + e - a$$
$$= p_6 - p_4$$

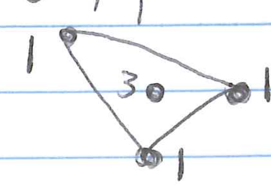
$$F_1 = d - e + f - g + h - i$$
$$= p_5 - p_6$$

$$F_2 = b - h + a - d + c - f$$
$$= p_4 - p_5$$

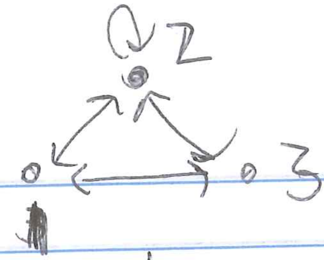
$$Q_D = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \Rightarrow Q = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

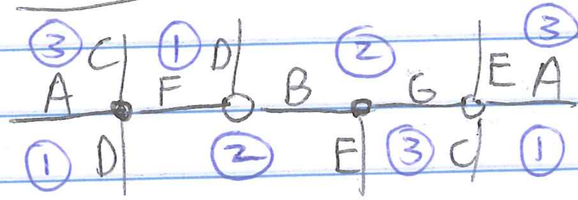
with $$(\ker Q)^T = \begin{bmatrix} 1 & 0 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{row-reduce}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \frac{1}{3} I \\ 0 & 0 & 1 & \end{array}\right]$$

rewrite in coords
$$\begin{array}{c} (1)-(2) \\ (2)-(3) \end{array} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}$$



4/6/15 (8) Example 3 SPP



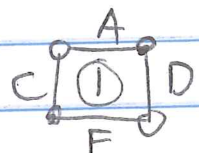


has perfect matching matrix

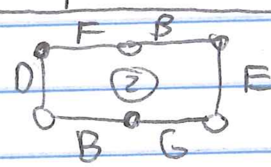
(see pg. 12 of Lecture 12)

$$M = \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$Q_F = (\ker M)^T = \begin{bmatrix} 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

Face 1 

$= D - F + C - A$
$= -p_1 + p_2 + p_3 - p_5$

Face 2 

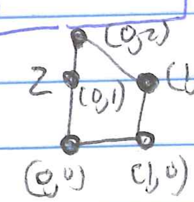
$= F - B + E - G + B - D$
$= F + E - G - D$
$= -2p_3 + p_4 + p_5$
(also $= -p_3 + p_6$ agrees up to Q_F)

$$\Rightarrow \quad Q_D = \begin{bmatrix} -1 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & -2 & 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow Q = \begin{bmatrix} 0 & 0 & 1 & -1 & -1 & 1 \\ \hline -1 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & -2 & 1 & 1 & 0 \end{bmatrix}$$

Rem: This is an example where $p_1^+ - p_1^-$ construction not sufficient.

$$(\ker Q)^T = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & [illegible] & 1 & [illegible] \end{bmatrix} \rightsquigarrow \begin{array}{c} (1)+(2) \\ (2)+(3) \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 2 & 1 \end{bmatrix}$$



4/6/15 (9)

For another e.g, see Sec. 3.6 of [Kennaway] for dP_1 (i.e. Somos-4) e.g.

We now turn to [Broomhead] for proof of the following result:

Claim: For a bipartite tiling on the torus, building the matching matrix M, and associated matrix Q (built from $Q_F = \text{Ker}(M)^T$ and Q_D)), then $(\text{Ker } Q)^T$ is of rowdim 3 and its columns are coplanar on $x+y+z=1$.

Lastly, projecting to the plane yields toric diagram Δ agreeing with the Newton polygon (including multiplicities) of Kasteleyn characteristic polynomial $K(z,w)$.

Before giving the proof, we rephrase this algorithm in terms of Broomhead's language of algebraic topology and commutative algebra.

1) cone $N^+ \subset \mathbb{N}^{Q_1}$ is generated by (0-1)-functions corresponding to perfect matchings.

$\Rightarrow$ By construction, matching matrix M has columns corresponding to generators of N^+, written in vector form.

2) However, the generators (as given) satisfy $\mathbb{Z}$-linear relations, i.e. they do not freely generate N^+.

4/6/15 (10) $Q_F = (\ker M)^T$ encodes these relations,

e.g. for $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$

N^+ gen'd by $p_1 = AE$, $p_2 = AF$, $p_3 = BE$, $p_4 = BF$, $p_5 = CD$

but we have the relation $p_1 p_4 = AEBF = p_2 p_3$
if we think of these as funcs on $\mathbb{Z}^{Q_1}$
so $AEBF = AFBE$ is shorthand for

function g s.t. $g(A)=1, g(B)=1, g(C)=0, g(D)=0,$
$g(E)=1, g(F)=1.$

Thus $(\ker Q_F)^T$ yields generators
which freely generate N^+

In the e.g. $(\ker Q_F)^T = $

	p_1	p_2	p_3	p_4	p_5	
	1	0	0	-1	0	v_1
	0	1	0	1	0	v_2
	0	0	1	1	0	v_3
	0	0	0	0	1	v_4

so N_+ freely generated by $\{v_1, v_2, v_3, v_4\}$ where

$p_1 = v_1 = AE$, $p_2 = v_2 = AF$, $p_3 = v_3 = BE$,
$p_4 = \frac{v_2 v_3}{v_1} = \frac{(AF)(BE)}{(AE)} = BF$, $p_5 = v_4 = CD$

$\Rightarrow$ $\boxed{v_1 = AE,\ v_2 = AF,\ v_3 = BE,\ v_4 = CD}$

4/6/15 (11)

3) We build an exact sequence

$$0 \to \mathbb{Z} \hookrightarrow \underset{\substack{\| \\ N_{in}}}{\mathbb{Z}^{Q_0}} \xrightarrow{d} N \twoheadrightarrow N_0 \to 0$$

where map d is the cochain map defined by

$df \in \mathbb{Z}^{Q_1}$ defined as $df(a) = f(h) - f(t)$

$\uparrow$ an arrow $\in Q_1$ $\underset{t}{\bullet} \longrightarrow \underset{h}{\bullet}$

$t, h \in Q_0$

In particular, to insure $\operatorname{im}(\mathbb{Z} \hookrightarrow \mathbb{Z}^{Q_0}) = \ker d$
we define $\mathbb{Z} \hookrightarrow \mathbb{Z}^{Q_0}$, $\lambda \longmapsto f$ s.t. $f(v) = \lambda$ for every $v \in Q_0$.

Clearly, $df = 0 = (\lambda - \lambda)$ on every arrow in this case.

$\operatorname{im} d \in N$ since $\mathbb{Z}^{Q_0} \xrightarrow{d} \mathbb{Z}^{Q_1} \xrightarrow{d} \mathbb{Z}^{Q_2}$
is a cochain complex satisfying $d^2 = 0$

hence any $g \in \operatorname{im}(\mathbb{Z}^{Q_0} \xrightarrow{d} \mathbb{Z}^{Q_1})$ also in $d^{-1}(0)$, $0 \in \mathbb{Z}^{Q_2}$

and N was defined as $d^{-1}(\mathbb{Z})$, $\mathbb{Z} \subset \mathbb{Z}^{Q_2}$.

4/6/15 (12) Thus all maps in this exact sequence are well-defined, including $N \twoheadrightarrow N_o$ which is the surjection onto $(\text{coker } d) = N/\text{im } d$.

<u>Rem:</u> Thinking of these as automorphisms of the path algebra (if we think of the related torus actions, i.e. one parameter subgroups) Broomhead refers to

$\mathbb{Z}^{Q_0}$ as N_{in} (inner automorphisms)

and $N/\text{im } d$ as N_o (outer automorphisms).

As Broomhead notes, physics literature also calls

N_{in} as <u>baryonic</u> symmetries and
N_o as <u>mesonic</u> symmetries

(see [Kennaway, Section 3.6.1])

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4) As discussed in (2), using $(\ker Q_F)^T$, we have a set of generators $\{v_1, \ldots, v_k\}$ that freely generates cone N^+.

Thus $N_o^+ := N^+ \cap (N_o \otimes_{\mathbb{Z}} \mathbb{R})$ obtained

by imposing new relations coming from $\text{im } d$.

(see pg. 16) saturation of the projection of the cone $N^+ \subset N$ into the $(2g+1)$-rank lattice N_o where $g = \text{genus}(Y)$ $= 2$ for torus

4/6/15 (13) e.g. continued, starting with $f \in \mathbb{Z}^{N_{in}}$ ($N_{in} \parallel Q_D$) defined by

$f(1) = \lambda_1$, $f(2) = \lambda_2$, then $df \in$ $d \subset N$ defined by

$$\begin{aligned}
f(A) &= \lambda_2 - \lambda_1 \\
f(B) &= \lambda_1 - \lambda_2 \\
f(C) &= \lambda_2 - \lambda_2 = 0 \\
f(D) &= \lambda_1 - \lambda_1 = 0 \\
f(E) &= \lambda_1 - \lambda_2 \\
f(F) &= \lambda_2 - \lambda_1
\end{aligned}$$

$\Rightarrow$ need $df = g \in N$ s.t. $g(A) = g(F) = -g(B) = -g(E)$ and $g(C) = g(D) = 0$.

in terms of our freely generating set
$v_1 = AE$, $v_2 = AF$, $v_3 = BE$, $v_4 = CD$

$\Rightarrow$ $g(v_1) = 0$, $g(v_2) = 2(\lambda_2 - \lambda_1)$
$g(v_3) = 2(\lambda_1 - \lambda_2)$, $g(v_4) = 0$

i.e. we quotient by functions g multiples of the form

$g(v_1)$	$g(v_2)$	$g(v_3)$	$g(4)$
0	1	−1	1

Notice this exactly matches Q_D for this e.g.
(we will explain why momentarily)

or

$g(p_1)$	$g(p_2)$	$g(p_3)$	$g(p_4)$	$g(p_5)$
0	1	−1	0	0

using larger (non-free) generating set for N^+.

4/6/15 (14) In conclusion, taking $\left(\text{Ker } Q_D|_{\text{gns } v_1,\dots,v_k}\right)^T$

yields rows which give generators for cone N_0^+ in terms of free generators $(v_1, \dots, v_k)$

e.g., cont.

$$\begin{array}{c} v_1 \ v_2 \ v_3 \ v_4 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} g_1 \\ g_2 \\ g_3 \end{matrix}$$

$g_1 := \; g_1(v_1) = 1, \; g_1(v_2) = 0, \; g_1(v_3) = 0, \; g_1(v_4) = 0$

$g_2 := \; g_2(v_1) = 0, \; g_2(v_2) = 1, \; g_2(v_3) = 1, \; g_2(v_4) = 0$

$g_3 := \; g_3(v_1) = 0, \; g_3(v_2) = 0, \; g_3(v_3) = 0, \; g_3(v_4) = 1$

g_1, g_2, g_3 generate $N_0^+ \subset \mathbb{Z}^{Q_1}$

We now rewrite g_1, g_2, g_3 in terms of $\{p_1, p_2, \dots, p_5\}$

$g_1(p_1) = 1, \; g_1(p_2) = 0, \; g_1(p_3) = 0, \; g_1(p_4) = g_1(v_2) + g_1(v_3) - g(v_1)$ ⟵ -1

$g_1(p_5) = g_1(v_4) = 0$

Similarly, $g_2(p_1) = 0, \; g_2(p_2) = 1, \; g_2(p_3) = 1, \; g_2(p_4) = 2, \; g_2(p_5) = 0$

$g_3(p_1) = 0, \; g_3(p_2) = 0, \; g_3(p_3) = 0, \; g_3(p_4) = 0, \; g_3(p_5) = 0$

We could also obtain g_1, g_2, g_3 in terms of the p_i's directly by taking $(\text{Ker } Q)^T$ where $Q = \left[\dfrac{Q_F}{Q_D}\right]$ with columns given by the $p_1, \dots, p_m$'s.

4/6/15 (15) Left to show

- Show the set $(\text{im } d) \subset N \subset \mathbb{Z}^{Q_1}$ is the $\mathbb{Z}$-linear combination of $F_1, \dots, F_{|Q_0|} \in \mathbb{Z}^{Q_1}$ where F_i given by

$$\begin{cases} F_i(e) = 1 & \text{if } e = \\ F_i(e) = -1 & \text{if } e = \\ F_i(e) = 0 & \text{otherwise} \end{cases}$$

- Show ker Q (or equivalently $(\ker Q_{d|_{v_i\text{'s}}})$) has 3 generators whose sum is a constant function.

(Equivalently, want to show that N_0^+ is a cone whose 3 generators are coplanar.)

- terms of $K(z, w)$ indeed correspond to the coordinates of these generators.

Phrased in this way the first part is clear since each $F_i \in \mathbb{Z}^{Q_1}$ defined by $F_i = dg_i$ where $g_i \in \mathbb{Z}^{Q_0}$ satisfies

$$g_i(j) = 1 \text{ for } j = i$$
$$g_i(j) = 0 \text{ for } j \neq i.$$

Thus $\mathbb{Z}$-linear combos exactly im d.

4/6/15 (16) In fact this shows that we can always write an alternating sum of edges around a face as a <u>$\mathbb{Z}$-linear combo of perfect matchings</u> (even if $P_K^+ - P_K^-$ construction fails) since such a function in $(\text{im } d) \subset N^{\#}$ and the cone N^+ is generated by the perfect matchings.

For the second part, Broomhead constructs the short exact sequence

$$0 \to H^1(Y; \mathbb{Z}) \hookrightarrow N_0 \xrightarrow{\deg} \mathbb{Z} \to 0$$

where in our case, $Y =$ torus rather than an arbitrary Rieman surface

$\Rightarrow H^1(Y; \mathbb{Z}) \cong \mathbb{Z}^2$ in our case.

since perfect matchings are of degree 1 in N_0 and generate N_0^+,

$\ker\left(N_0 \xrightarrow{\deg} \mathbb{Z}\right)$ generated by

$\{P_2 - P_1, P_3 - P_1, \ldots, P_m - P_1\}$ where P_1 is chosen arbitrarily.

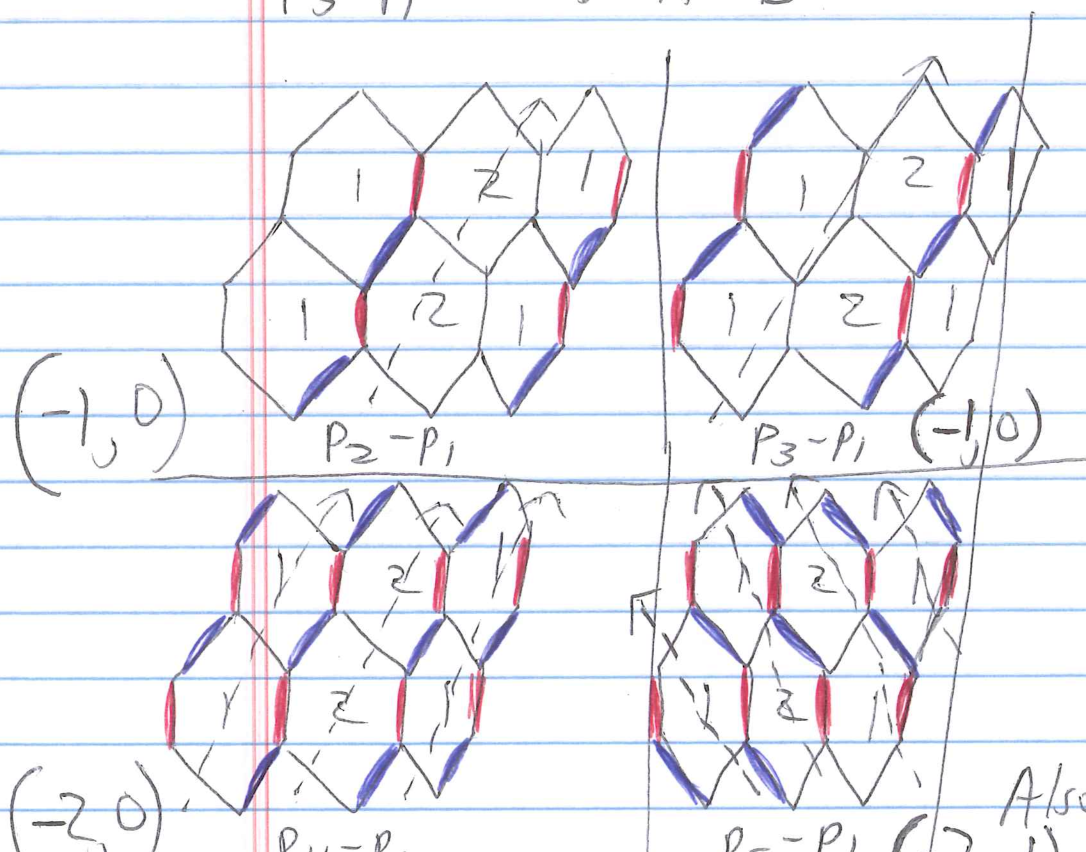
4/6/15 (17) e.g. cont. (see pg. 6)

$P_2 - P_1 = F - E$

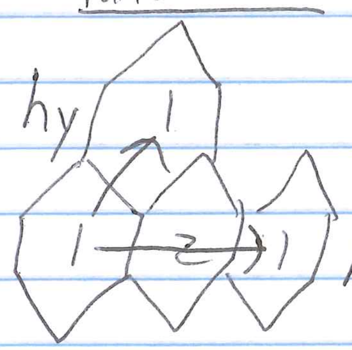
$P_3 - P_1 = B - A$

$P_4 - P_1 = B + F - A - E$

$P_5 - P_1 = C + D - A - E$

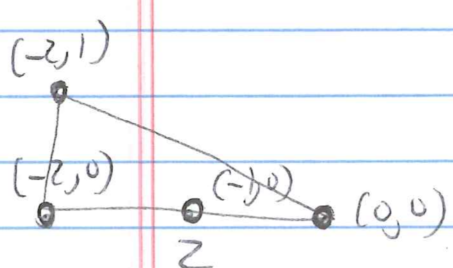


Called height functions



(h_x, h_y) noted in each case

Also $\boxed{P_1 - P_1 = (0, 0)}$ included



negative contribution for crossing  (red)

positive contribution for crossing  (blue)

4/6/15 (19) be a linear combination of fundamental cycles (rather than as contours around faces).

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we conclude that N_0 is a rank 3 lattice (rank $2g+1$ for general Riemann surf.)

perfect matchings all have degree 1 so their images in N_0 span a lattice polytope in a rank 2 affine sublattice.

N_0^+ is the cone on this polytope.

$\{P_1 - P_1, P_2 - P_1, \dots, P_m - P_1\}$ are each cocycles and the lattice polytope is the convex hull of all the relative cohomology classes (of the cocycles) with multiplicities.

Next time: zig-zags and a diffeent set of cocycles.

Master space $= (\text{Ker } Q_F)^\top \longleftrightarrow$ matching polytope

Moduli space $= \left(\text{ker} \begin{bmatrix} Q_F \\ Q_D \end{bmatrix}\right)^\top \longleftrightarrow$ matroid polytope

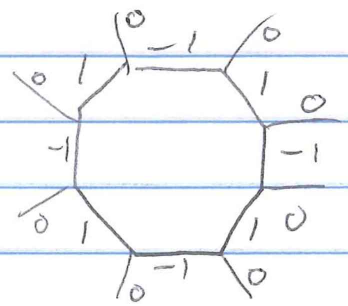
See "Matching polytopes, toric geometry, and the totally nonnegative Grassmannian" by Postnikov, Speyer, Williams

4/6/15 (18) We think of $H^1(Y;\mathbb{Z})$ as $\dfrac{\ker d: \mathbb{Z}^{Q_1} \to \mathbb{Z}^{Q_2}}{\operatorname{Im} d: \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_1}}$

which on the level on functions of edges of a bipartite tiling (rather than on arrows of the quiver) is the quotient

$$\frac{\{\mathbb{Z}\text{-functions that sum to zero at every vertex}\}}{\{\mathbb{Z}\text{-functions that alternate around faces of the tiling}\}}$$

e.g.,



and zero every other edge is zero at every vertex, but is in $\operatorname{Im} d: \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_1}$.

$$N_0 = N / \operatorname{im} d: \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_1} \cong \mathbb{Z}^{Q_1} / \operatorname{im} d$$

so $\ker: N_0 \xrightarrow{\deg} \mathbb{Z} = \ker: \mathbb{Z}^{Q_1}/\operatorname{im} d \xrightarrow{d} \mathbb{Z}^{Q_2}$

indeed is $H^1(Y;\mathbb{Z})$.

In the case of the torus, $H^1(Y;\mathbb{Z}) \cong \mathbb{Z}^2$ are the two directions of fundamental cycles.

In other words, we want $\mathbb{Z}$-linear combo of perfect matchings to