4/8/15

Lecture 20-21: Zig-Zag Symmetries and consistency conditions

We follow Secs 3–4 of [Broomhead]

To Review: Given quiver and potential (Q, W) we have superpotential algebra $A = \mathbb{C}Q/\partial W$.

$N \subset \mathbb{Z}^{Q_1}$ is the one-parameter subgroup lattice of a complex torus $\mathbb{T} \leq \mathrm{Aut}(A)$ of global symmetries (i.e. acting by multiplication on arrows of A & homogeneously on each term of W)

$\mathbb{Z}^{Q_0} \cong N_{in}$ is one-paramete subgroup lattice of T_{in} of invertible elements of A

i.e. $T_{in} = \left\{ \sum_{i \in Q_0} t_i e_i : t_i \in \mathbb{C}^* \right\}$

we have exact sequence $0 \to \mathbb{Z} \hookrightarrow N_{in} \xrightarrow{d} N \twoheadrightarrow N_0 \to 0$

where $N_0 = N/\operatorname{im} d$ (rank 3 by short exact seq. $0 \to \underbrace{H^1(Y;\mathbb{Z})}_{\cong \mathbb{Z}^2} \to N_0 \to \mathbb{Z} \to 0$).

cone $N^+ \subset N$ (with nonnegative integer values) generated by perfect matchings

N_0^+ = saturation of the projection of N^+ into the rank 3 lattice N_0

4/8/15 ② i.e. $N_0^+ = N_0 \cap (\text{image of } N^+ \text{ in } N_0 \otimes_{\mathbb{Z}} \mathbb{R})$

images of perfect matchings (generating N^+) span a ~~[illegible]~~ lattice polytope in a rank 2 affine sublattice.

N_0^+ = cone on this polytope.

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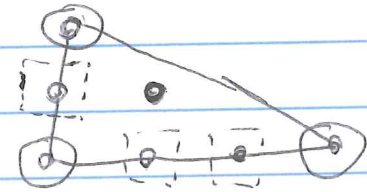
cone N^+ describes a normal affine toric variety X

& cone N_0^+ describes " " $X_0 = X / T_{in}$

Today, we wish to show that certain perfect matchings are more fundamental than the others.

<u>Def:</u> We say a <u>perfect matching</u> p of the <u>bipartite tiling</u> (associated to (Q, W)) is <u>external</u> if its image in N_0 lies on the boundary (i.e. a facet) of the polygon given by the cross-section w/ degree one elements in N_0^+.

It is called <u>extremal</u> if its image in N_0 is a vertex of the polygon.

e.g. 

circled = extremal & external

boxed = external only

Under a certain condition called "geometric consistency", we get:

4/8/15 3 Claim: When projecting from N^+ to N_0^+, if p is an extremal perf. matching, then the image of p has multiplicity 1.

Claim: The external perf. matchings, on the other hand, have multiplicities given by binomial coeffs.

Claim: The real cone $(N_0^+)_{\mathbb{R}}$ is generated over $\mathbb{R}^+$ by the images of ~~external~~ perf. matchings

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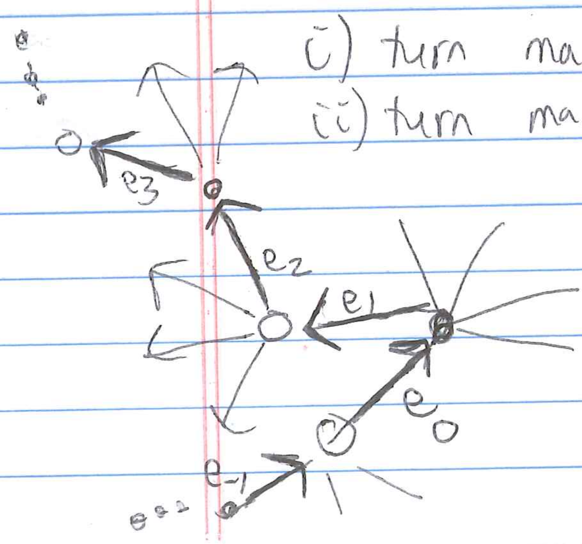
To prove these results, it is easiest to introduce the technology of zig-zag paths.

we work with the bipartite tiling.

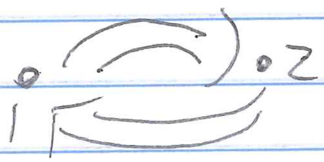
A zig-zag path is a doubly-infinite sequence of edges in the universal cover that starts with a single edge with a choice of orientation and propogates by the following two rules

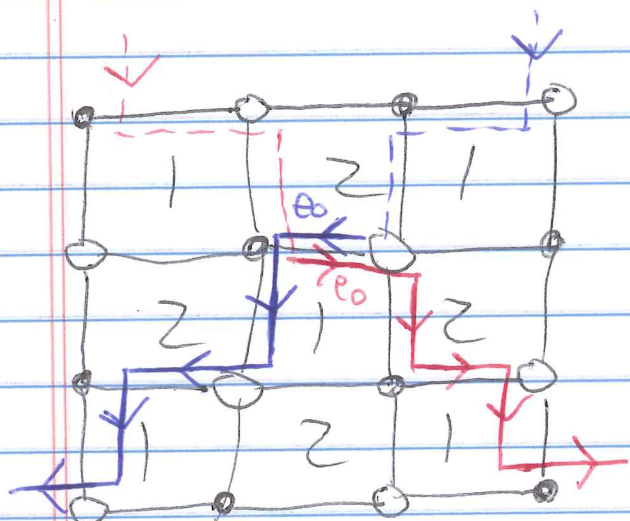
i) turn maximally left at • vertex
ii) turn maximally right at ∘ vertex

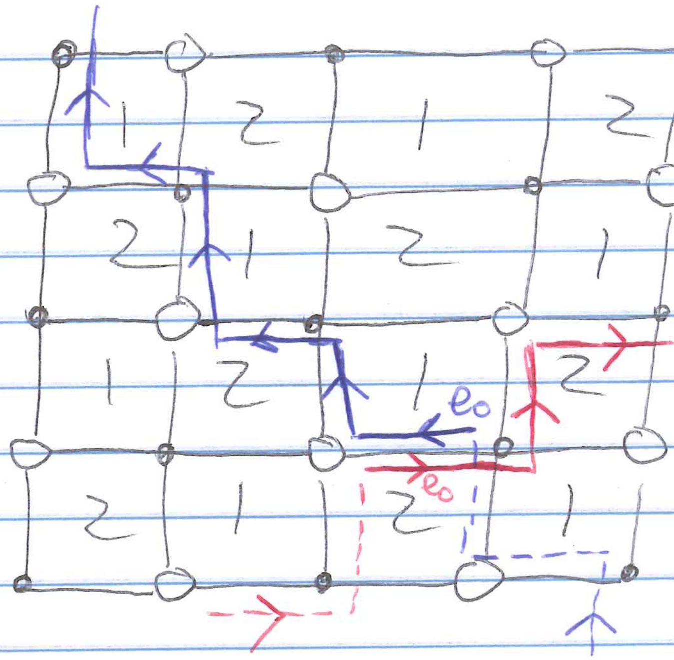


[begin with arbitrary choice of e_0 in this direction]

4/8/15 ④ For every initial choice of e_0, two possible zig-zag paths up to <u>choosing direction</u>.

e.g. 

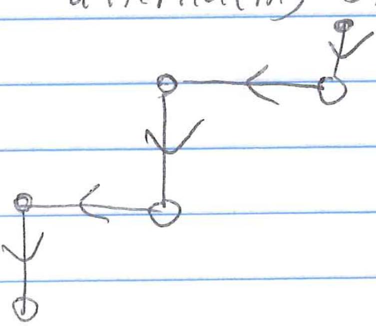




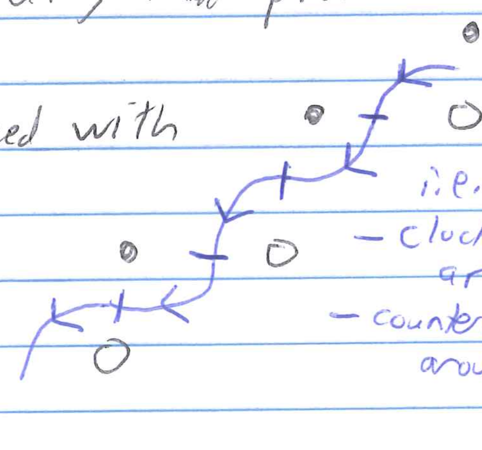
<u>Rem 1</u>: Such zig-zag paths in bijection with two other combinatorial paths

- alternating strand diagram pieces
- train tracks

For alternating strand diagram pieces



replaced with



i.e.
- clockwise around ●
- counterclockwise around ○

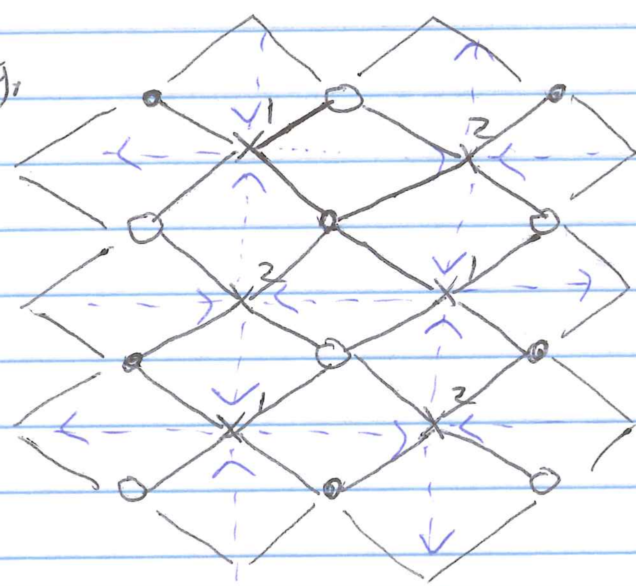
[Due to Kenyon – Schlenker]

4/8/15 (5) For train tracks, we make a "quad graph" whose vertices $= \left\{ \begin{array}{c} \text{vertices of} \\ \text{bip. tiling} \\ \bullet \ \circ \end{array} \right\} \sqcup \left\{ \begin{array}{c} \text{vertices of} \\ \text{dual quiver} \\ \times \end{array} \right\}$

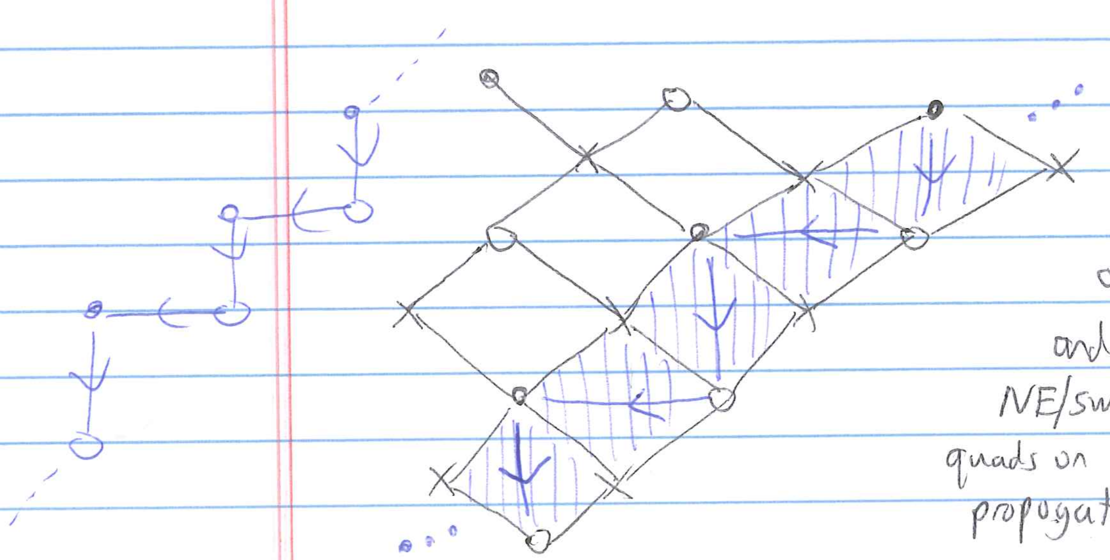
equivalently, vertices $= Q_0 \sqcup Q_2$.

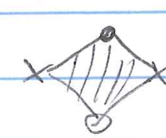
edges connect vertices $v \in Q_0$ and $f \in Q_2$
$\iff v \in \partial f$

e.g.



Then zig-zag path $\longleftrightarrow$ train track



In general, a train track starts with a choice of quadrilateral  and a choice NE/SW or NW/SE to color quads on two opposite sides and propogate.

4/8/15 6 <u>Def</u>: A dimer model (corresp. to (Q,W)) is <u>geometrically consistent</u> $\Leftrightarrow$

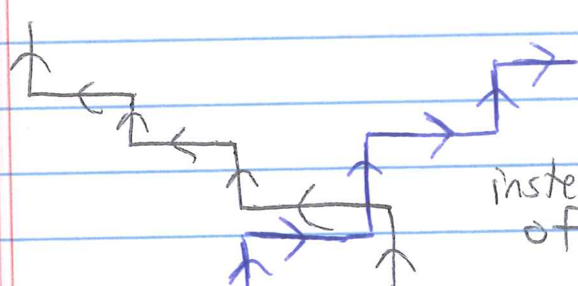
a) No zig-zag path intersects itself on the universal cover

b) If α and β are not parallel, then they intersect exactly once in the universal cover.

c) If $[\alpha], [\beta] \in H_1(Y;\mathbb{Z})$ are linearly dependent, then α and β are parallel in the universal cover.

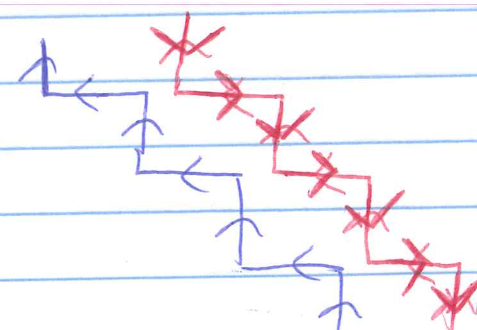
Here, homology classes $[\alpha]$ and $[\beta]$ refer to α and β's projections to the fundamental domain.

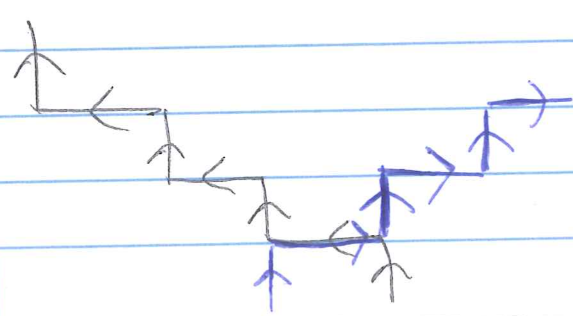
We also consider perturbations to allow us to consider transverse intersections, i.e. we don't allow two zig-zag paths to have a real interval in common.



instead of

&





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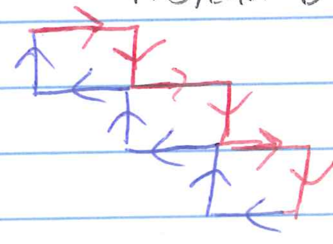
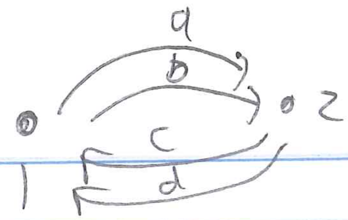
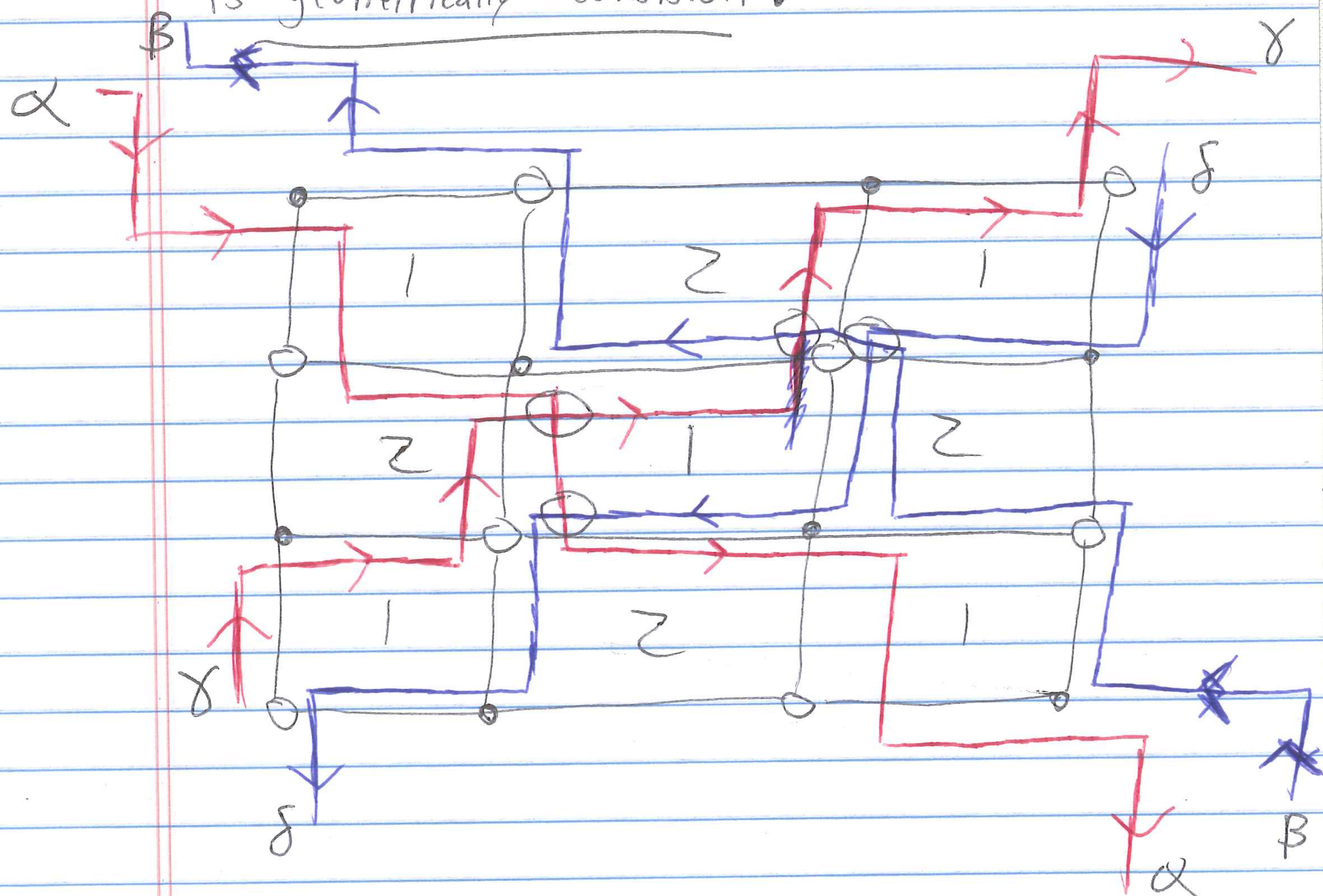


4/8/15 (7) e.g.



$W = acbd - adbc$

is geometrically consistent:



$[\alpha] = (1,0)$ $\qquad$ α & β parallel (technically <u>antiparallel</u>)

$[\beta] = (-1,0)$ $\qquad$ γ & δ parallel

$[\gamma] = (0,1)$ $\qquad$ α & γ, δ one intersection

$[\delta] = (0,-1)$ $\qquad$ β & γ, δ one intersection

Any other zig-zag path would have one of these four homologies and be parallel.

4/8/15 ⑥ **Def:** A dimer model (corresp. to (Q, W)) is <u>geometrically consistent</u> $\Leftrightarrow$

a) No zig-zag path intersects itself on the universal cover.

b) If α and β are not parallel, then they intersect exactly once in the universal cover

c) If $[\alpha], [\beta] \in H_1(Y; \mathbb{Z})$ are linearly dependent, then α and β are parallel in the universal cover.

Here homology classes $[\alpha]$ and $[\beta]$ refer to α's and β's projections to the fund. domain.

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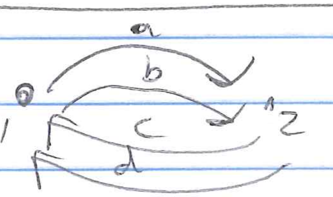
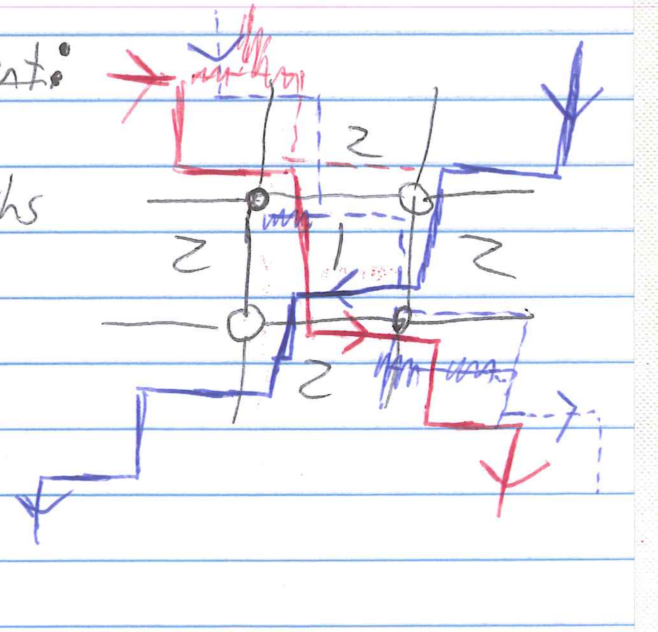
e.g.



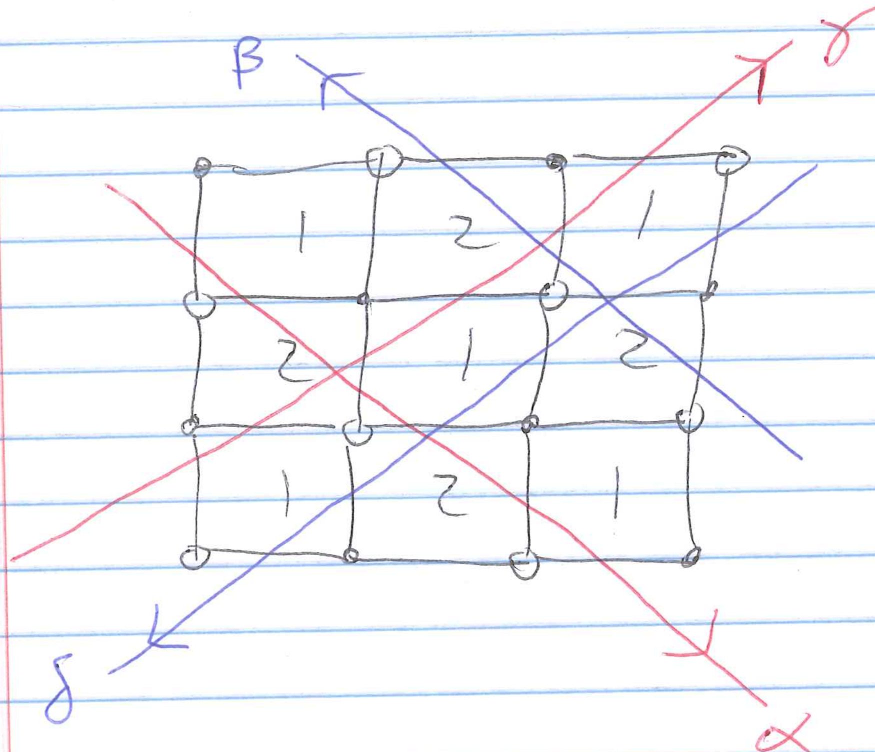
$W = acbd - adbc$

is <u>geometrically consistent</u>:

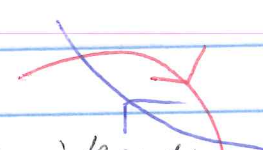
4 parallel zig-zag paths



4/8/15 (8) As alternating strand diagrams, there can actually straighten-out to geometric lines



Rem: if we had tweaked the zig-zag paths differently, we would have seen



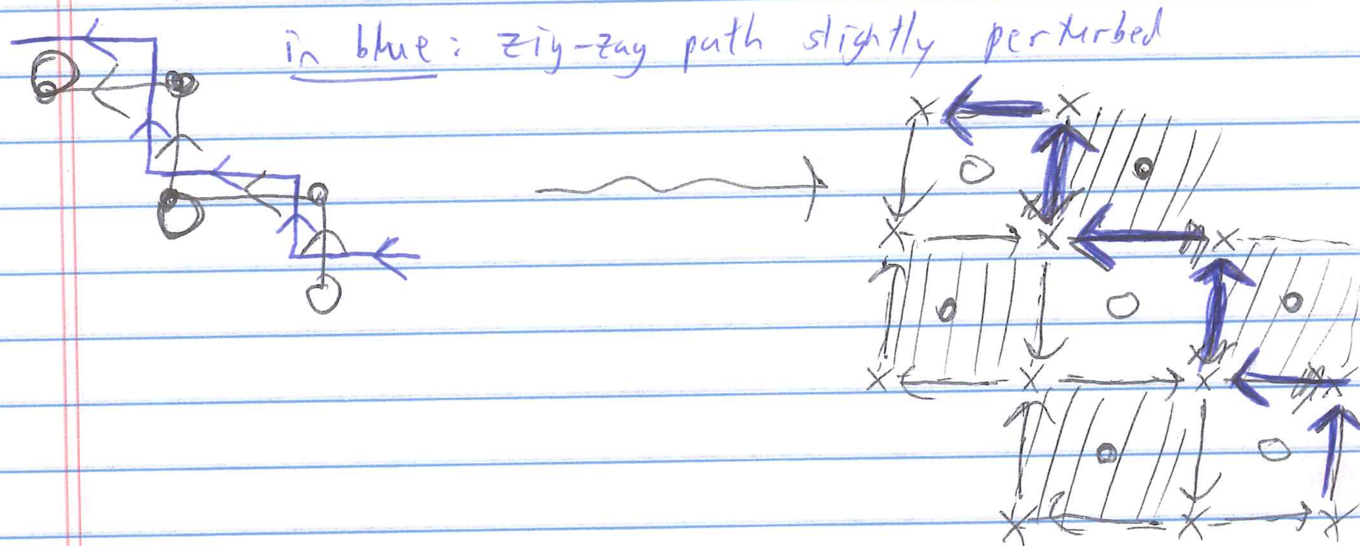
For our purposes, such a double-intersection is allowed, but

is not.

Rem: In terms of train tracks, global consistency $\iff$
- no train track self-intersects
- no two train tracks have more than 1 intersection.

4/8/15 (9) we note that on the level of the unfolded periodic quiver (rather than the bipartite tiling), the zig-zag paths look like

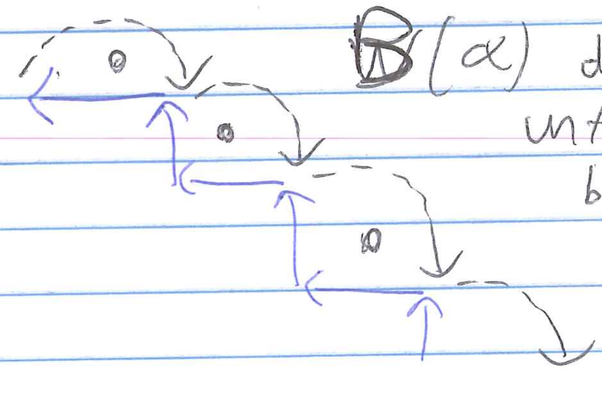
in blue: zig-zag path slightly perturbed



Thus it makes sense to define

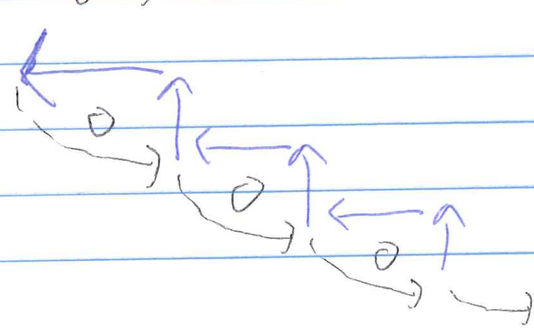
- white boundary flow $W(\alpha)$
- & black boundary flow $B(\alpha)$

of a zig-zag path α.



$B(\alpha)$ defined as path in unfolded quiver traversing black faces incident to zig-zag path w/o using the two arrows of the ZZ path

$W(\alpha)$ defined analogously.

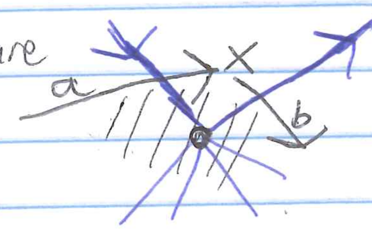


Rem: In homology,
$[W(\alpha)] = [B(\alpha)] = -[\alpha]$ in $H_1(\Sigma; \mathbb{Z})$.

4/8/15 (10) For the time being, we will consider zig-zag paths on the unfolded quiver rather than the bipartite tiling.

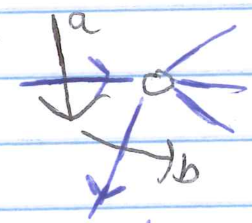
Claim: A zig-zag path does not intersect its white boundary flow nor its black boundary flow. (By intersect, we mean "share an arrow with.")

To see this, we show that if zig-zag path α shares an arrow with the boundary of a face $f \in Q_2$, then it must share exactly two arrows, either ab



maximal left

or



maximal right

Broomhead calls these two arrows zig-zag pair OR zag-zig pair.

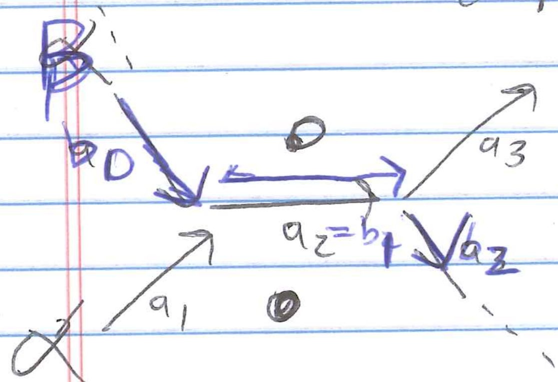
Firstly, if arrow a in zig-zag path α borders $f \in Q_2 \longleftrightarrow$ black vertex then ab goes around • clockwise (b being the next step of α) while bc (c being double-deck) would go around b's ○-neighbor counterclockwise so out of that local config, only steps ab border f.

The question is if α can come back to face f again later.

4/8/15 (11) Assume α contains $\vec{a_1}\ \vec{a_2} \cdots \vec{a_k}\ \vec{a_{k+1}}$ bordering black face f [only these 4 arrows]

with distance $k \geq 2$ assumed to be minimal.

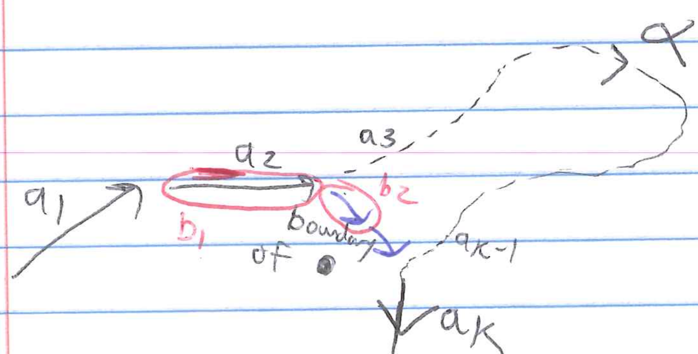
Let β be the unique zig-zag path using arrow $\vec{a_2} = \vec{b_1}$ and continuing w/ the other orientation



i) a_k can't be a_1 or a_2, because geometric consistency $\Rightarrow$ α has no self-intersection

ii) Further a_k can't be b_2 because geometric consistency $\Rightarrow$ α & β do not intersect twice.

Thus around $\bullet f$ we see



compare

$q = a_3 a_4 \cdots a_{k-1}$

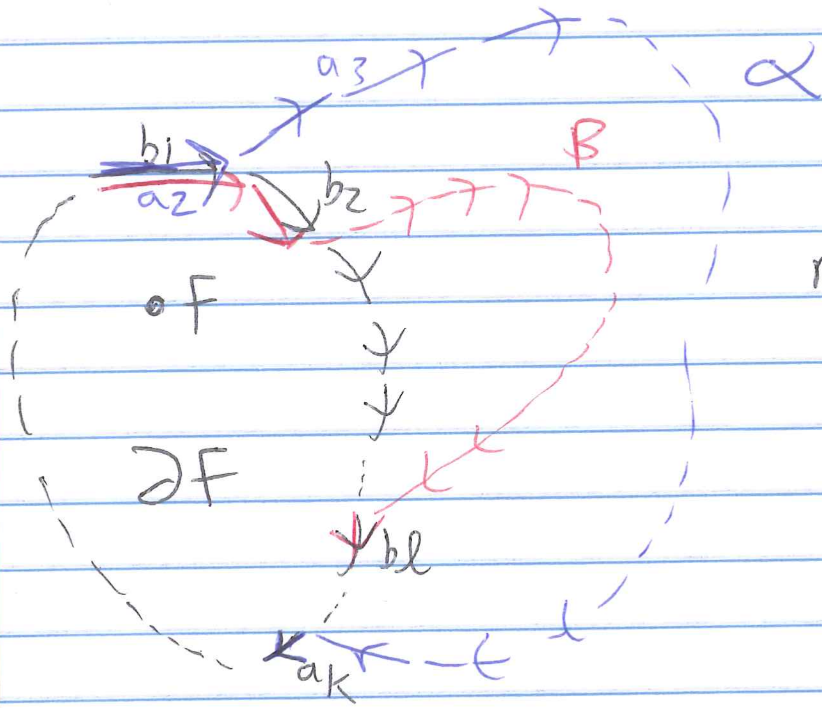
to $p = \underbrace{b_2 c_3 \cdots c_m}_{\text{boundary of } \bullet}$

pq^{-1} intersects β in b_2

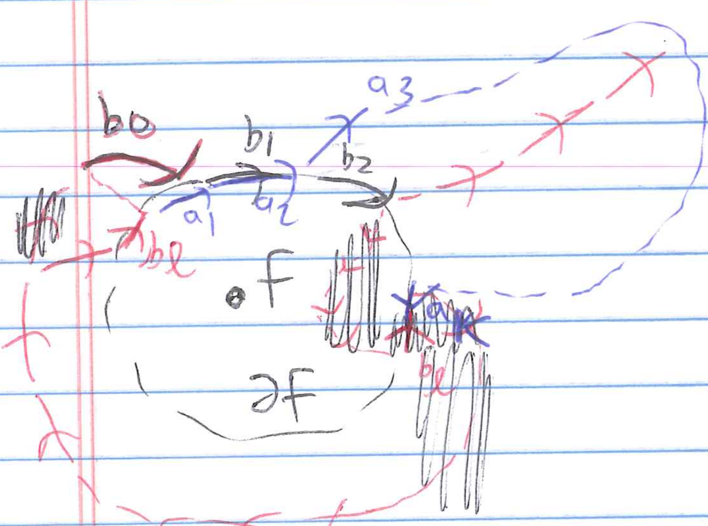
we claim that $\Rightarrow$ pq^{-1} must intersect β in a second arrow.

α & β cannot intersect in q $\Rightarrow$ β must intersect with p.

4/8/15 (12) However then we can essentially repeat the above argument:

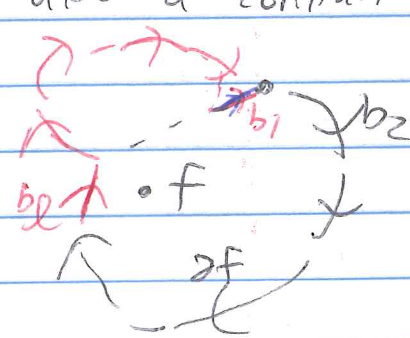


not allowed by minimality of k



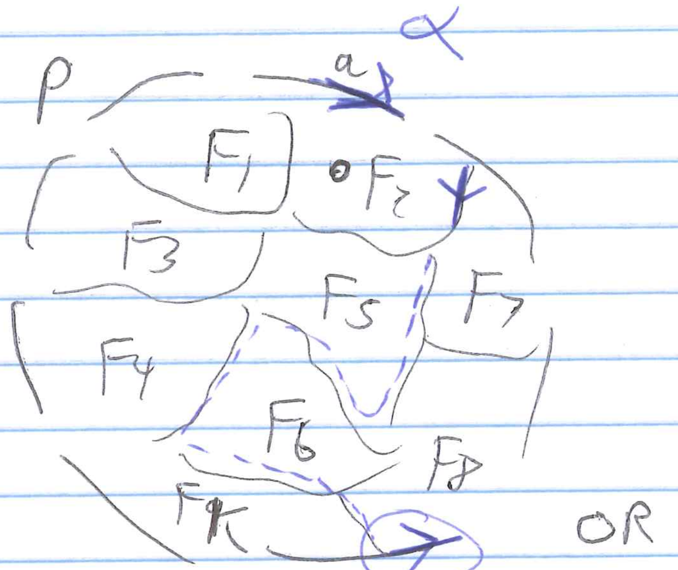
α & loop only intersect in $a_2 = b_1$

also a contradiction



4/8/15 (13) Finally, we need that if p is a (finite) simple closed path and p intersects zig-zag path α in one arrow, then p & α must intersect in another arrow as well:

p borders a certain finite number of faces in plane



if α & p intersect at arrow a, then α turns and cuts between two faces inside of p before or after

OR immediately continues along p

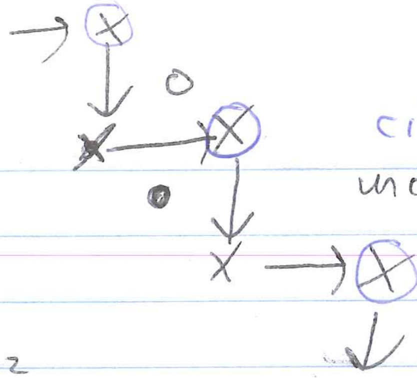
There are only a finite # arrows inside and α does not intersect itself $\Rightarrow$ α must eventually exit and will again border p as it does.

Consequences: Going back to page 9

Left hand side & right hand sides of zig-zag path α well-defined.

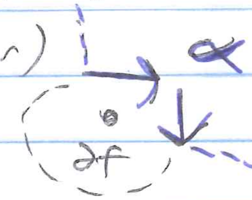
Splits unfolded periodic quiver into two halves. vertices i, j equiv $\Leftrightarrow$ $\exists$ finite path from i to j not intersecting α.

4/8/15 (14) we also say



Given a face $\in Q_2$ in unfolded periodic quiver define $X(F) = \{\alpha \mid \alpha$ zig-zag path intersecting boundary of $f\}$

we proved earlier that such α's intersect in a single zig-zag (or zag-zig pair)

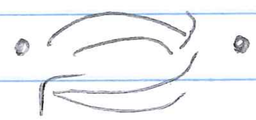


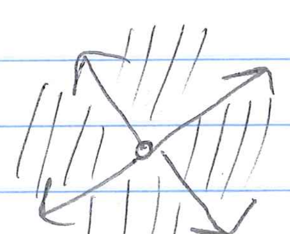
each $\alpha \mapsto [\alpha] \in H^1(Y;\mathbb{Z})$. (nonzero)

<u>Def:</u> <u>Local zig-zag fan</u> at face f is complete fan of strongly convex rational polyhedral cones in $H^1(Y;\mathbb{Z})$ whose rays are generated by the $[\alpha]$'s in $X(f)$

<u>Def:</u> Global zig-zag fan generated by all $[\alpha]$'s <u>in $X(f)$'s for all faces</u> f of unfolded quiver. (Although can think of finite quiver now since faces with same labels give same homologies.)

e.g.





<u>Construction:</u> Given a <u>$\overset{\text{2-dim}}{\text{cone}}$</u> σ of the <u>global</u> <u>zig-zag fan</u>, for a given face f, <u>local zig-zag fan</u> is a coarsening $\Rightarrow \sigma$ contained in some $\overset{\text{2-dim}}{\text{cone}}$ σ_f.

$\sigma_f \longleftrightarrow$ unique arrow in ∂f (zig-zag pair)

4/8/15 (15) $\sigma_F \longleftrightarrow$ unique arrow $a \in \partial F$

$P_F(\sigma) \in \mathbb{Z}^{Q_1} :=$ char func $= \begin{cases} 1 & \text{on } a \\ 0 & \text{o.w.} \end{cases}$

Def: $P(\sigma) := \frac{1}{2} \sum_{F \in Q_2} P_F(\sigma)$.

Claim: $P(\sigma)$ is a perfect matching.

Pf: Any arrow a is in the boundary of exactly 2 faces $\{f_W, f_B\}$

$P(\sigma)(a) = \frac{1}{2} P_{f_B}(\sigma)(a) + \frac{1}{2} P_{f_W}(\sigma)(a)$

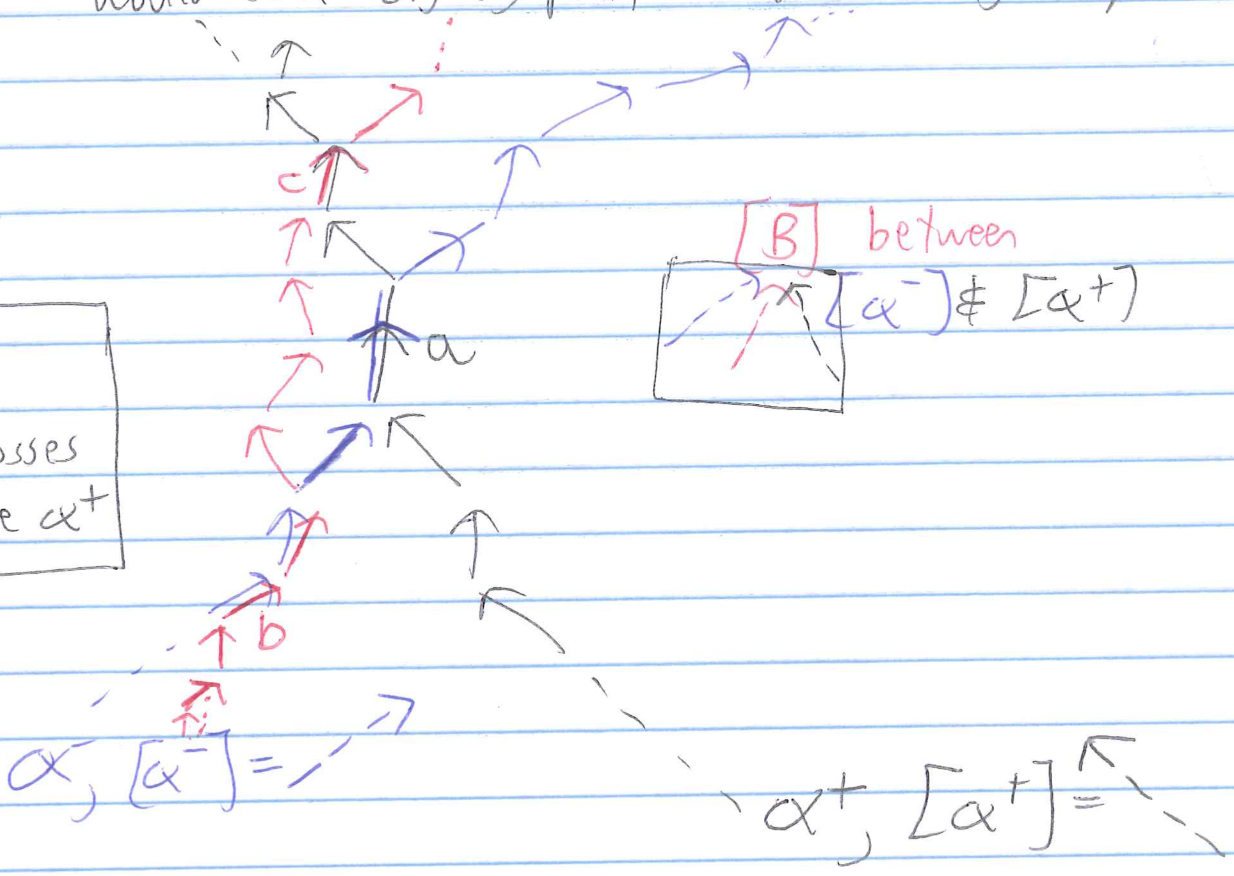
Let α^+ and α^- be the 2 zig-zag paths through arrow a (up to parallel translations of fund. domain)

Let $\mathcal{E}(f_B)$ be the local zig-zag fan of face f_B
$\mathcal{E}(f_W)$ " " f_W

Since $X(f_B)$ and $X(f_W)$ both contain α^+ & α^-
$\mathcal{E}(f_B)$ and $\mathcal{E}(f_W)$ both contain the rays generated by $[\alpha^+]$ & $[\alpha^-] \in H^1(Y;\mathbb{Z})$.

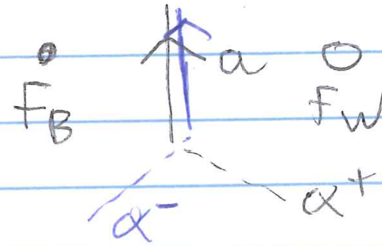
4/8/15 (16) Furthermore, $[\alpha^+]$ & $[\alpha^-]$ (rays associated to) span 2-dim cones in $\mathcal{E}(f_B)$ and $\mathcal{E}(f_W)$ since otherwise there would be a zig-zag path β with this global picture

W.l.o.g., say β crosses α^- before α^+



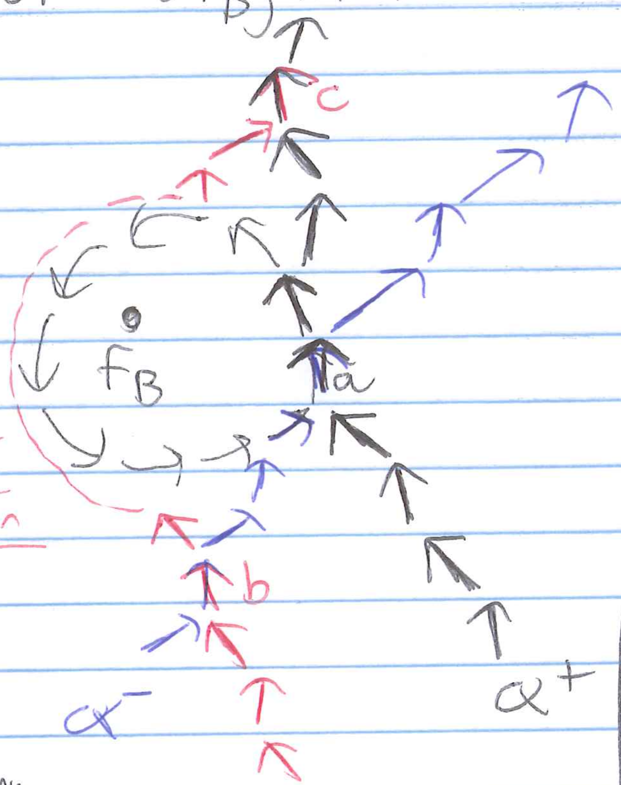
Since β not parallel to α^+ nor α^-, must intersect both, say in b and c.

However $\mathcal{E}(f_B)$ and $\mathcal{E}(f_W)$ only will contain ray $\leftrightarrow [\beta]$ if $\beta \in X(f_B)$ (resp. $X(f_W)$), i.e. if β intersects boundary of f_B (resp. f_W).



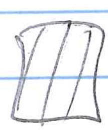
4/8/15 17 since ∂F_W is to the right of both α^+ & α^-,
and β in the vicinity of arrow a,
impossible for β to intersect ∂F_W
$\Rightarrow [\beta]$ not in $\mathcal{E}(F_W)$.

For ∂F_β, this is to the left of both α^+ & α^-

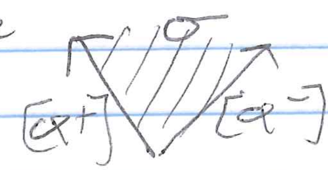


Could not intersect ∂F_β in correct direction between b & c

even intersecting in wrong direction would push β to the right of α^- & would then stay to the right of α^- & cross α^+ before a.



So $\mathcal{E}(F_\beta)$ & $\mathcal{E}(F_W)$ both contain 2-dim cone



determined by arrow a
"dual to a"

$\Rightarrow \mathcal{E}(F_\beta) \to G\mathbb{Z}\mathbb{Z}$-fan
$\sigma \longmapsto$ refinement

$\mathcal{E}(F_W)$
$\sigma \longmapsto$ " "

same maps

$$\Rightarrow \boxed{\frac{1}{2}\sum_{F \in Q_2} P_F(\sigma) = \sum_{\substack{F \in Q_2 \\ F \text{ black}}} P_F(\sigma)}$$

4/8/15

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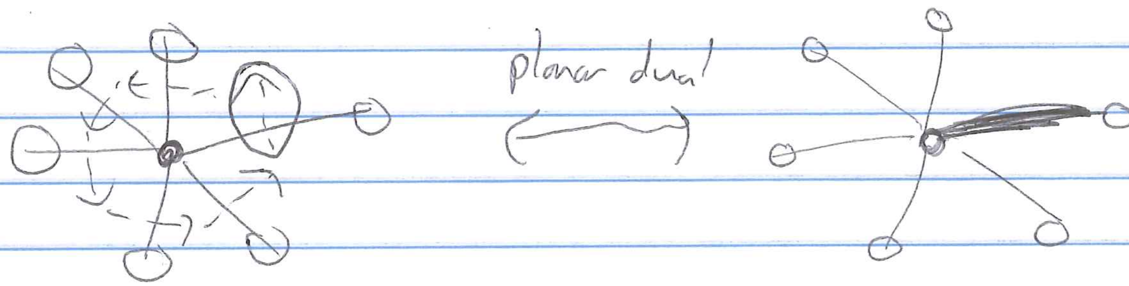
So we restrict our attention to black faces.

$$P_f(\sigma)\Big|_{\partial f} = \begin{cases} 1 & \text{on a single arrow } a \leftrightarrow \sigma_f \\ 0 & \text{every other arrow} \end{cases}$$

$\Rightarrow$ $\boxed{P(\sigma)} = \sum_{\substack{f \in Q_2 \\ f \text{ black}}} P_f(\sigma)$ evaluates to 1 on a single arrow in boundary of each face

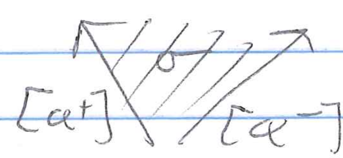
e.g.



Becomes a perfect matching.

Claim: For every σ in Global zig-zag fan, $P(\sigma)$ is an extremal P. M. and extremal P. M's are exactly the images of such $P(\sigma)$'s for all σ's.

Firstly, consider the picture



we show $P(\sigma)$ is the unique perf. matching s.t.

$P(\sigma)$(black boundary flow α^+) = $P(\sigma)$(white b. f. α^+)

= $P(\sigma)$(black b. f. α^-) = $P(\sigma)$(white b. f. α^-) = 0.

4/8/15 (19) By construction, $P(\sigma)(a) = \begin{cases} 1 & \text{for unique } a \leftrightarrow \sigma_f \text{ in } \partial f \\ 0 & \text{o.w.} \end{cases}$

$\alpha^+ \cap \partial f$ = zig-zag or zag-zig pair

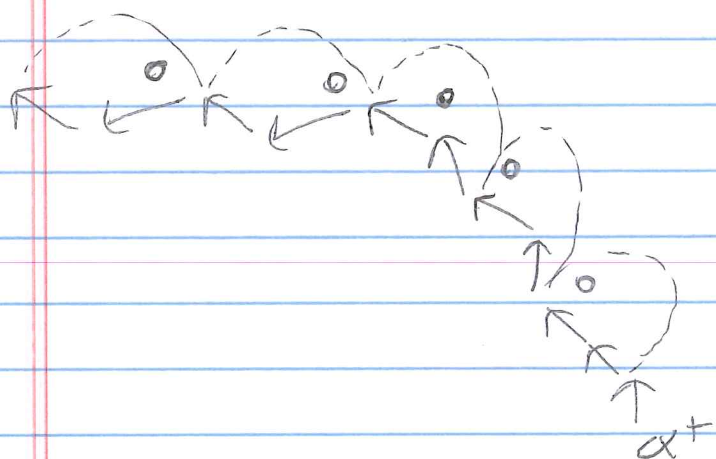
& black/white boundary flow contains remainder of ∂f

similar for α^-.

$\Rightarrow$ $P(\sigma)$ indeed evaluates to zero as desired.

To see that $P(\sigma)$ is the unique P.M. with this property, we note that α^+ & α^- are not parallel so their black boundary flows have at least one vertex in common.

I omit rest of the argument today.



Cor: image of $P(\sigma)$ in $N_0^+ = 0$ on two lin ind classes of homology. $\Rightarrow$ lies on zero-dim facet $\Rightarrow$ extremal.

Uniqueness of this $P(\sigma)$ $\Rightarrow$ multiplicity $= 1$.

Rem: Using global zig-zag fan, real cone $(N_0^+)_{\mathbb{R}}$ generated over $\mathbb{R}^+$ by images in N_0^+ of $P(\sigma)$ for all 2-dim cones σ.