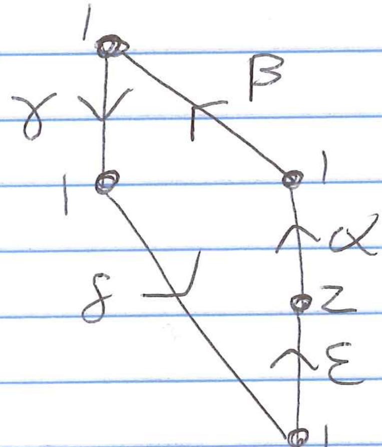
4/15/15 Lecture 23: Toric Duality, Seiberg Duality, and Mutation of quivers with potentials

First: one more note on SPP e.g.

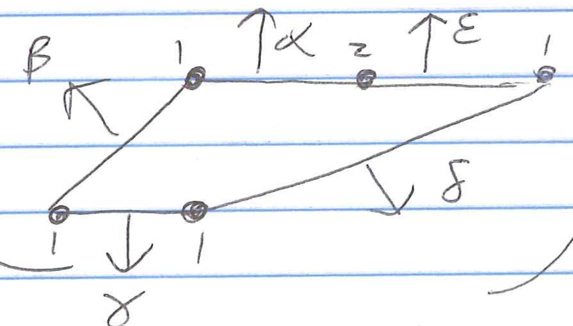
$[\varepsilon]=[\alpha]=(0,1)$, $[\beta]=(-1,1)$, $[\gamma]=(0,-1)$, $[\delta]=(1,-2)$

In Goncharov-Kenyon point of view, these $[\eta]$'s correspond to primitive sides of toric diagram Δ.

e.g.



(Rem: In Hanany-Vegh set-up, normal vectors



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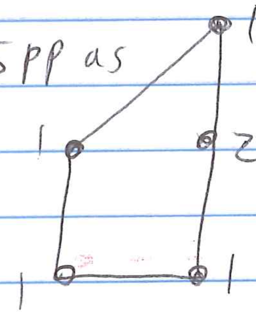
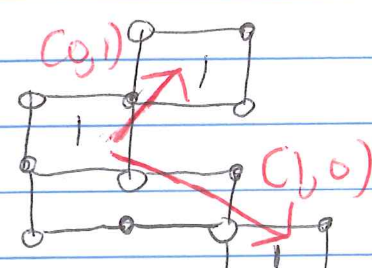
In the past, we drew Δ for SPP as

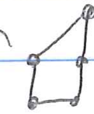


which agrees with above up to $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ & letting bottom right corner $=(0,0)$.

Rem: If we would have defined fundamental coordinates instead as



, we would have gotten $\Delta=$



from $[\alpha],\ldots,[\varepsilon]$ directly.

4/25/15 2 In general, external (& non-extremal) perfect matchings arise when distinct zig-zag paths have the same homology coordinates, e.g. $[\alpha]=[\varepsilon]$.

Let us now take a journey back to 2000!

"D-Brane Gauge Theories from Toric Singularities and Toric Duality" by Feng-Hanany-He (arXiv:0003085)

<u>Forward Procedure:</u> Gauge Theories $\rightarrow$ Toric Data

Input = Quiver Q and Superpotential W

Ouput = matrix $G_t = \begin{pmatrix} a_1 & a_2 & & a_m \\ b_1 & b_2 & \cdots & b_m \\ 1 & 1 & & 1 \end{pmatrix}$ where

$(a_1, b_1), \ldots, (a_m, b_m)$ are the lattice points in the toric diagram with possible repetition.

(see Lectures 19-20)

<u>Inverse Procedure:</u> Toric Data $\rightarrow$ Quiver Gauge Theory

1) Embed $\triangle$ into a triangular toric diagram, which corresponds to an orbifold $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ via $(\omega, \omega^{-1}, 1)$, $(1, w, w^{-1})$ for the two generators of $G = \mathbb{Z}_n \times \mathbb{Z}_n$.

4/15/15 (3)

2) Fine-tune FI (Fayet-Iliopoulos) parameters

Recall that in the Forward Algorithm, we let

$\{p_1, \dots, p_m\}$ = perfect matchings of biparitite tiling,

$M = |Q_1| \times m$ incidence matrix (of edges in perfect matchings)

$Q_F = (\operatorname{Ker} M)^T$, $Q_D = |Q_0| \times m$ matrix whose rows $\leftrightarrow$ cycles around each face in bipartite tiling

$G_t := \left(\operatorname{Ker}\begin{bmatrix} Q_F \\ \hline Q_D \end{bmatrix}\right)^T$, i.e. solutions to $Q\vec{x} = \vec{0}$ (written as row vectors)

We now instead consider solutions to $Q\vec{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hline \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}$ where $n = |Q_0| = \#$ faces in the bipartite tiling

(or we can let a subset of the ξ_i's still equal zero)

e.g. $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ has

$$Q = \begin{array}{c} \begin{array}{ccccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 \end{array} \\ \left[\begin{array}{ccccccccc} 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ \hline 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \end{array}$$

Consider the solution space

$$Q\vec{x}^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \hline \xi_1 \\ 0 \\ 0 \end{bmatrix}$$

one such solution is

$$\begin{array}{c} \begin{array}{ccccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 \end{array} \\ \left[\begin{array}{ccccccccc} 0 & 0 & 0 & 0 & 0 & \xi_1 & \xi_1 & 0 & \xi_1 \end{array}\right] \end{array}$$

4/15/15 (4) This is called "giving non-zero VEV's" to perfect matchings P_6, P_7, and P_9 (vacuum expectation values)

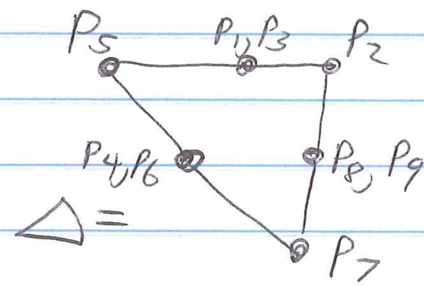
note that this solution was not unique

e.g. $\begin{array}{ccccccccc} & & & P_4 & & & P_7 & P_8 & \\ \end{array}$
$\left[0 \ 0 \ 0 \ -r_1 \ 0 \ 0 \ r_1 \ -r_1 \ 0 \right]$ also satisfies

$$Q\vec{x}^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \hline r_1 \\ 0 \\ 0 \end{bmatrix}$$ and would given non-zero VEV's to P_4, P_7, and P_8

Moral: Original columns of $G_t = (\ker Q)^T$ were

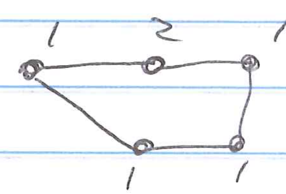
$$\begin{array}{ccccccccc} P_1 & P_2 & P_3 & P_4 & P_5 & P_6 & P_7 & P_8 & P_9 \end{array}$$
$$\begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \longleftrightarrow \Delta =$$



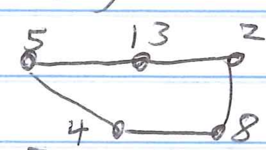
"Turning on FI parameter" and Producing "non-zero VEV's"

leads to deletion of columns P_6, P_7, P_9 (alternatively P_4, P_7, P_8)

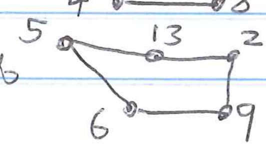
Either way, we get new toric diagram (agrees with that of SPP e.g.)



$r_1 > 0$ & $r_2 = r_3 = 0$ leads to



$r_1 < 0$ (i.e. $-r_1 > 0$) & $r_2 = r_3 = 0$ leads to

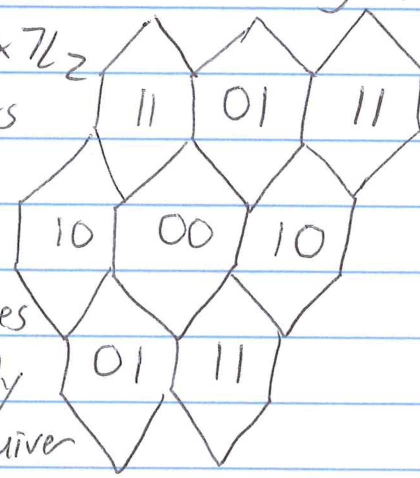


4/15/15 ⑤ Back in 2000 (if I understand correctly),

it would be necessary to solve for FI-parameters whose deletions would yield desired toric diagram

then matrices Q_F, Q_D and related matrices could be used to bootstrap quiver Gauge theory (i.e. F-terms and D-terms) from theory with certain nonzero FI parameters. $\Rightarrow$ (quiver and superpotential)

Rem: With modern point of view, taking dimer model for $\mathbb{C}^3/\mathbb{Z}_2\times\mathbb{Z}_2$



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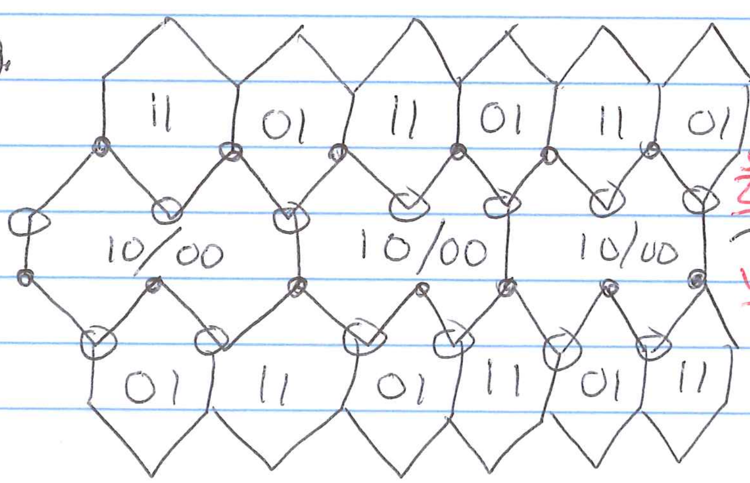
turning on FI parameters corresponds to

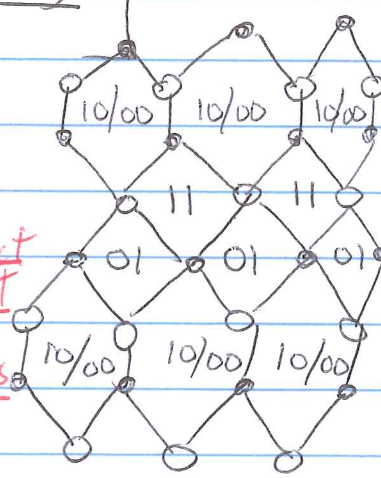
- deleting an edge/merging faces
  OR equivalently
- contracting an arrow of quiver

[Called partial resolution (of orbifold singularity) or Higgsing]

e.g.



contract 2-valent ~ vertices



SPP

4/15/15 6 There are a number of ambiguities where non-uniqueness creeps in during the Inverse Procedure:

- F-D ambiguity: If we wish to find matrix $Q_t = \begin{bmatrix} Q_F \\ Q_D \end{bmatrix}$ s.t. $(\text{Ker } Q_t)^T = G_t$ where $G_t = \begin{bmatrix} a_1 & \cdots & a_m \\ b_1 & & b_m \\ 1 & & 1 \end{bmatrix}$, ambiguous which matrix Q_t to use and how to split into rows of Q_F & rows of Q_D.

- Repetition ambiguity: ~~when~~ columns of G_t might repeat but we do not nec. see this from toric diagram as lattice polygon w/o multiplicities

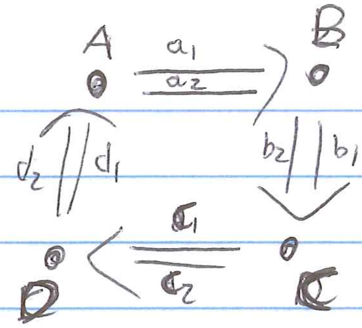
- Inverse algorithm involves computing nullspaces and choosing bases at many steps.

Def'n: Two quiver gauge theories (Q, W) are said to be toric dual to each other if they both have same toric diagram Δ (suppressing multiplicities). I.e. they are obtainable from Δ by the Inverse Procedure.

Physicists say "These two different gauge theories flow to the same universality class in the IR" (low energy)

Geometrically, they have same toric moduli space.

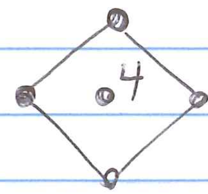
4/15/15 (7) Example



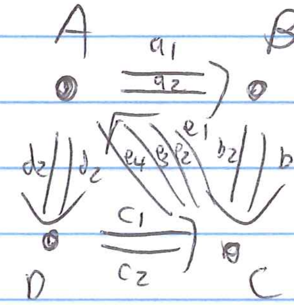
$$W = \epsilon^{ij}\epsilon^{kl} a_i b_k c_j d_l$$

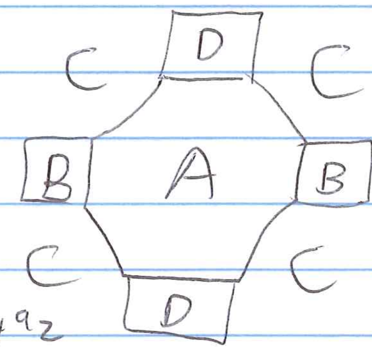
C	D	C	D
B	A	B	A
C	D	C	D
B	A	B	A

Toric Diagram is



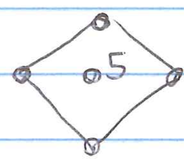
is toric dual to





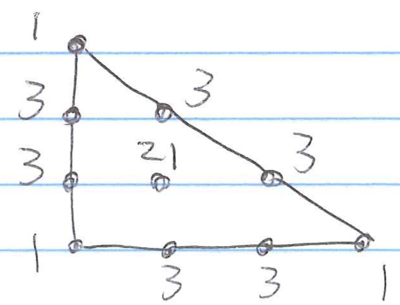
$$W = c_1 e_1 d_1 + c_2 e_2 d_2 + b_1 e_3 a_1 + b_2 e_4 a_2 - c_1 e_3 d_2 - c_2 e_4 d_1 - b_1 e_1 a_2 - b_2 e_2 a_1$$

with Toric Diagram



In "Phase Structure of D-brane Gauge Theories and Toric Duality" by Feng-Hanany-He (arXiv:0104259)

Toric Duality explained by applying partial resolutions to the orbifold theory $\mathbb{C}^3/\mathbb{Z}_3\times\mathbb{Z}_3$



We list examples up to $GL_2(\mathbb{Z})$ & translations

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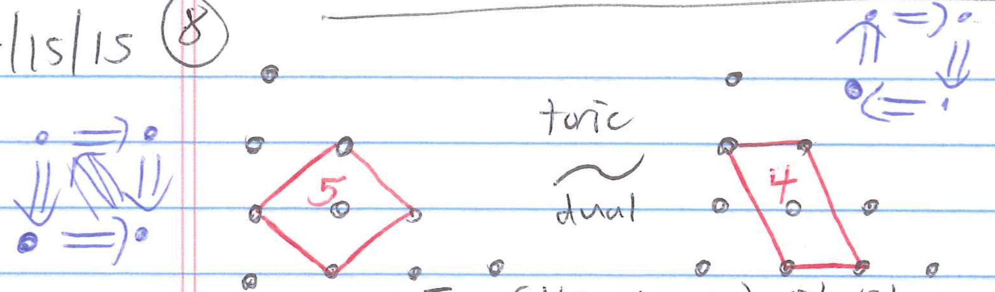
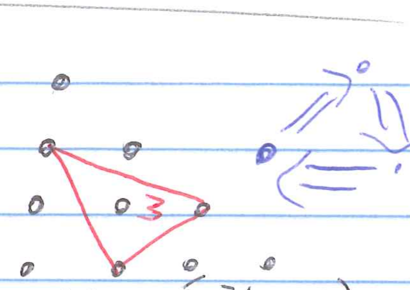
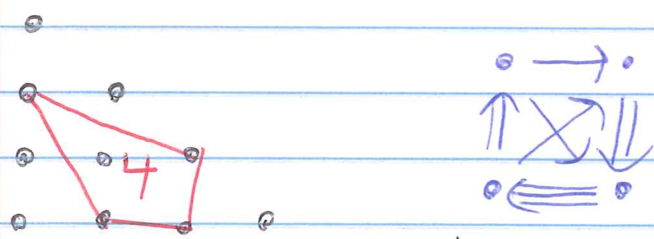


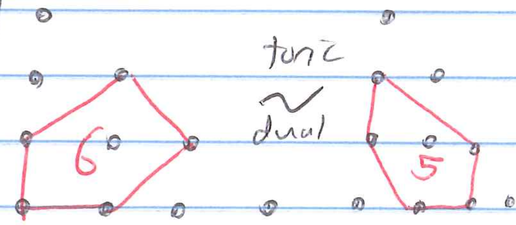
F_0 (Hirzebruch) $\mathbb{P}^1 \times \mathbb{P}^1$



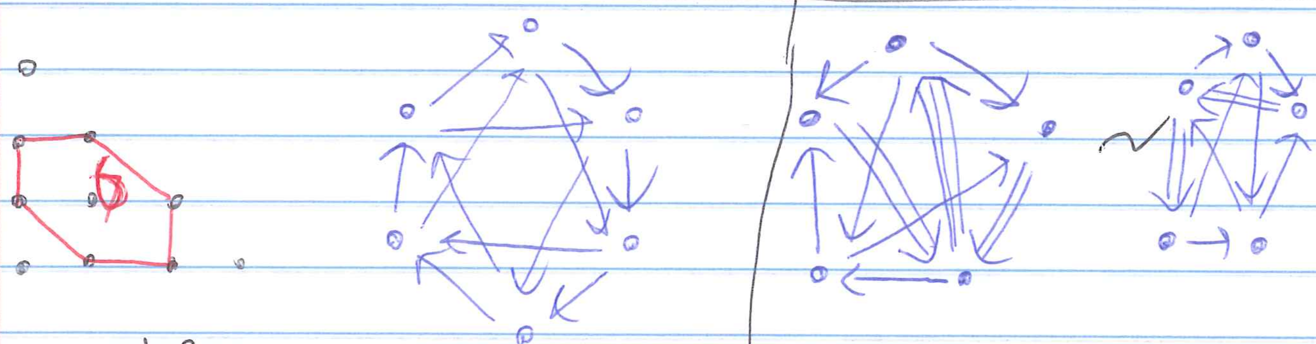
dP_0 $(\mathbb{C}^3/(1,1,1))$



dP_1 (Somos-4)



dP_2 ("Somos-5")



dP_3

New explanation in "Toric Duality is Seiberg Duality" by Beasley and Plesser (arXiv:0109053)

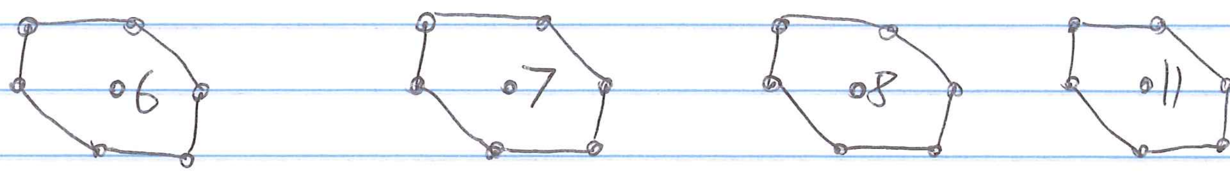
Important example: dP_3 (complex cone over a del Pezzo surface, i.e. a blow-up of $\mathbb{P}^2$ at 3 generic pts)

As above, this is a partial resolution of $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$.

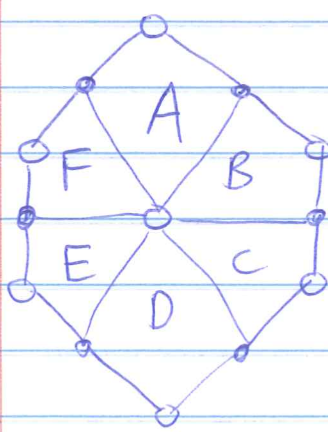
1602 cones in the space of FI parameters corresponding to partial resolutions leading to this singularity.

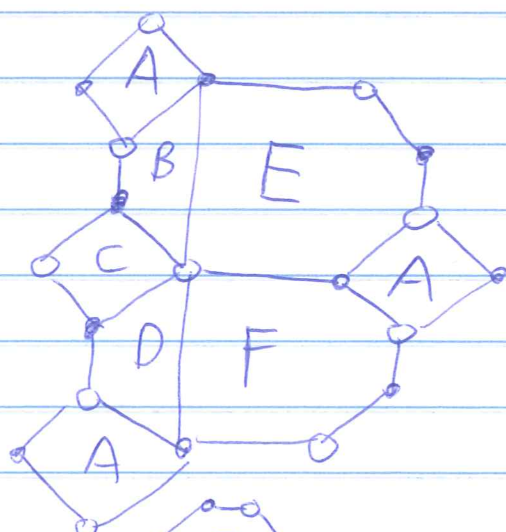
4/15/15 (9) Claim: Varying the FI parameters lead to four distinct $\mathcal{N}=1$ $SU(N)^6$ [6 vertices] gauge-theories that in the IR correspond to D3-branes transverse to this singularity. [Here $\mathcal{N}=1$ is the degree of supersymmetry. Since it is $\mathcal{N}=1$ rather than $\mathcal{N}=2$, quiver allowed to have directed arrows (i.e. "chiral fields")]

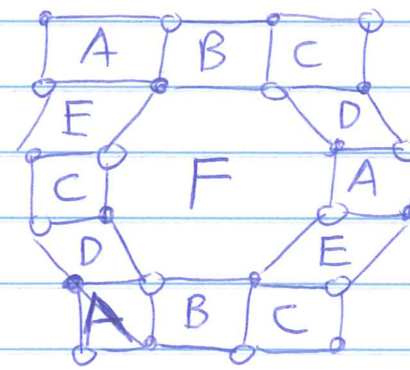
Four possible toric diagrams (w/ multiplicities)



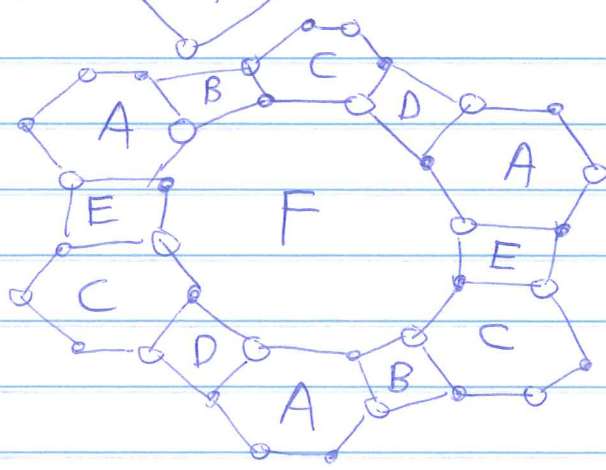
(superpotentials suppressed)







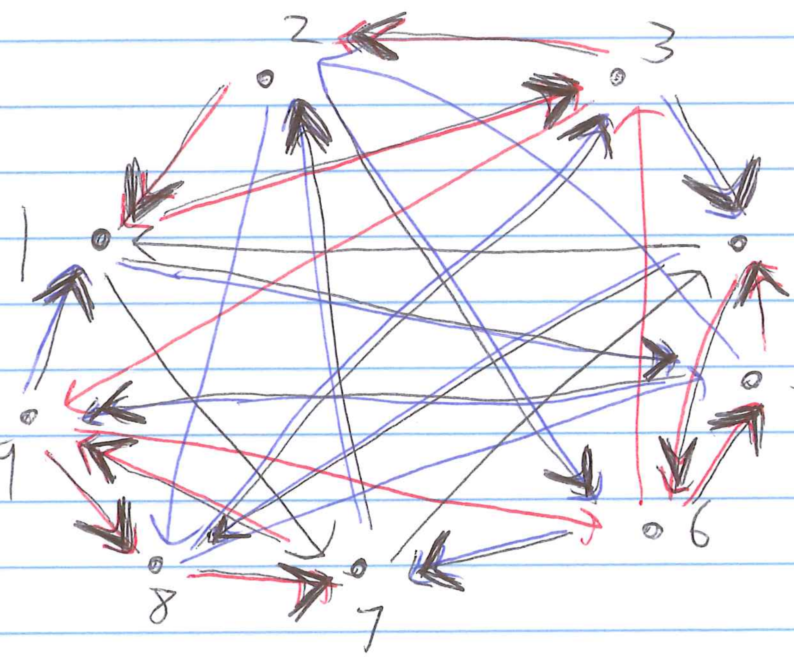
Modern Point of view in terms of dimers



4/15/15 (10) These four "toric phases" of dP_3 can all be obtained from $\mathbb{C}^3/\mathbb{Z}_3\times\mathbb{Z}_3$ by partial resolution (i.e. turning on FI parameters/contracting certain arrows)

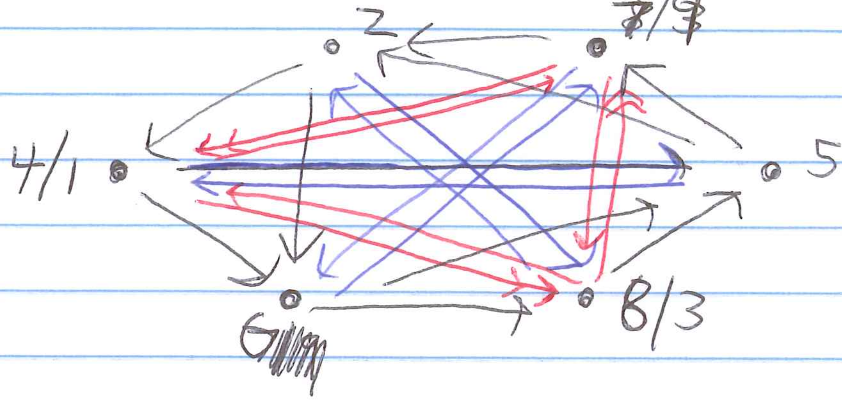
$$W = Z_{17}(X_{72}Y_{21} - Y_{79}X_{91}) + Z_{28}(X_{83}Y_{32} - Y_{87}X_{72})$$
$$+ Z_{39}(X_{91}Y_{13} - Y_{98}X_{83}) + Z_{41}(X_{15}Y_{54} - Y_{13}X_{34})$$
$$+ Z_{52}(X_{26}Y_{65} - Y_{21}X_{15}) + Z_{63}(X_{34}Y_{46} - Y_{32}X_{26})$$
$$+ Z_{74}(X_{48}Y_{87} - Y_{46}X_{67}) + Z_{85}(X_{59}Y_{98} - Y_{54}X_{48})$$
$$+ Z_{96}(X_{67}Y_{79} - Y_{65}X_{59})$$

$Q =$



(I) X_{83}, Y_{79}, Z_{41}	(II) X_{67}, Y_{87}, Z_{53}
(III) X_{15}, Y_{32}, Z_{17}	(IV) X_{26}, Y_{21}, Z_{28}

4/15/15 (11) Example (I):



$$\widetilde{W}_1 = Z_{47}(X_{72}Y_{24} - X_{74}) + Z_{28}(Y_{82} - Y_{87}X_{72})$$

$$+ Z_{87}(X_{74}Y_{48} - Y_{78} - X_{88}) + (X_{45}Y_{54} - Y_{43}X_{34})$$

$$+ \dots + Z_{76}(X_{67}Y_{77} - X_{65}X_{57})$$

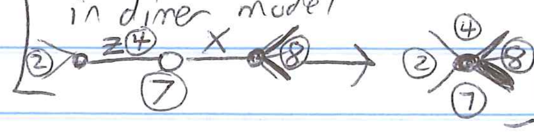
Since $\widetilde{W}_1$ has degree two terms corresponding to 2-cycles in the quiver, these are called "massive terms" and can be "integrated out".

e.g. $-Z_{47}X_{74}$ is a massive term. $+X_{74}Y_{48}Z_{87}$

cyclic rotation

$+Z_{47}X_{72}Y_{24}$ and $+Z_{87}X_{74}Y_{48}$

are the other two terms involving Z_{47} or X_{74}. Consequently, we delete 2-cycle $Z_{47}X_{74}$ from quiver and merge the other two terms involving Z_{47} or X_{74} as $\underline{+Y_{48}Z_{87}X_{72}Y_{24}}$. [in dimer model



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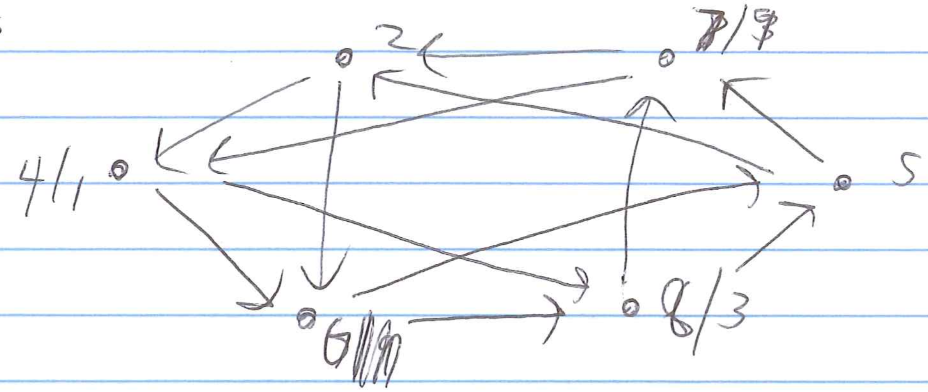
4/15/15 (12) Proceeding in this way for all 2-cycles / massive terms yields



and $W_1 = Y_{24} Y_{46} Z_{68} Z_{85} X_{57} X_{72}$

$+ X_{26} Y_{65} Z_{52} + X_{48} Y_{87} Z_{74}$

$- X_{72} X_{26} Z_{68} Z_{87} \cancel{X_{74}} - Y_{24} Y_{46} Y_{65} Z_{52}$

$- X_{48} Z_{85} X_{57} Z_{74}$

The other three toric phases can be obtained similarly.

However, as a big leap forward, Beasley & Plesser (Section 2.3) also describe how

"Seiberg $N=1$ duality" [Electric-magnetic duality in supersymmetric non-Abelian Gauge theories 1995], which relates Superconformal Quantum Chromodynamic (SQCD)-like theories flowing to the same conformal fixed point in the deep IR, can also explain the relationship between these four toric phases.

"Toric Duality" by Beasley and Plesser
is Seiberg Duality
arXiv: 0109053

If we evaluate these anomaly coefficients perturbatively in each of the four models[7], summing over the $6N^2$ gluinos of $\mathcal{R}$-charge 1 and two sets of $6N^2$ matter fermions of $\mathcal{R}$-charges $-\frac{2}{3}$ and $-\frac{1}{3}$, we see that $a = c$ and $c \sim 2N^2$.

If we perform the same calculation of the central charge in the $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ orbifold gauge theory, we find $c \sim 4N^2$. As observed by Gubser [19], in such a situation where we deform the orbifold gauge theory in the UV and flow to the IR dP_3 theory, we expect a relation from the AdS/CFT correspondence

$$\frac{c_{IR}}{c_{UV}} = \frac{1/\mathrm{Vol}(H)}{1/\mathrm{Vol}(S^5/(\mathbb{Z}_3 \times \mathbb{Z}_3))} \tag{2.18}$$

between ratios of central charges and respective volumes of horizons. We have just computed $c_{IR} : c_{UV}$ as $1 : 2$. In section 3, we will compute the ratio of volumes $\mathrm{Vol}(S^5/(\mathbb{Z}_3 \times \mathbb{Z}_3)) : \mathrm{Vol}(H)$, which we find to be $1 : 2$ as well.

2.3. $\mathcal{N} = 1$ duality in the four models.

In this section, we show that in fact the four models are related by the $\mathcal{N} = 1$ duality of [11]. This duality, of course, relates SQCD-like theories flowing to the same conformal fixed point in the deep IR. Moreover, it provides a mapping of chiral operators between the two models that can be used to map deformations of one into "dual" deformations of the other, preserving the property that the physics at extremely low energies is identical. What we mean by $\mathcal{N} = 1$ duality in the context of our models is the following. Consider, in any of the models, a factor of the gauge group under which a total of $2N$ chiral multiplets transform in the fundamental representation. If we deform the theory by turning off the superpotential couplings for these fields, as well as the gauge coupling for any other factors in the gauge group under which they are charged, we isolate a SQCD-like theory. In this deformed theory, [11] predicts an equivalent description in terms of "magnetic" variables transforming under a magnetic gauge group $SU(N)$ (hence the insistence on $2N$ fundamentals, since we wish to remain with quivers of this type). One can then turn on the remaining couplings, rewritten in terms of the magnetic variables, returning to a dual description of the original theory. One of the most striking properties of the $\mathcal{N} = 1$ duality is the fact that such mappings of deformations lead to agreement between two dual models. We will see that this is indeed the case.

[7] We believe that only in Model I is there a scale at which this perturbative analysis is valid, and it seems to be a coincidence that the other models agree.

So in order to apply the $\mathcal{N}=1$ duality to our models with a product gauge-group and superpotential, we start in Model I by going to a point[8] corresponding to a pure $SU(N)$ SQCD theory, for which all factors of the gauge-group but one, suppose the first, decouple and the superpotential is turned off. In this pure $SU(N)$ theory we apply the usual $\mathcal{N}=1$ transformation to obtain a dual description.

Thus we define composite mesonic fields (μ an arbitrary mass scale)

$$\begin{aligned} \mu M_{25} &= X_{21}X_{15}, \\ \mu M_{26} &= X_{21}X_{16}, \\ \mu M_{45} &= X_{41}X_{15}, \\ \mu M_{46} &= X_{41}X_{16}, \end{aligned} \tag{2.19}$$

and introduce the dual chiral fields $\tilde{X}_{12}$, $\tilde{X}_{14}$, $\tilde{X}_{51}$, and $\tilde{X}_{61}$. The dual $SU(N)$ SQCD model possesses a cubic superpotential

$$\lambda\Big\{\mathrm{tr}\Big[M_{46}\tilde{X}_{61}\tilde{X}_{14}\Big] - \mathrm{tr}\Big[M_{45}\tilde{X}_{51}\tilde{X}_{14}\Big] + \mathrm{tr}\Big[M_{25}\tilde{X}_{51}\tilde{X}_{12}\Big] - \mathrm{tr}\Big[M_{26}\tilde{X}_{61}\tilde{X}_{12}\Big]\Big\}. \tag{2.20}$$

The relative signs in (2.20) are fixed by the presence in the original SQCD theory of a global $SU(2)\times SU(2)$ symmetry under which X_{51} and X_{61} (resp. X_{12} and X_{14}) transform as doublets.

We now restore the original superpotential and the remaining gauge couplings. In the dual model this leads to the superpotential

$$\begin{aligned} \widetilde{W}_I = {} & h_1\mathrm{tr}\Big[X_{32}X_{26}X_{63}\Big] + h_2\mu\,\mathrm{tr}\Big[M_{45}X_{54}\Big] + h_3\mu\,\mathrm{tr}\Big[X_{34}X_{42}M_{26}X_{65}X_{53}\Big] \\ & - h_4\mu\,\mathrm{tr}\Big[M_{46}X_{63}X_{34}\Big] - h_5\mu\,\mathrm{tr}\Big[M_{25}X_{53}X_{32}\Big] - h_6\mathrm{tr}\Big[X_{54}X_{42}X_{26}X_{65}\Big] \\ & + \lambda\Big\{\mathrm{tr}\Big[M_{46}\tilde{X}_{61}\tilde{X}_{14}\Big] - \mathrm{tr}\Big[M_{45}\tilde{X}_{51}\tilde{X}_{14}\Big] + \mathrm{tr}\Big[M_{25}\tilde{X}_{51}\tilde{X}_{12}\Big] - \mathrm{tr}\Big[M_{26}\tilde{X}_{61}\tilde{X}_{12}\Big]\Big\}. \end{aligned} \tag{2.21}$$

We now integrate-out the massive fields M_{45} and X_{54} from (2.21) to obtain an equivalent effective potential for the low-energy dual theory

$$\begin{aligned} \widetilde{W}_I^{eff} = {} & h_1\mathrm{tr}\Big[X_{32}X_{26}X_{63}\Big] + h_3\mu\,\mathrm{tr}\Big[X_{34}X_{42}M_{26}X_{65}X_{53}\Big] \\ & - h_4\mu\,\mathrm{tr}\Big[M_{46}X_{63}X_{34}\Big] - h_5\mu\,\mathrm{tr}\Big[M_{25}X_{53}X_{32}\Big] - 2\frac{h_6\lambda}{h_2\mu}\,\mathrm{tr}\Big[\tilde{X}_{51}\tilde{X}_{14}X_{42}X_{26}X_{65}\Big] \\ & + \lambda\mathrm{tr}\Big[M_{46}\tilde{X}_{61}\tilde{X}_{14}\Big] + \lambda\mathrm{tr}\Big[M_{25}\tilde{X}_{51}\tilde{X}_{12}\Big] - \lambda\mathrm{tr}\Big[M_{26}\tilde{X}_{61}\tilde{X}_{12}\Big]. \end{aligned} \tag{2.22}$$

[8] Note that this point can only be reached via relevant deformations of the theory, and hence lies away from the critical manifold of interest. However, the dual SQCD theories will possess dual relevant deformations which may be used to reach the critical manifold.

We observe that $\widetilde{W}_I^{eff}$ is the superpotential for Model II and that the representation contents of the models agree. Thus deforming back to the original model, we have found that Model I is dual to Model II. We also observe that the fundamental chiral superfields Y_{26}, Y_{46}, and X_{25} in Model II arise under the duality from the mesonic fields M_{26}, M_{46}, and M_{25} in Model I. Under the inverse duality transformation, we of course obtain Model I again. In this case we find that the fundamental field X_{54} in Model I arises from a mesonic composite field in Model II.

We now consider the relation between Model II and Model III, with similar reasoning. In this case, we consider the SQCD point in the deformation space of Model II for which all $SU(N)$ factors decouple but the one labelled '5' and for which the superpotential is turned off. At this point, we again define the composite mesonic fields

$$\begin{aligned}
\mu\, M_{21} &= X_{25}X_{51}\,, \\
\mu\, M_{23} &= X_{25}X_{53}\,, \\
\mu\, M_{61} &= X_{65}X_{51}\,, \\
\mu\, M_{63} &= X_{65}X_{53}\,,
\end{aligned} \tag{2.23}$$

and the dual fields $\tilde{X}_{15}$, $\tilde{X}_{35}$, $\tilde{X}_{52}$, and $\tilde{X}_{56}$. The dual superpotential $\widetilde{W}_{II}$ then is

$$\begin{aligned}
\widetilde{W}_{II} = h_1\mathrm{tr}\Big[Y_{46}Y_{61}X_{14}\Big] + h_2\mu\,\mathrm{tr}\Big[M_{21}X_{12}\Big] - h_3\mathrm{tr}\Big[Y_{46}X_{63}X_{34}\Big] - h_4\mu\,\mathrm{tr}\Big[M_{23}X_{32}\Big] \\
+ h_5\mathrm{tr}\Big[X_{32}X_{26}X_{63}\Big] - h_6\mathrm{tr}\Big[X_{12}Y_{26}Y_{61}\Big] + h_7\mu\,\mathrm{tr}\Big[X_{34}X_{42}Y_{26}M_{63}\Big] - h_8\mu\,\mathrm{tr}\Big[X_{14}X_{42}X_{26}M_{61}\Big] \\
+ \lambda\Big\{\mathrm{tr}\Big[M_{23}\tilde{X}_{35}\tilde{X}_{52}\Big] - \mathrm{tr}\Big[M_{63}X_{35}X_{56}\Big] + \mathrm{tr}\Big[M_{61}X_{15}X_{56}\Big] - \mathrm{tr}\Big[M_{21}\tilde{X}_{15}\tilde{X}_{52}\Big]\Big\}\,.
\end{aligned} \tag{2.24}$$

Integrating-out the massive fields M_{21}, X_{12}, M_{23}, and X_{32} yields a low-energy effective superpotential

$$\begin{aligned}
\widetilde{W}_{II}^{eff} = h_1\mathrm{tr}\Big[Y_{46}Y_{61}X_{14}\Big] - h_3\mathrm{tr}\Big[Y_{46}X_{63}X_{34}\Big] + \lambda\mathrm{tr}\Big[M_{61}X_{15}X_{56}\Big] - \lambda\mathrm{tr}\Big[M_{63}X_{35}X_{56}\Big] \\
+ h_7\mu\,\mathrm{tr}\Big[X_{34}X_{42}Y_{26}M_{63}\Big] - h_8\mu\,\mathrm{tr}\Big[X_{14}X_{42}X_{26}M_{61}\Big] \\
+ 2\frac{h_5\lambda}{h_4\mu}\,\mathrm{tr}\Big[\tilde{X}_{35}\tilde{X}_{52}X_{26}X_{63}\Big] - 2\frac{h_6\lambda}{h_2\mu}\,\mathrm{tr}\Big[\tilde{X}_{15}\tilde{X}_{52}Y_{26}Y_{61}\Big]\,.
\end{aligned} \tag{2.25}$$

We again see that this effective superpotential is the superpotential of Model III and the representation contents agree. Further, we note that the fundamental superfields X_{61} and Y_{63} of Model III arise under the duality from the composite fields M_{61} and M_{63} of Model

II. Under the inverse duality transformation, the fields X_{32} and X_{12} in Model II arise as composite mesons in Model III.

Finally, to relate Model III to Model IV, we decouple all the $SU(N)$ factors of the gauge group in Model II except factor '2' and turn off the superpotential. At this point we define the composite mesonic fields

$$\begin{aligned} \mu\, M_{46} &= X_{42}X_{26}\,, \\ \mu\, M_{56} &= X_{52}X_{26}\,, \\ \mu\, N_{46} &= X_{42}Y_{26}\,, \\ \mu\, N_{56} &= X_{52}Y_{26}\,, \end{aligned} \tag{2.26}$$

and introduce the dual fields $\tilde{X}_{24}$, $\tilde{X}_{25}$, $\tilde{X}_{62}$, and $\tilde{Y}_{62}$. The corresponding dual superpotential is

$$\begin{aligned} \widetilde{W}_{III} = {} & h_1\mathrm{tr}\Big[X_{56}X_{61}X_{15}\Big] - h_2\mathrm{tr}\Big[X_{56}Y_{63}X_{35}\Big] + h_3\mathrm{tr}\Big[Y_{46}Y_{61}X_{14}\Big] - h_4\mathrm{tr}\Big[Y_{46}X_{63}X_{34}\Big] \\ & + h_5\mu\,\mathrm{tr}\Big[N_{46}Y_{63}X_{34}\Big] - h_6\mu\,\mathrm{tr}\Big[N_{56}Y_{61}X_{15}\Big] \\ & + h_7\mu\,\mathrm{tr}\Big[M_{56}X_{63}X_{35}\Big] - h_8\mu\,\mathrm{tr}\Big[M_{46}X_{61}X_{14}\Big] \\ & + \lambda\Big\{\mathrm{tr}\Big[M_{46}\tilde{Y}_{62}\tilde{X}_{24}\Big] - \mathrm{tr}\Big[M_{56}\tilde{Y}_{62}\tilde{X}_{25}\Big] + \mathrm{tr}\Big[N_{56}\tilde{X}_{62}\tilde{X}_{25}\Big] - \mathrm{tr}\Big[N_{46}\tilde{X}_{62}\tilde{X}_{24}\Big]\Big\}\,. \end{aligned} \tag{2.27}$$

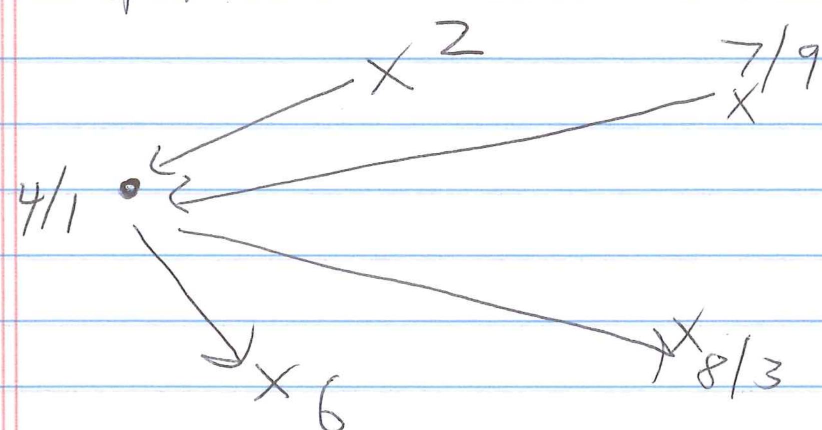
We identify $\widetilde{W}_{III}$ with the superpotential of Model IV and note that the representations also agree. We also note that in this case the fundamental fields Y_{56}, Z_{56}, X_{46}, and Z_{46} of Model IV have appeared from the composite fields M_{56}, N_{56}, M_{46}, and N_{46}. Under the inverse duality transformation, we reproduce Model III. None of the fundamental fields in Model III arise as composites in Model IV.

So, by going to various points at which we can apply the $\mathcal{N}=1$ duality, we see that the four models do describe the same family of conformal theories. In [15] the authors mention two other pairs of "toric dual" theories, describing D3-branes transverse to a cone over a del Pezzo surface dP_2 which is the blowup of $\mathbb{P}^2$ at two points and to a cone over a Hirzebruch surface $\mathbb{F}_0$ (better known as $\mathbb{P}^1 \times \mathbb{P}^1$). We will indicate how these theories are related via $\mathcal{N}=1$ duality in Section 5. We are led to conjecture that all instances of "toric duality" are in fact of this form.

4/15/15 (13) As described, start in Model I and

"a pure SU(N) SQCD theory for which all factors of the gauge-group but one... decouple and the superpotential is turned off"

<u>In other words</u>: isolate vertex 1 and its incident arrows in the quiver. Locally there are no cycles so can ignore/turn-off superpotential



"define composite mesonic fields (μ an arbitrary mass scale)"

$$\mu M_{28} = Y_{24} X_{48}$$

$$\mu M_{78} = Z_{74} X_{48}$$

$$\mu M_{26} = Y_{24} Y_{46}$$

$$\mu M_{76} = Z_{74} Y_{46}$$

"introduce dual chiral fields" $\tilde{X}_{42}, \tilde{X}_{84}, \tilde{Z}_{47}, \tilde{Y}_{64}$

4/15/15 (14) "dual SU(N) SQCD model possesses a cubic superpotential"

$$\lambda\Big\{ \mathrm{tr}\left[M_{28}\tilde{X}_{84}\tilde{Y}_{42}\right] - \mathrm{tr}\left[M_{78}X_{84}\tilde{Z}_{47}\right] - \mathrm{tr}\left[M_{26}\tilde{Y}_{64}\tilde{X}_{42}\right] + \mathrm{tr}\left[M_{76}Y_{64}\tilde{Z}_{47}\right]\Big\}$$

"the relative signs are fixed by the prescence in the original SQCD theory of a global SU(2)×SU(2) symmetry under which [certain arrows] transform as doublets"

"We now restore the original superpotential and the remaining gauge couplings,"

"We now integrate-out the massive fields... to obtain an equivalent effective potential for the low-energy dual theory",

---

Point 1: Seiberg duality is special case of mutation of quivers with potential [Derksen-Weyman-Zelevinsky] (that also works with 2-cycles and 1-cycles).

Point 2: In modern dimer teminology, corresponds to urban renewal / square move + contracting 2-valent vertices

Point 3: The toric phases related by Seiberg dualities.

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Description of Seiberg Duality (from physics)

From "**Brane Dimers and Quiver Gauges Theories (2005)** by Franco, Hanany, Kennaway, Wegh, and Wecht:

After picking a node to dualize at: "Reverse the direction of all arrows entering or exiting the dualized node. This is because Seiberg duality requires that the dual quarks transform in the conjugate flavor representations to the originals. ...

Next, draw in ... bifundamentals which correspond to composite (mesonic) operators. ... the Seiberg mesons are promoted to the fields in the bifundamental representation of the gauge group. ...

It is possible that this will make some fields massive, in which case the appropriate fields should then be integrated out."

Description of Seiberg Duality (rephrased combinatorially)

Pick a vertex j of the quiver Q (equiv. face of the brane tiling $\mathcal{T}_Q$) at which to mutate. Then, reverse the direction of all arrows incident to j, i.e. $A_{ij} \longrightarrow A^*_{ji}$. Next, for every two-path $i \longrightarrow j \longrightarrow k$, *"meson"*, in Q draw in a new arrow $i \longrightarrow k$, *"the Seiberg mesons are promoted to the fields"*. Let Q' denote this new quiver.

We similarly alter the superpotential W to get W'. For every 2-path $i \longrightarrow j \longrightarrow k$ in Q, we replace any appearance of the product $A_{ij}A_{jk}$ in W with the singleton A_{ik}, and add or subtract a new degree 3-term, $A_{ik}A^*_{kj}A^*_{ji}$.

It is possible, that this will make some of the terms of W' of degree two, *"massive"*, in which case there should be an associated 2-cycle in the mutated quiver Q' that can be deleted, *"the appropriate fields should then be integrated out"*.

This is in fact Mutation of Quivers with potential from cluster algebras (as defined by Derksen-Weyman-Zelevinsky)!

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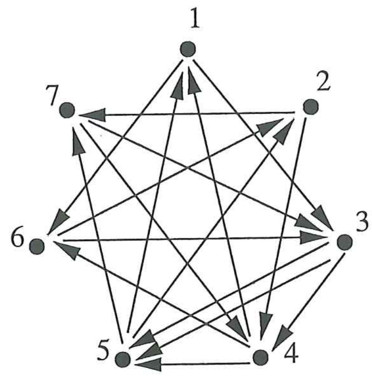
Description of Seiberg Duality (on the Brane Tiling)

In the special case, that we are mutating at a vertex with two arrows in and out, a **toric vertex**, this corresponds to a Urban Renewal of a square face in the brane tiling.

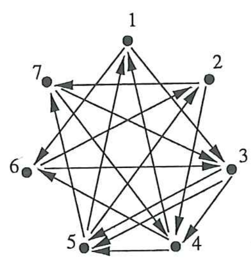


Example ($Q_7^{(2,3)}$): with potential

$$
\begin{aligned}
W &= A_{13}A_{34}A_{41} + A_{16}A_{63}A_{35}A_{51} + A_{35}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\
&- A_{16}A_{62}A_{24}A_{41} - A_{34}A_{46}A_{63} - A_{13}A_{35}A_{51} - A_{27}A_{73}A_{35}A_{52} - A_{45}A_{57}A_{74}.
\end{aligned}
$$

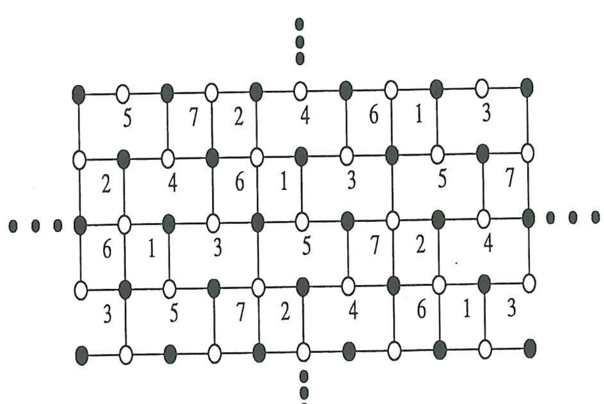
Consider the corresponding Brane Tiling $\mathcal{T}_7^{(2,3)}$ and mutation of (Q, W) at the toric vertex labeled 1. **(Associated to Gale-Robinson Sequence)**

Description of Seiberg Duality (on the Brane Tiling)



Example ($Q_7^{(2,3)}$): with potential

$$
\begin{aligned}
W &= A_{13}A_{34}A_{41} + A_{16}A_{63}A_{35}^{(V)}A_{51} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\
&- A_{16}A_{62}A_{24}A_{41} - A_{34}A_{46}A_{63} - A_{13}A_{35}^{(H)}A_{51} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74}.
\end{aligned}
$$



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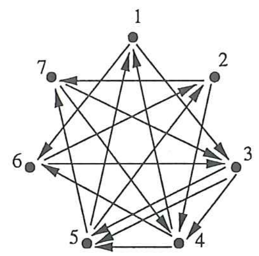
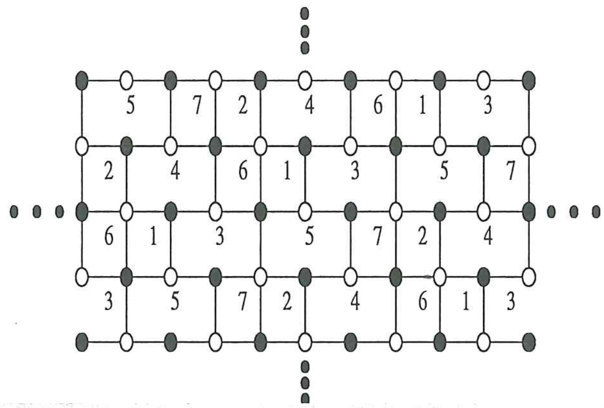
Description of Seiberg Duality (on the Brane Tiling)

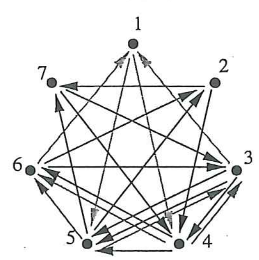


Example $(Q_7^{(2,3)})$: Rotate potential terms containing 1

$$\begin{aligned} W &= A_{41}A_{13}A_{34} + A_{51}A_{16}A_{63}A_{35}^{(V)} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\ &- A_{41}A_{16}A_{62}A_{24} - A_{34}A_{46}A_{63} - A_{51}A_{13}A_{35}^{(H)} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74}. \end{aligned}$$

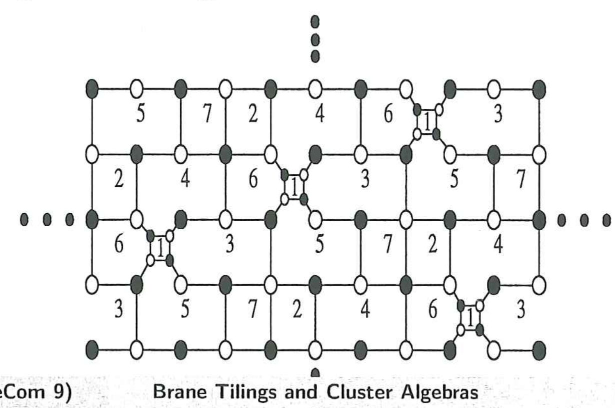


Description of Seiberg Duality (on the Brane Tiling)



Example $(Q_7^{(2,3)})$: Mutating at 1 yields

$$\begin{aligned} W' &= A_{43}A_{34} + A_{56}A_{63}A_{35}^{(V)} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\ &- A_{46}^{(D)}A_{62}A_{24} - A_{34}A_{46}A_{63} - A_{53}^{(H)}A_{35}^{(H)} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74} \\ &+ A_{14}^{*}A_{46}^{(D)}A_{61}^{*} + A_{15}^{*}A_{53}^{(H)}A_{31}^{*} - A_{14}^{*}A_{43}A_{31}^{*} - A_{15}^{*}A_{56}A_{61}^{*}. \end{aligned}$$



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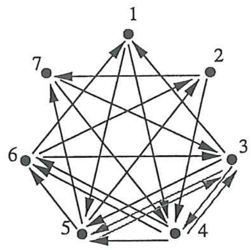
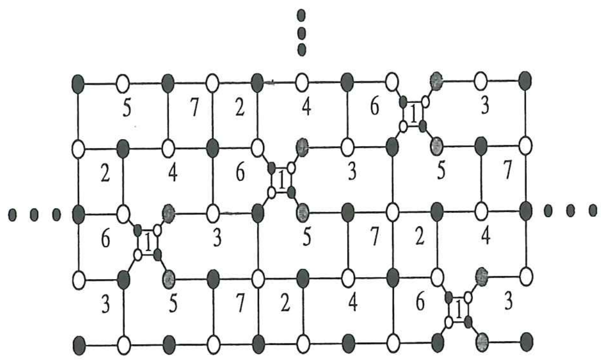
Description of Seiberg Duality (on the Brane Tiling)

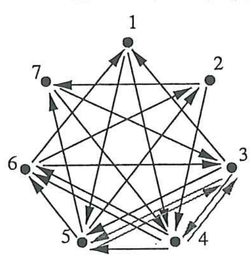


Example ($Q_7^{(2,3)}$): Highlighting Massive terms

$$\begin{aligned} W' &= A_{43}A_{34} + A_{56}A_{63}A^{(V)}_{35} + A^{(H)}_{35}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\ &- A^{(D)}_{46}A_{62}A_{24} - A_{34}A_{46}A_{63} - A^{(H)}_{53}A^{(H)}_{35} - A_{27}A_{73}A^{(V)}_{35}A_{52} - A_{45}A_{57}A_{74} \\ &+ A^*_{14}A^{(D)}_{46}A^*_{61} + A^*_{15}A^{(H)}_{53}A^*_{31} - A^*_{14}A_{43}A^*_{31} - A^*_{15}A_{56}A^*_{61}. \end{aligned}$$

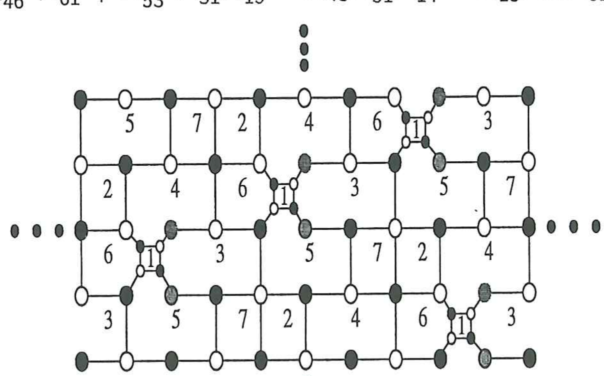


Description of Seiberg Duality (on the Brane Tiling)



Example ($Q_7^{(2,3)}$): Highlighting complementary terms

$$\begin{aligned} W' &= A_{43}A_{34} + A_{56}A_{63}A^{(V)}_{35} + A^{(H)}_{35}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\ &- A^{(D)}_{46}A_{62}A_{24} - A_{34}A_{46}A_{63} - A^{(H)}_{53}A^{(H)}_{35} - A_{27}A_{73}A^{(V)}_{35}A_{52} - A_{45}A_{57}A_{74} \\ &+ A^*_{14}A^{(D)}_{46}A^*_{61} + A^{(H)}_{53}A^*_{31}A^*_{15} - A_{43}A^*_{31}A^*_{14} - A^*_{15}A_{56}A^*_{61}. \end{aligned}$$



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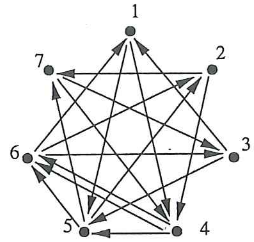
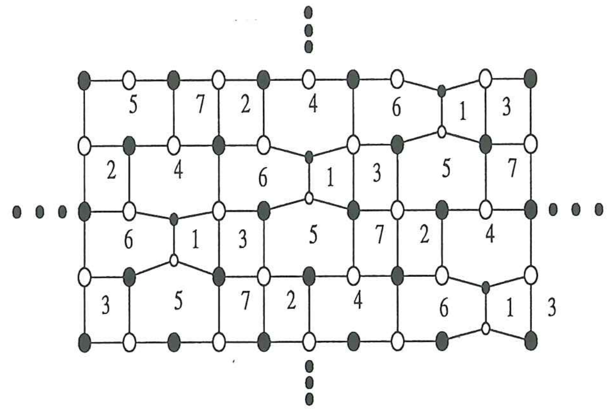
Description of Seiberg Duality (on the Brane Tiling)

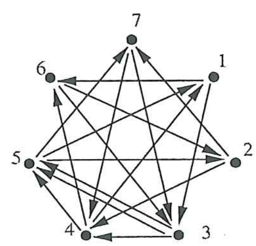


Example $(Q_7^{(2,3)})$: Reduces the potential to

$$\begin{aligned} W'' &= A_{56}A_{63}A_{35}^{(V)} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} - A_{46}^{(D)}A_{62}A_{24} - A_{27}A_{73}A_{35}^{(V)}A_{52} \\ &- A_{45}A_{57}A_{74} + A_{14}^{*}A_{46}^{(D)}A_{61}^{*} - A_{15}^{*}A_{56}A_{61}^{*} - A_{46}A_{63}A_{31}^{*}A_{14}^{*} + A_{31}^{*}A_{15}^{*}A_{57}A_{73}. \end{aligned}$$

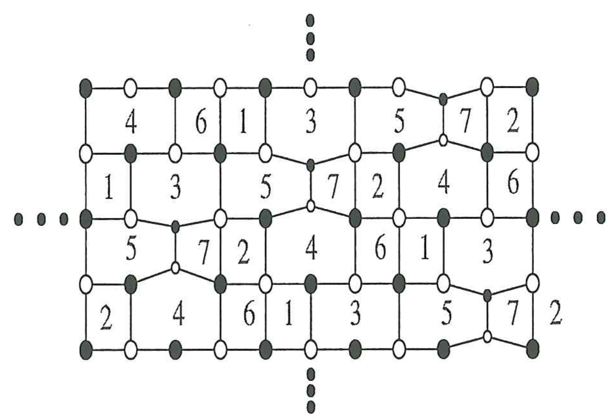


Description of Seiberg Duality (on the Brane Tiling)



Example $(Q_7^{(2,3)})$: If we cyclically permute vertices

$$\begin{aligned} W'' &= A_{45}A_{52}A_{24}^{(V)} + A_{13}A_{34}A_{41} + A_{16}A_{63}A_{35}A_{51} - A_{35}^{(D)}A_{51}A_{13} - A_{16}A_{62}A_{24}^{(V)}A_{41} \\ &- A_{34}A_{46}A_{63} + A_{73}^{*}A_{35}^{(D)}A_{57}^{*} - A_{74}^{*}A_{45}A_{57}^{*} - A_{35}A_{52}A_{27}^{*}A_{73}^{*} + A_{27}^{*}A_{74}^{*}A_{46}A_{62}. \end{aligned}$$



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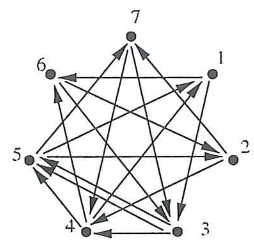
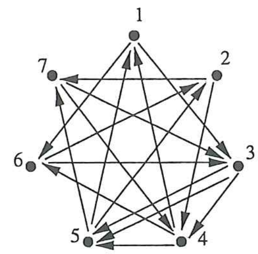
Description of Seiberg Duality (on the Brane Tiling)

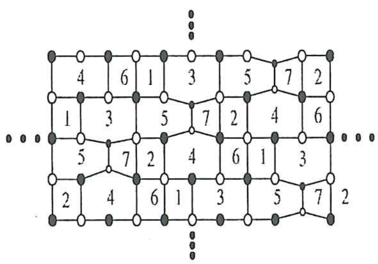


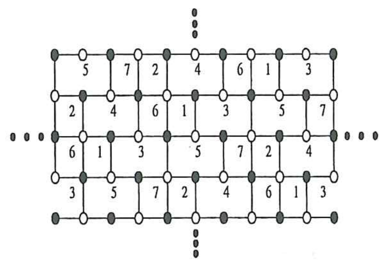


Example ($Q_7^{(2,3)}$): $\longleftarrow$

The cyclic permutation yields the **original** Brane Tiling and (Q, W)!

$$\begin{aligned} W'' &= A_{45}A_{52}A_{24}^{(V)} + A_{13}A_{34}A_{41} + A_{16}A_{63}A_{35}A_{51} - A_{35}^{(D)}A_{51}A_{13} - A_{16}A_{62}A_{24}^{(V)}A_{41} \\ &- A_{34}A_{46}A_{63} + A_{73}^{*}A_{35}^{(D)}A_{57}^{*} - A_{74}^{*}A_{45}A_{57}^{*} - A_{35}A_{52}A_{27}^{*}A_{73}^{*} + A_{27}^{*}A_{74}^{*}A_{46}A_{62} \\ W &= A_{13}A_{34}A_{41} + A_{16}A_{63}A_{35}^{(V)}A_{51} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\ &- A_{16}A_{62}A_{24}A_{41} - A_{34}A_{46}A_{63} - A_{13}A_{35}^{(H)}A_{51} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74}. \end{aligned}$$





$\longleftarrow$