Algebraic consistency and Towards

4/20/15 **Lecture 24: Pyramid Partition Functions**

We begin by summarizing all of the consistency relations we have discussed.

[See Section 5 of "Dimer models and the special McKay correspondence" by Ishii & Ueda]

Def 5.1 A dimer model is consistent if

- there is no homologically trivial zig zag path
- no zigzag path has a self-intersection on the univ. cover
- no two zig zag paths on the univ. cover intersect each other twice or more in the same direction.

Def 5.2 A dimer model is isoradial if the edges of the associated quad graph are all of equal length. (Equivalently, the faces of the dimer model are all polygons inscribed in circles of a fixed radius using a flat metric on the torus.)

Thm 5.3 (appears in [Kenyon-Schlenker]) A dimer model is isoradial $\Leftrightarrow$ zig-zag paths behave like straight lines, I.e.

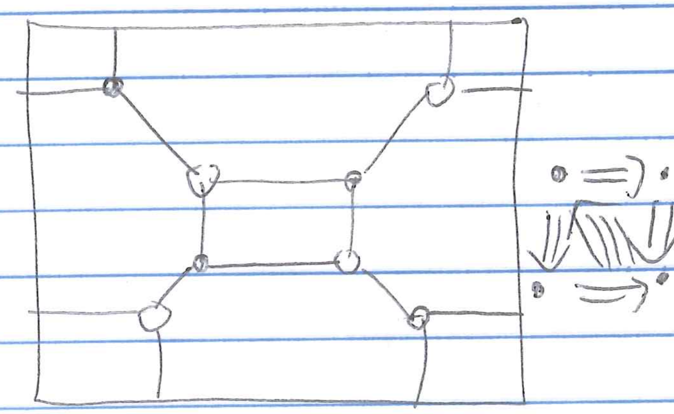
- every zig-zag path is a simple closed curve
- two zig zag pairs on univ. cover intersect at most once.

② 4/20/15 Cor 5.4 Isoradial dimer models are consistent

Converse is false: e.g.

consistent but

not isoradial



Def 5.5 (Gulotta) A dimer model is properly ordered if

- no homologically trivial zigzag path
- no zigzag path has a self-intersection on univ. cover
- no two zig-zag paths with the same homology class incident to a common vertex of bipartite tiling
- cyclic order of zig-zag paths incident to a vertex (i.e. the local zigzag fan) agrees with cyclic order of the homology classes.

Prop 5.6 (Appears in earlier [IU] paper) A dimer model is consistent $\Leftrightarrow$ properly ordered.

(3) 4/20/15 Remark: Broomhead's definition of "geometrically consistent" agrees with "isoradial"

but his use of "consistent" (defined in terms of R-symmetry) agrees with today's use of "consistent".

Remark: Broomhead also introduces a concept called "algebraic consistency", that we discuss shortly, and proves

geom consistency $\Rightarrow$ algebraic consistency

Rem: Bocklandt (in "Consistency Conditions for Dimer Models" arXiv:1104.1592)

extends this to prove

consistency $\Rightarrow$ algebraically consistent

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and in fact proves for a dimer model on a torus,

consistency $\Leftrightarrow$ algebraic consistent
(in terms of R-charge)

as well as several other equivalent conditions, including

consist. $\Leftrightarrow$ alg. consist. $\Leftrightarrow$ cancellative $\Leftrightarrow$ A_Q is a NCCR (noncommutative crepant resolution) of its center.
$A_Q = \mathbb{C}Q/\partial W$

4/20/15 (4)

We return to these algebraic definitions shortly.

But with these as motivation, we note

Thm 1.1 [Ishii-Ueda] Let G be a consistent dimer model with associated characteristic polygon Δ, i.e. Δ = Newton polygon of $\det K_G(z_1, z_2)$. Let c be an external vertex of Δ (corresponds to an extremal perfect matching as we have discussed) and $\Delta' = \Delta - c$.

Then there is an explicit algorithm to remove edges of G to obtain dimer model G' s.t.

- G' is consistent
- characteristic polygon of G' is Δ'

Rem: More explicitly algorithm but related to partial resolution/Higgsing discussed last week.

Cor 1.2 (Starting with McKay Quiver & Potential) For any lattice polygon Δ, there is some consistent dimer model whose characteristic polygon is Δ.

In contrast, Thm 1.1 of [Ueda-Yamazaki]: If Δ is a convex lattice polygon, a dimer model G obtained by linear Hanany-Vegh (equiv. Goncharov-Kenyon) is isoradial and Δ is the characteristic polygon of G up to translation.

4/20/15 5 We now define Algebraic Consistency:

Recall we have cochain maps

$$\mathbb{Z}^{Q_2} \xrightarrow{d} \mathbb{Z}^{Q_1} \xrightarrow{d} \mathbb{Z}^{Q_0}$$

(Q_0 = vertices of quiver
Q_1 = edges
Q_2 = superpotential terms)

where $d(c) = \sum_{a \in c} a$ for $c \in Q_2$

$d(a) = h(a) - t(a)$ for $a \in Q_1$

define a new map $\mathbb{Z} \xleftarrow{e} \mathbb{Z}^{Q_2}$ by $e(c) = 1 \ \forall c \in Q_2$

and let $M = \mathbb{Z}^{Q_1} / d e^{-1}(0) = \operatorname{coker} \bar{d}$ for

liner combinations of superpotential terms whose coeffs sum to 0

$\bar{d} : e^{-1}(0) \xrightarrow{\bar{d}} \mathbb{Z}^{Q_1}$, $e^{-1}(0) \subseteq \mathbb{Z}^{Q_2}$

[Rem: By definition, W always in $e^{-1}(0)$]

Let $M_{ij}^{+} = \dfrac{d^{-1}(i-j) \cap \mathbb{N}^{Q_1}}{d e^{-1}(0)}$ For any two vertices $i, j \in Q_0$

and define $B_Q = \left(\bigoplus_{i,j \in Q_0} \operatorname{Span}(M_{ij}^{+}) \right) \subset \left\{ \begin{array}{c} n \times n \text{ matrices} \\ \text{with entries from} \\ \mathbb{C}[M] \end{array} \right\}$ $n = |Q_0|$

<u>Def:</u> A dimer model is <u>algebraically consistent</u>

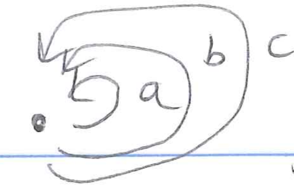
$\Leftrightarrow$ $\tau : A_Q \to B_Q$ defined by

$A_Q = \mathbb{C}Q / \partial W$, $\operatorname{Span}(M_{ij})$

$a \longmapsto a^{\tau} = \overset{\bullet}{i} \to \overset{\bullet}{j}$

is an <u>isomorphism</u>.

(rather than simply a homomorphism)

4/20/15 (6) <u>Example 1:</u> $Q =$  , $W = abc - acb$

$M_{11}^+ = \dfrac{d^{-1}(0) \cap \mathbb{N}^{Q_1}}{de^{-1}(0)} = \dfrac{\mathbb{N}^{Q_1}}{de^{-1}(0)}$ since $da = db = dc = v_1 - v_1 = 0$

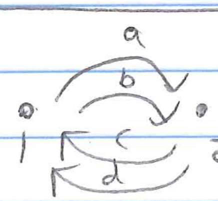
and since only 2 terms in Q_2, $e^{-1}(0) = \{kW : k \in \mathbb{Z}\}$

$\Rightarrow$ the image $de^{-1}(0) = 0$ since $dW = a+b+c-a-c-b = 0$

Thus $M_{11}^+ = \mathbb{N}^{Q_1}$ and $B = \mathrm{Span}(M_{11}^+)$ since only one vertex in Q_0

has basis $\{a, b, c\}$

$\Rightarrow B \cong \mathbb{C}[a,b,c]$ agreeing with $A \cong \mathbb{C}Q/\partial W$.

<u>Example 2:</u> $Q =$  $W = acbd - adbc$

$M_{11}^+ = M_{22}^+ = \dfrac{d^{-1}(0) \cap \mathbb{N}^{Q_1}}{de^{-1}(0)}$.

Again, only 2 terms in Q_2 and they both have the same arrows in each so $e^{-1}(0) = \{kW : k \in \mathbb{Z}\}$ and $de^{-1}(0) = 0$ since $dW = a+c+b+d-a-d-b-c = 0$.

This time however $d^{-1}(0) \cap \mathbb{N}^{Q_1}$ more interesting.

4/20/15 (7) $d^{-1}(0) \cap \mathbb{N}^{Q_1} = \langle a+c, a+d, b+c, b+d \rangle$ ← $\mathbb{N}$-linear combos

$$M_{12}^+ = \frac{d^{-1}(v_1 - v_2) \cap \mathbb{N}^{Q_1}}{de^{-1}(0)} = d^{-1}(v_1 - v_2) \cap \mathbb{N}^{Q_1}$$

$$= a \text{ or } b + \langle a+c, a+d, b+c, b+d \rangle$$

Similarly, $M_{21}^+ = c$ or $d + \langle a+c, a+d, b+c, b+d \rangle$

Let $Z = \mathrm{Span}_{\mathbb{C}} \langle a+c, a+d, b+c, b+d \rangle$

$$Z = \mathrm{Span}_{\mathbb{C}} \langle \underset{\substack{\| \\ x_1}}{t^{a+c}}, \underset{\substack{\| \\ x_2}}{t^{a+d}}, \underset{\substack{\| \\ x_3}}{t^{b+c}}, \underset{\substack{\| \\ x_4}}{t^{b+d}} \rangle \cong \frac{\mathbb{C}[x_1, x_2, x_3, x_4]}{(x_1 x_4 - x_2 x_3)}$$

$$\text{so } B_Q = \left\{ \begin{bmatrix} C_{11} & aC_{12} + bC_{12}' \\ cC_{21} + dC_{21}' & C_{22} \end{bmatrix} : C_{11}, C_{12}, C_{12}', C_{21}, C_{21}', C_{22} \in Z \right\}$$

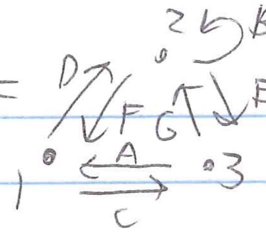
$\cong A_Q$ with

$$\begin{aligned} &\quad M_{11}^+ \; M_{22}^+ \\ x_1 &= ac + ca \\ x_2 &= ad + da \\ x_3 &= bc + cb \\ x_4 &= bd + db \end{aligned} \qquad x_1 x_4 - x_2 x_3 = 0 \text{ again}$$

and we can also multiply by singletons $a, b, c,$ or d.

We will momentarily see an alternative construction of $A_Q \cong B_Q$ for this e.g.

4/20/15 (8)

Example 3 (SPP): $Q =$ 

$W = ADCF - BFD + BEG - ACGE$

$$\forall i \in Q_0,\ M_{ii}^+ = \frac{d^{-1}(0) \cap \mathbb{N}^{Q_1}}{d e^{-1}(0)} = \frac{\langle D+F, E+G, A+C, B, A+D+E, C+G+F \rangle}{d\left\langle \begin{matrix} ADCF - BFD, \\ ADCF - ACGE, \\ BEG - BFD \end{matrix} \right\rangle}$$

$$= \frac{\langle D+F, E+G, A+C, B, A+D+E, C+G+F \rangle}{\langle A+C-B,\ D+F-E-G,\ E+G-F+D \rangle}$$

$$= \langle \underset{z}{D+F},\ \underset{t}{A+C},\ \underset{x}{A+D+E},\ \underset{y}{C+G+F} \rangle$$

Let $Z = \mathrm{Span}_{\mathbb{C}}(M_{ii}^+) = \mathbb{C}[x,y,z,t]/(xy - z^2 t)$

using $z = \varepsilon^{D+F} \sim \varepsilon^{E+G}$ from quotient

Again, Z = center of B_Q and this equation

$xy - z^2 t$ is origin of name "Suspended Pinch Point".

We omit the computation of M_{ij}^+ for $i \neq j$ in this case but do indeed get

$B_Q \cong A_Q$ in this example.

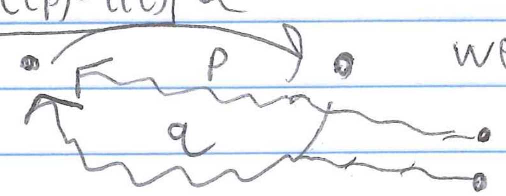
4/20/15 (9) In general, $M_{ii}^+ = \dfrac{d^{-1}(0)\cap \mathbb{N}^{Q_1}}{de^{-1}(0)}$ for any $i \in Q_0$

$Z = \text{Span}_{\mathbb{C}}\, M_{ii}^+$ is <u>the center of B_Q</u>.

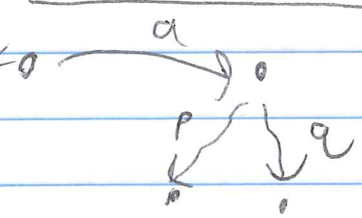
so when we write that A_Q is a NCCR in the sense of Van den Bergh over its center, its center is Z in the case of an (algebraic) consistent dimer model.

The last algebraic condition is "cancellative" or <u>A_Q is a cancellation algebra.</u>

<u>Def:</u> "$\mathbb{C}_Q/\partial W$ (A_Q) is a <u>cancellation algebra</u> if for <u>every arrow a</u> and any two paths p, q s.t. $\boxed{h(a)=t(p)=t(q)}$ we have <u>$pa = qa \Rightarrow p = q$ in A_Q</u>



and for any paths p, q s.t. $\boxed{t(a)=h(p)=h(q)}$ we have <u>$ap = aq \Rightarrow p = q$ in A_Q</u>



<u>Rem:</u> Though we won't use it, Broomhead & others (such as Davison) prove that an algebraic consistent (cancellation) dimer model is a Calabi-Yau-3 algebra.

4/20/15 Putting all of the equivalent algebraic conditions together allows us to restate a result of [Mozgovoy – Reineke]

See "On the noncommutative Donaldson–Thomas invariants arising from Brane Tilings" (arXiv:0809.0117)

(Essentially) Lemma 4.11 of [Mozgovoy–Reineke]

A dimer model on a torus is consistent

$\Leftrightarrow$ it is cancellative

$\Leftrightarrow$ two paths in Q with the same start point are equivalent in $A = \mathbb{C}Q/\partial W$ if and only if ~~the~~ their weights are equal

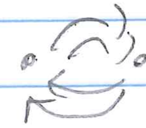
Ker Q_F in N (in Broomhead's notation)

Fin. gen. free abelian gp

[Here, by weights we define $wt: Q_1 \to \Lambda$ so that W is Λ-homogeneous (in other words, wt is a global symmetry)]

$\Leftrightarrow$ if two paths in Q with the same start point are weakly equiv. then they are equivalent in $A = \mathbb{C}Q/\partial W$

[Here, weakly equivalently means allowing both a & a^{-1} for every arrow $a \in Q_1$]

We shortly turn our attention back to  and "Non-commutative Donaldson–Thomas theory and the conifold" by Szendrői arXiv: 0705.349