4/29/15

Lecture 27: Proofs of Speyer's Octahedron Rec. Combinatorial Interpretation

Proof 1: Kuo Condensation

We begin by summarizing Eric Kuo's 2003 paper "Applications of Graphical Condensation for Enumerating Matchings and Tilings" (arXiv: 0304090)

Thm 2.1 Let $G = (V, E)$ be a planar bipartite graph with $V = B \sqcup W$, $|B| = |W|$, with a specific planar embedding in mind so there is a cyclic ordering of edges around each vertex.

called a "plane" or "planar ribbon" graph

Let vertices $\overset{\bullet}{a}$, $\overset{\circ}{b}$, $\overset{\bullet}{c}$, and $\overset{\circ}{d}$ appear in a cyclic order on a face around G (possibly the infinite face).

Then $$\boxed{M(G)\,M(G-\{a,b,c,d\}) = M(G-\{a,b\})\,M(G-\{c,d\}) + M(G-\{a,d\})\,M(G-\{b,c\})}$$

where $M(G') = \#$ perfect matchings in G'.

Also $G' = G - \{a,b\}$ means delete vertices a & b from G along with any incident edges.

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To prove this theorem, Kuo considers the superposition of <u>a matching of G</u> with <u>a matching of $G-\{a,b,c,d\}$</u>

versus a superposition of <u>matchings of $G-\{a,b\}$ and $G-\{c,d\}$</u>

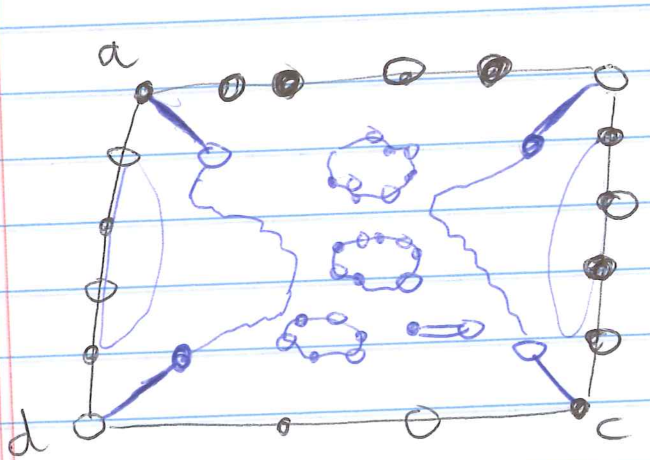
OR a superposition of <u>matchings of $G-\{a,d\}$ and $G-\{b,c\}$</u>

Each of these superpositions is a collection of edges (possibly with duplicates) so that vertices <u>a,b,c,d</u> are incident to <u>exactly one edge</u>

but every other vertex (in $G-\{a,b,c,d\}$) is incident to <u>exactly two edges</u> in the collection.

Schematically, such superpositions look like

[without loss of generality consider a,b,c,d to be on the unbounded face]

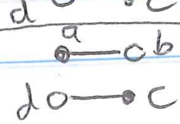


i.e.
- doubled edges
- 2-cycles
- paths with endpoints two out of a, b, c, d

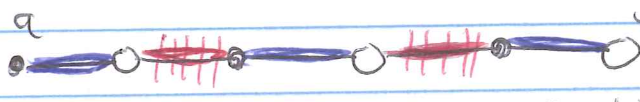
The paths either go

(as pictured)

OR



4/29/15 (3) Note that these two paths are both of odd length so splits into two perfect matchings

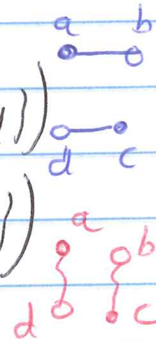


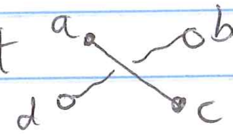
where one is incident to $\{a, b\}$ and one is not.

can be split into 2 different perfect matchings

we thus obtain a 2^k-to-2^k map (where $k = \#$ cycles with ≥ 4 vertices)

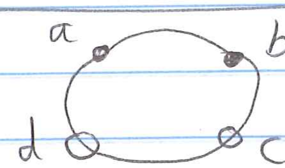
$$M(G)\,M(G-\{a,b,c,d\}) \longrightarrow M(G-\{a,b\})\,M(G-\{c,d\}) \sqcup M(G-\{a,d\})\,M(G-\{b,c\})$$



Rem: Cannot connect  because then the paths would have to intersect in a vertex of degree > 2 $\Rightarrow\!\Leftarrow$.

Variant (Thm 2.3) Assume $|B| = |W|$ &

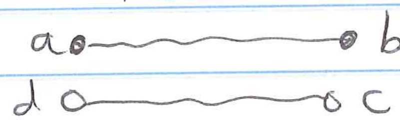
[notice the colors are different than above]



$$M(G)\,M(G-\{a,b,c,d\}) = M(G-\{a,d\})\,M(G-\{b,c\}) - M(G-\{a,c\})\,M(G-\{b,d\})$$

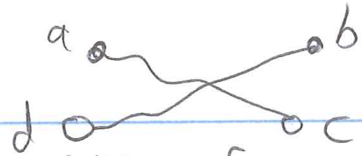
PF Sketch: Superposition involves either

— even length paths



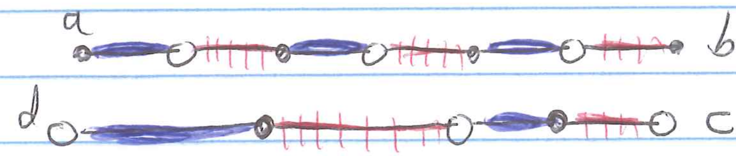
OR — odd length paths

4/29/15 (4) Again, paths



impossible since no vertices of degree >2.

even length paths



decompose as matchings as one in $G-\{a,d\}$
& one in $G-\{b,c\}$

OR

one in $G-\{a,c\}$
& one in $G-\{b,d\}$

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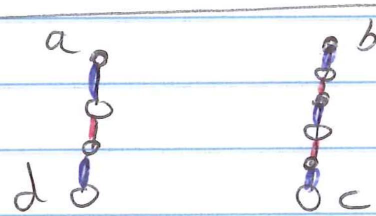
odd length paths



decompose as matchings as one in $G-\{a,d\}$
& one in $G-\{b,c\}$

OR

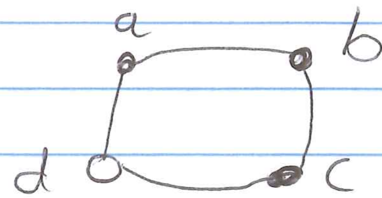
one in G
& one in $G-\{a,b,c,d\}$

Besides the cycles, we again have a bijection, but rearranged,

$$M(G-\{a,d\})\,M(G-\{b,c\}) \longmapsto M(G-\{a,c\})\,M(G-\{b,d\})$$
$$\sqcup\ M(G-\{a,b,c,d\})\,M(G)$$

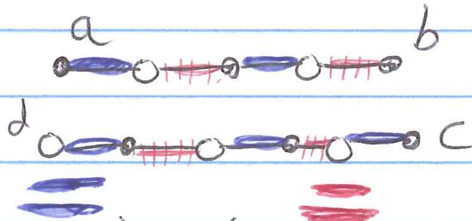
4/29/15 5 Lastly, there are $\boxed{\text{unbalanced}}$ versions of Kuo Condensation

Thm 2.4 $\quad |B| = |W| + 1$

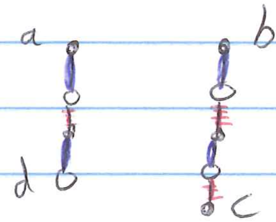


$$M(G-\{b\})\,M(G-\{a,c,d\}) = M(G-\{a\})\,M(G-\{b,c,d\}) + M(G-\{c\})\,M(G-\{a,b,d\})$$

Pf sketch:

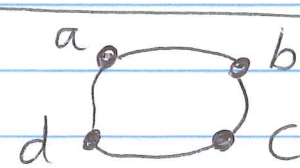


in $M(G-\{b\})\,M(G-\{a,c,d\})$ or $M(G-\{a\})\,M(G-\{b,c,d\})$

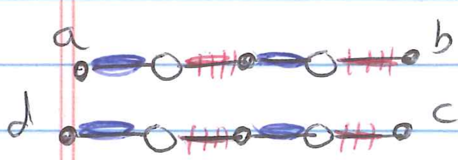


in $M(G-\{b\})\,M(G-\{a,c,d\})$ or $M(G-\{c\})\,M(G-\{a,b,d\})$

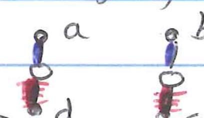
Thm 2.5 $\quad |B| = |W| + 2$



$$M(G-\{a,c\})\,M(G-\{b,d\}) = M(G-\{a,b\})\,M(G-\{c,d\}) + M(G-\{a,d\})\,M(G-\{b,c\})$$



in $M(G-\{b,c\})\,M(G-\{a,d\})$ or $M(G-\{a,c\})\,M(G-\{b,d\})$



in $M(G-\{a,c\})\,M(G-\{b,d\})$

or $M(G-\{a,b\})\,M(G-\{c,d\})$

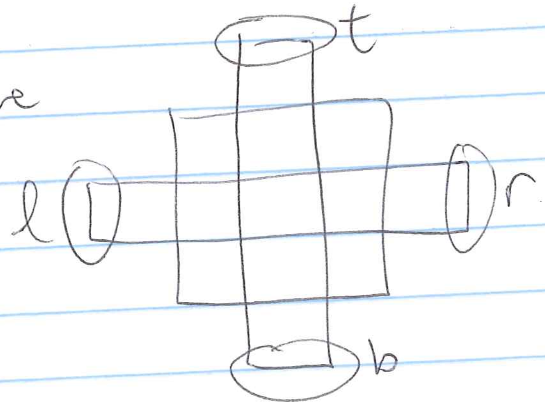
4/29/15 ⑧ Kuo also discusses weighted Aztec Diamonds (Thm 5.5)

$$m(A_n)\, m(A_{n-2}) = m(A_{n-1})^2 + m(A_{n-1})^2$$

up to including some extra edges (with weights)

$$W(A_n)\, W(A_{middle}) = l \cdot r\, W(A_{top})\, W(A_{bot})$$
$$+ t \cdot b\, W(A_{left})\, W(A_{right})$$

where



Pf is analogous to above. We will discuss a more general case later and its proof then.

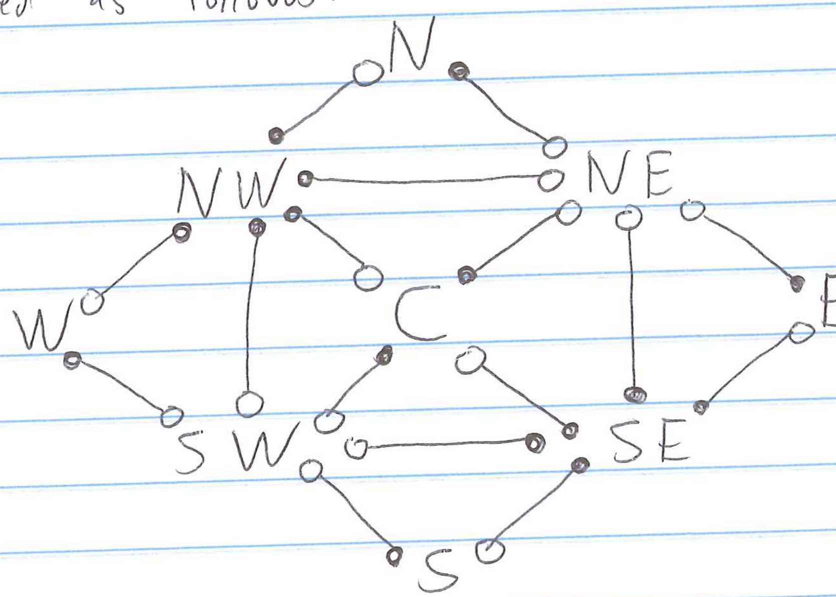
4/29/15 ⑦ Speyer merges some of these cases into one Condensation Theorem

<u>Thm</u>: Let G be a bipartite planar graph with vertices partitioned into nine sets (disjointly):

$$V(G) = C \sqcup N \sqcup NE \sqcup E \sqcup SE \sqcup S \sqcup SW \sqcup W \sqcup NW$$

satisfying the following conditions:

1) Edges bridging between two of these regions must be colored as follows:



In other words, edges incident to NW or SE (and another region) must be black,
edges " " NE or SW must be white

Note also that two regions can be connected by an edge only if the pair is an ordinal direction and adjacent cardinal direction where we also consider C to be a "cardinal direction"

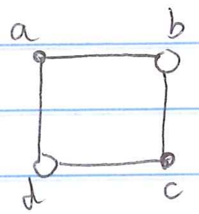
4/29/15 (8) 2) The regions NW & SE contain one more black vertex than white vertex

The regions NE & SW " " white vertex than black vertex

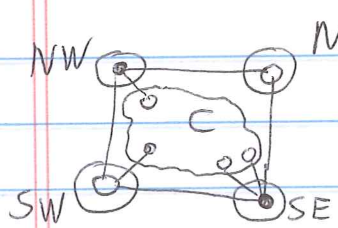
The other five regions are balanced.

Then
$$\boxed{\begin{aligned} M(G)M(C) &= M(W\cup NW\cup SW\cup C)\,M(E\cup NE\cup SE\cup C)\,M(S)\,M(N) \\ &+ M(S\cup SW\cup SE\cup C)\,M(N\cup NW\cup NE\cup C)\,M(W)\,M(E) \end{aligned}}$$

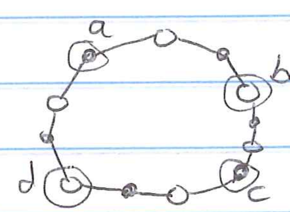
Rem: Here are some examples how Speyer's version specializes to Kuo's:

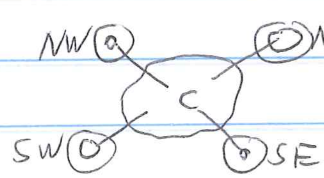
e.g. 1) 

is a 4-face. Let $NW=\{a\}$, $NE=\{b\}$, $SW=\{d\}$, $SE=\{c\}$, $C=G-\{a,b,c,d\}$, $N=S=E=W=\emptyset$



$$M(G)M(G-\{a,b,c,d\}) = M(G-\{b,c\})M(G-\{a,d\}) + M(G-\{a,b\})M(G-\{c,d\}).$$

Variant: $\{a,b,c,d\}$ on >4-face, e.g. 

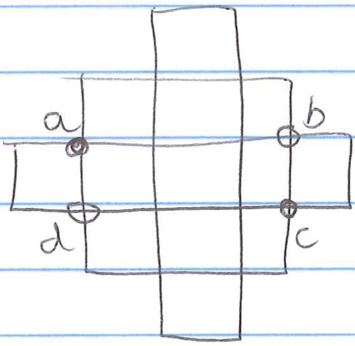
assign regions just as before but now 

with no connections between ordinal directions.

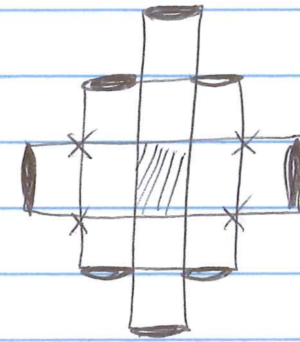
Combination of these ⟹ Thm 2.1 in general.

4/29/15 (9) Weighted Aztec Diamond case could be treated as ordinary Kuo Condensation (Thm 2.1) or via Speyer's reformulation?

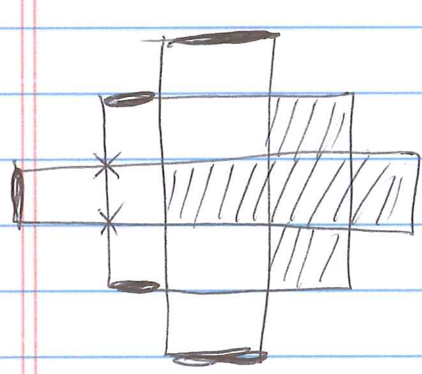
e.g.

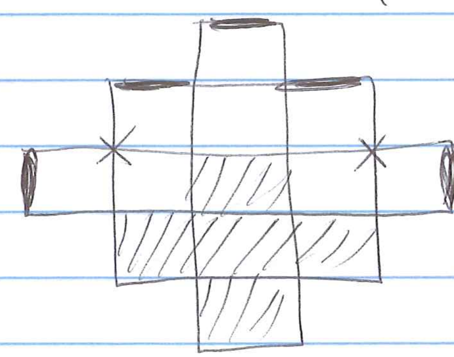


$G - \{a,b,c,d\}$



$G - \{a,d\}$ ($G - \{b,c\}$ analogous) $G - \{a,b\}$ ($G - \{c,d\}$ analogous)





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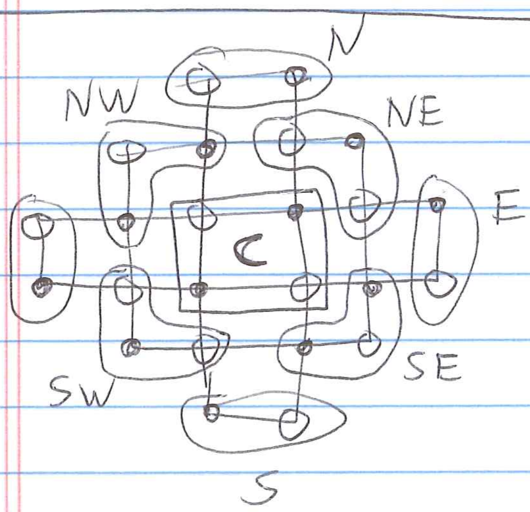
versus



$$\boxed{\begin{aligned} m(G)\, m(C) = {} & m(N \cup NW \cup NE \cup C)\\ & m(S \cup SW \cup SE \cup C)\\ & m(E)\, m(W)\\ + {} & m(E \cup NE \cup SE \cup C)\\ & m(W \cup NW \cup SW \cup C)\\ & m(N)\, m(S) \end{aligned}}$$

4/29/15 10 <u>We will prove Speyer's formulation of condensation shortly.</u>

First, we describe how it proves his Octahedron Recurrence combinatorial interpretation.

Consider $G(n,i,j)$'s as constructed last lecture.
By abuse of notation, let $V(n,i,j)$ denote the vertices of $G(n,i,j)$.

To avoid a boundary case, assume n large enough so that $(n-2,i,j) \notin \mathcal{C}$.

<u>Claim 1:</u> $V(n-2,i,j) = V(n-1,i+1,j) \cap V(n-1,i-1,j)$
$= V(n-1,i,j+1) \cap V(n-1,i,j-1)$.

<u>Claim 2:</u> $V(n,i,j) = V(n-1,i+1,j) \cup V(n-1,i-1,j)$
$\cup V(n-1,i,j+1) \cup V(n-1,i,j-1)$.

Let $C = G(n-2,i,j)$.

Let $NE = G(n-1,i+1,j) \cap G(n-1,i,j+1) - C$

Let $NW = G(n-1,i-1,j) \cap G(n-1,i,j+1) - C$

Let $N = G(n-1,i,j+1) - (NE \cup NW \cup C)$

Define the remaining regions analogously.

4/29/15 (11) <u>Claim 3:</u> Region **N** (and analogously regions **E**, **W**, and **S**) has a unique perfect matching.

<u>Claim 4:</u> Regions **NE** and **SW** have one more black vertex than white. Regions **NW** and **SE** have one more white vertex.

<u>Claim 5:</u> The adjacencies between the nine regions is as needed for the hypotheses.

<u>Result:</u> $F(n,i,j)\,F(n-2,i,j) =$

$$a \cdot c \cdot f(n-1,i,j+1) \cdot F(n-1,i,j-1)$$

$$+\ b \cdot d \cdot f(n-1,i+1,j) \cdot f(n-1,i-1,j)$$

where a,b,c,d are weights of edges in the unique perfect matchings of regions E, N, W, S, resp.

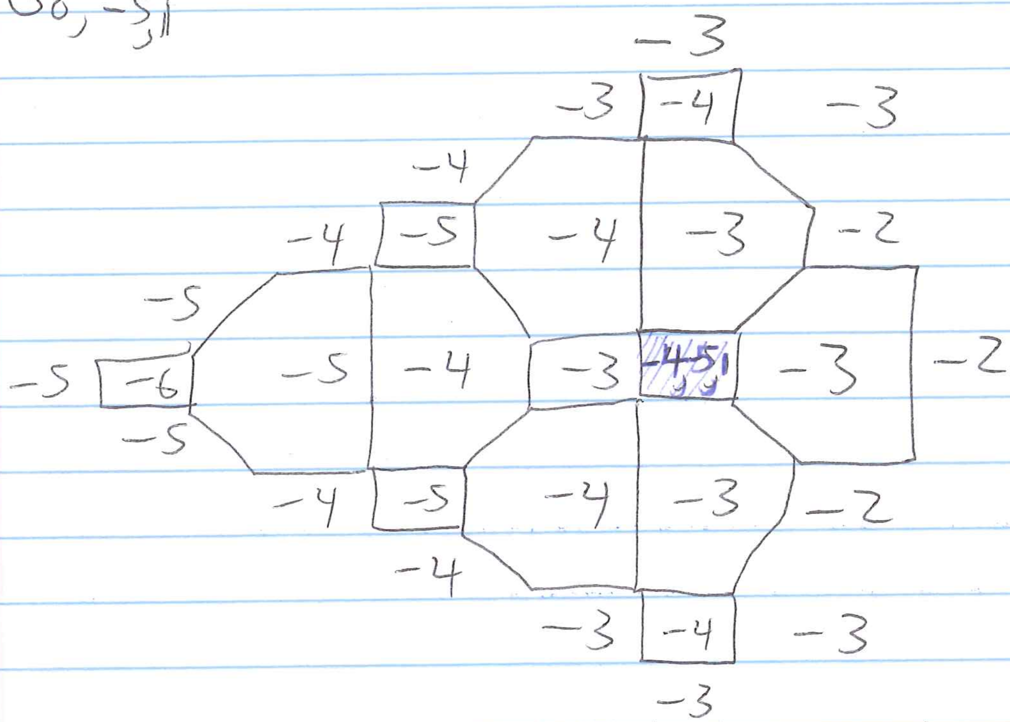
Technically, Speyer proves this by letting face weights go to 1 and introducing a different edge weighting, but a,c,b,d exactly compensate for cancelling out face weights for faces in G but not a particular region. $G = G(n,i,j)$

ie.  contributes ~~TEM~~ $\overset{x-0}{\omega}$ to face weight, as does

 , etc.

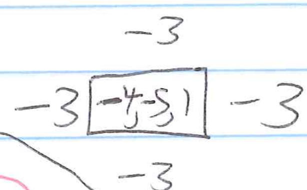
4/29/15 (12) Example (Somos-4)

$G_{0,-5,1} =$



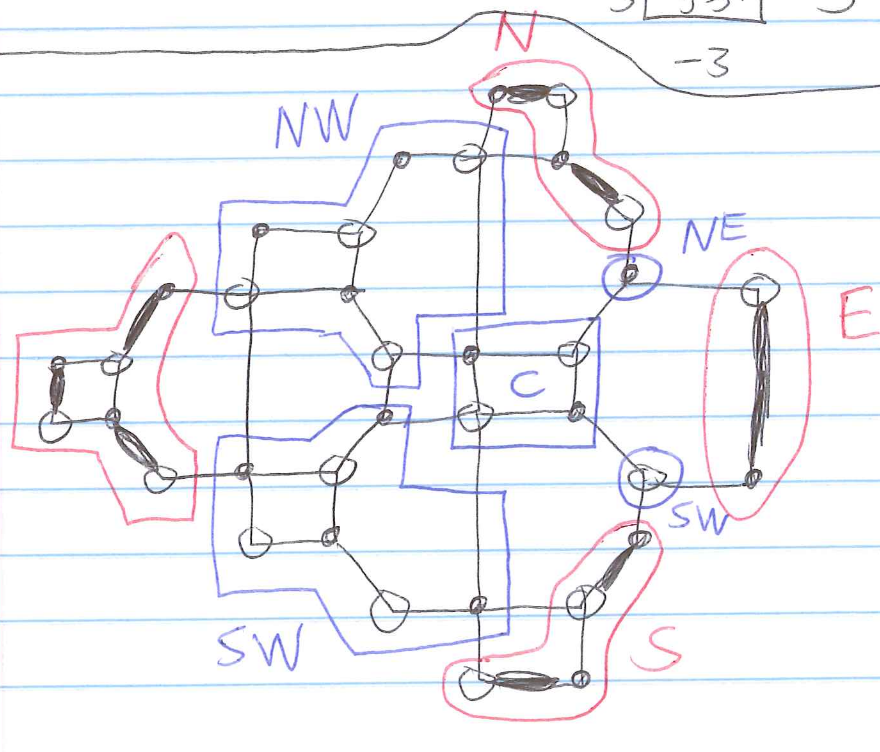
while we consider $G_{-2,-5,1}$, $G_{-1,-5,0}$, $G_{-1,-5,2}$, $G_{-1,-4,1}$, $G_{-1,-6,1}$

e.g. $G_{-2,-5,1} =$



$= C$

=>



4/29/15 (13) Verifying Claims

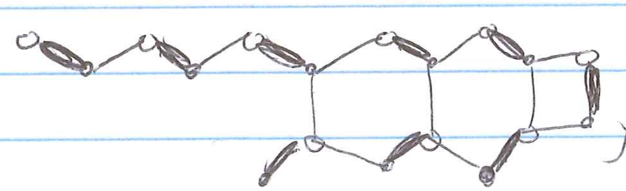
Claim 1: $G(n-2, i, j)$ includes faces (n', i', j') s.t. taxicab distance away from $(n-2, i, j)$ less than the difference $(n-2) - n'$

$G(n-1, i\pm 1, j)$ or $G(n-1, i, j\pm 1)$ similar but as if one step down further is possible.
But will go E/W or N/S hence why intersections match up.

Claim 2: Similarly $G(n, i, j)$ compared w/ $G(n-1, i\pm 1, j)$ or $G(n-1, i, j\pm 1)$ allows one more step down so their union yields everything.

Claim 3: On border so is at best a "thickened path",

e.g.,

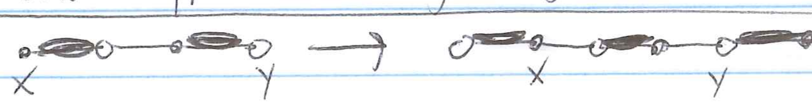


possible.

Claim 4 & 5: By Claim 3, this unique perfect matching is a path which we can augment on both sides using one vertex of opposite colors using vertices in the adjacent ordinal regions.

Pairing all these off & the fact that G, C, E, W, S, N are colored balanced yields the right adjacencies & counts.

Augmentation:



4/29/15 (14) <u>Sketch of Pf of Speyer's Condensation Thm</u>

<u>Similar to Kuo</u>: Under the hypotheses,

superimposing a matching on G and C leads to

disjoint union

$$\boxed{\gamma_1 \sqcup \gamma_2 \sqcup M_C \sqcup \bigsqcup_{q \in X} M_q}$$

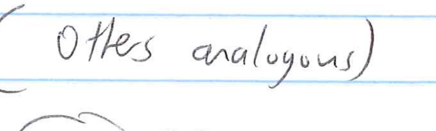
where <u>γ_1 is a N-join, γ_2 is a S-join</u>

<u>OR γ_1 is a E-join, γ_2 is a W-join</u>)

M_C = disjoint union of cycles in C

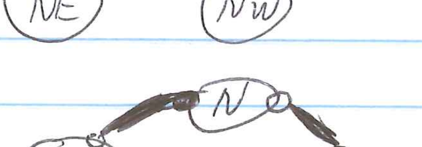
X = eight cardinal & ordinal regions
and M_q = matching entirely inside q
q = N, NE, etc.

A <u>N-join</u> is (Others analogous)

1) a path 

through (C)

2) a single edge 

OR 3) a pair of edges 

$$LHS = M(G)\,M(C). \quad RHS = \underbrace{\quad\quad}_{\text{N-join \& S-join}} + \underbrace{\quad\quad}_{\text{E-join \& W-join}}$$