# Lecture 28: Proofs of Speyer's Octahedron Recurrence
## Combinatorial Interpretation II

**Theorem:** Build $G(n,i,j)$'s from infinite graph $\mathcal{G}_h$ as discussed last week. For $(n_0, i_0, j_0)$ in $\mathcal{U}$, i.e. $n_0 = h(i_0, j_0)$, let $F(n_0, i_0, j_0) = x_{ij}$.

Assume the other $F(n,i,j)$'s satisfy the oct. rec.

$$F(n,i,j)\,F(n-2,i,j) = F(n-1,i+1,j)\,F(n-1,i-1,j) + F(n-1,i,j+1)\,F(n-1,i,j-1).$$

Then for $n > h(i,j)$, we have

$$F(n,i,j) = \sum_{\substack{M \text{ a perf.} \\ \text{matching of} \\ G(n,i,j)}} w(M)$$

using the face weight (as a Laurent monomial in $x_{ij}$'s) discussed last week

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Fix $(n_1, i_1, j_1)$ such that $n_1 \geq h(i_1, j_1)$.

**Proof 2 (by Urban Renewal):**

Let $\mathcal{U} = \{(n,i,j) : n+i+j \equiv 0 \bmod 2 \ \& \ h(i,j) < n\}$
("upper half-space")

$C_{(n_1, i_1, j_1)} = \{(n,i,j) : n+i+j \equiv 0 \bmod 2 \ \& \ n \leq n_1 - |i-i_1| - |j-j_1|\}$
(closed cone)

5/4/15 ② We prove the Theorem by induction on the number of faces in $\mathcal{U} \cap C_{(n_1, i_1, j_1)}$.

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If $\mathcal{U} \cap C_{(n_1, i_1, j_1)} = \emptyset$ (and we assume $n_1 \geq h(i_1, j_1)$)

then $n_1 = h(i_1, j_1)$, i.e. $(n_1, i_1, j_1) \in \mathcal{C}$ and

$f(n_1, i_1, j_1) = x_{i_1, j_1}$ by hypothesis.

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Otherwise, define $\tilde{h} : \mathbb{Z}^2 \to \mathbb{Z}$ by

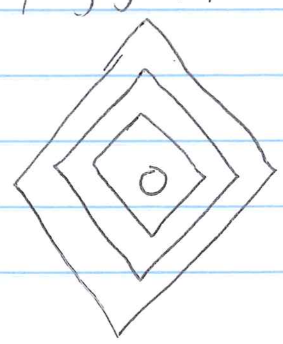
$$\tilde{h}(i,j) = \min\left( h(i,j),\ n_1 - |i - i_1| - |j - j_1| \right)$$

e.g. $h(i,j) = \begin{cases} 0 & \text{if } i+j \equiv 0 \bmod 2 \\ -1 & \text{if } i+j \equiv 1 \bmod 2 \end{cases}$

then $\tilde{h}(i,j)$ relative to $(n_1, i_1, j_1) = (0,0,0)$

| | | | | | |
|---|---|---|---|---|---|
| $-4$ | $-3$ | $-4$ | | | |
| $-3$ | $-2, 0, 2$ | $-3, 1, 2$ | $-4$ | | |
| $-2$ | $-1, 0, 1$ | $-2, 1, 1$ | $-3, 2, 1$ | $-4$ | |
| $-1, -1, 0$ | $0, 0, 0$ | $-1, 1, 0$ | $-2, 2, 0$ | $-3, 3, 0$ | $-4$ |
| $-2$ | $-1, 0, -1$ | $-2$ | $-3$ | $-4$ | |

i.e.

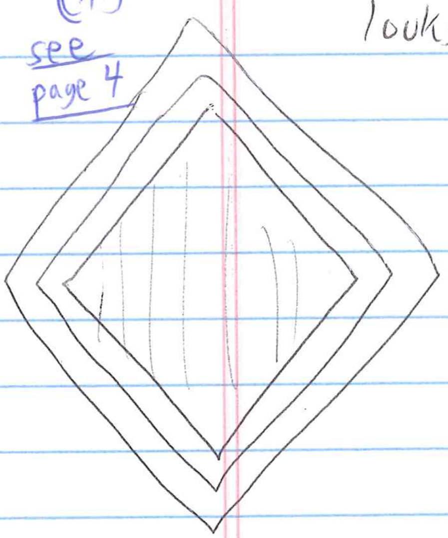


5/4/15 ③ on the other hand, $\tilde{h}(i,j)$ (relative to $(n_1, i_1, j_1) = (-2,1,1)$) looks like

| | | | |
|---|---|---|---|
| -6 | -5 | -4 | -5 |
| -5 | -4 | -3, 1, 2 | -4 |
| -4 | -3, 0, 1 | -2, 1, 1 | -3, 2, 1 |
| -5, -1, 0 | -4, 0, 0 | -3, 1, 0 | -4 |
| -6 | -5, 0, -1 | -4, 1, -1 | -5 |

while $\tilde{h}(i,j)$ (relative to $(n, i, j) = (+2,1,1)$) looks like

(*) see page 4



| | | | | | | |
|---|---|---|---|---|---|---|
| -4 | -3 | -2 | -1 | -2 | -3 | -4 |
| -3 | -2 | -1 | 0, 1, 3 | -1 | -2 | -3 |
| -2 | -1 | 0, 0, 2 | -1, 1, 2 | 0, 2, 2 | -1 | -2 |
| -1 | 0, -1, 1 | -1, 0, 1 | 0, 1, 1 | -1, 2, 1 | 0, 3, 1 | -1 |
| -2 | -1 | 0, 0, 0 | -1, 1, 0 | 0, 2, 0 | -1 | -2 |
| -3 | -2 | -1 | 0, 1, -1 | -1 | -2 | -3 |
| -4 | -3 | -2 | -1 | -2 | -3 | -4 |
| -5 | -4 | -3 | -2 | -3 | -4 | -5 |

- In particular, if we compute $\tilde{h}(i,j)$ relative to an $(n_1, i_1, j_1)$ under $cl$ then $\tilde{h}$ decreases monotonically with taxi-cab distance from $(i_1, j_1)$.

- However, if $(n_1, i_1, j_1)$ is above $cl$, then $\tilde{h}$ defines a plateau near $(i_1, j_1)$ where $\tilde{h} = h$ and outside monotonically decreases.

5/4/14 (4) since we assumed that $h:\mathbb{Z}^2\to\mathbb{Z}$ was a <u>Speyer height func.</u>

we have $\displaystyle\lim_{|i|+|j|\to\infty} h(i,j)+|i|+|j| \to \infty$

$\Rightarrow$ for $|i-i_1|+|j-j_1|$ large enough, $h(i,j) \geq n_1 - |i-i_1| - |j-j_1|$

$\Rightarrow$ $\tilde{h}(i,j)$ agrees with $h(i,j)$ for <u>at most a finite region</u> around $(i_1,j_1)$ and then monotonically decreases.

<u>Rem:</u> $\tilde{h}(i,j)$ is not a height function for this reason, but still satisfies

- $\tilde{h}(i,j) \equiv i+j \bmod 2$ and
- $\tilde{h}(i,j) \pm 1 = \tilde{h}(i\pm1,j) = \tilde{h}(i,j\pm1)$.

Speyer calls $\tilde{h}$ a <u>pseudo-height function.</u>

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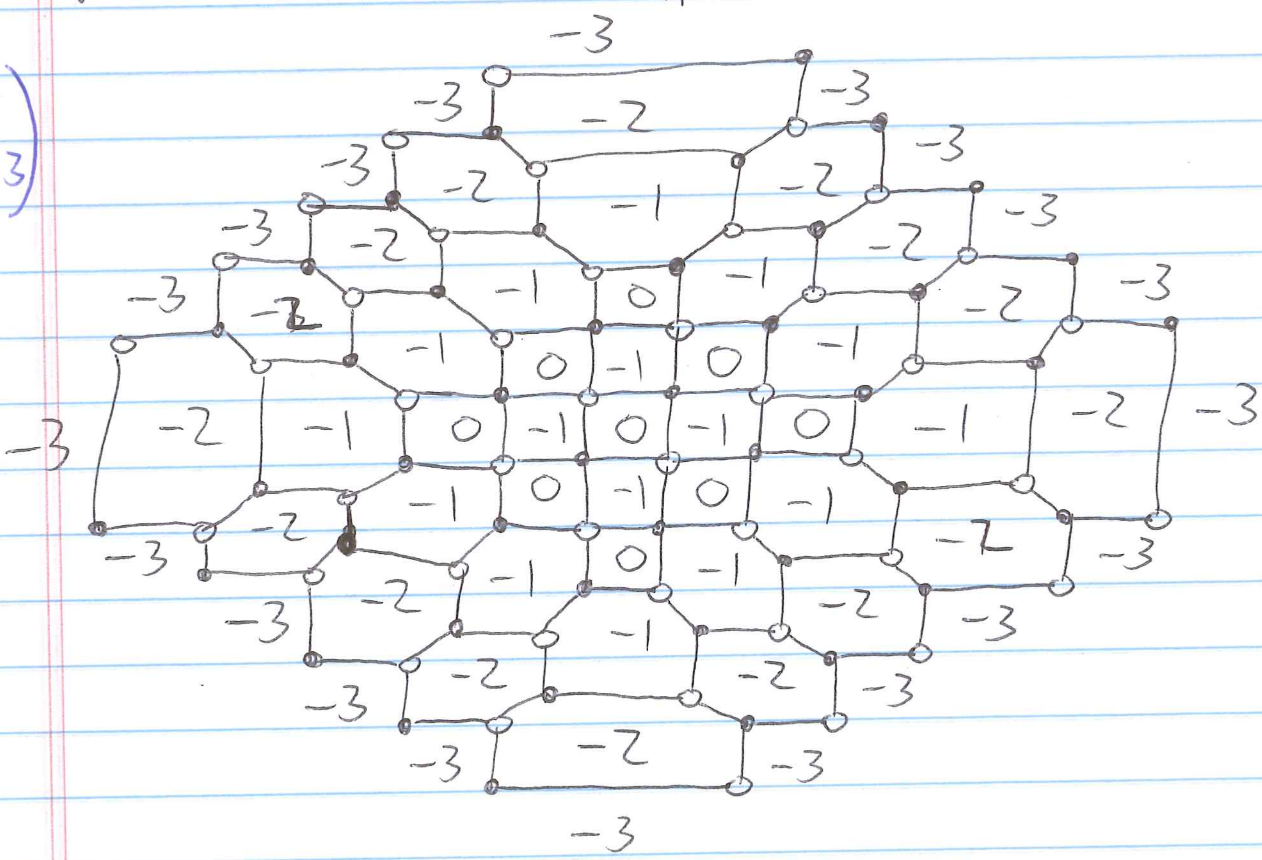
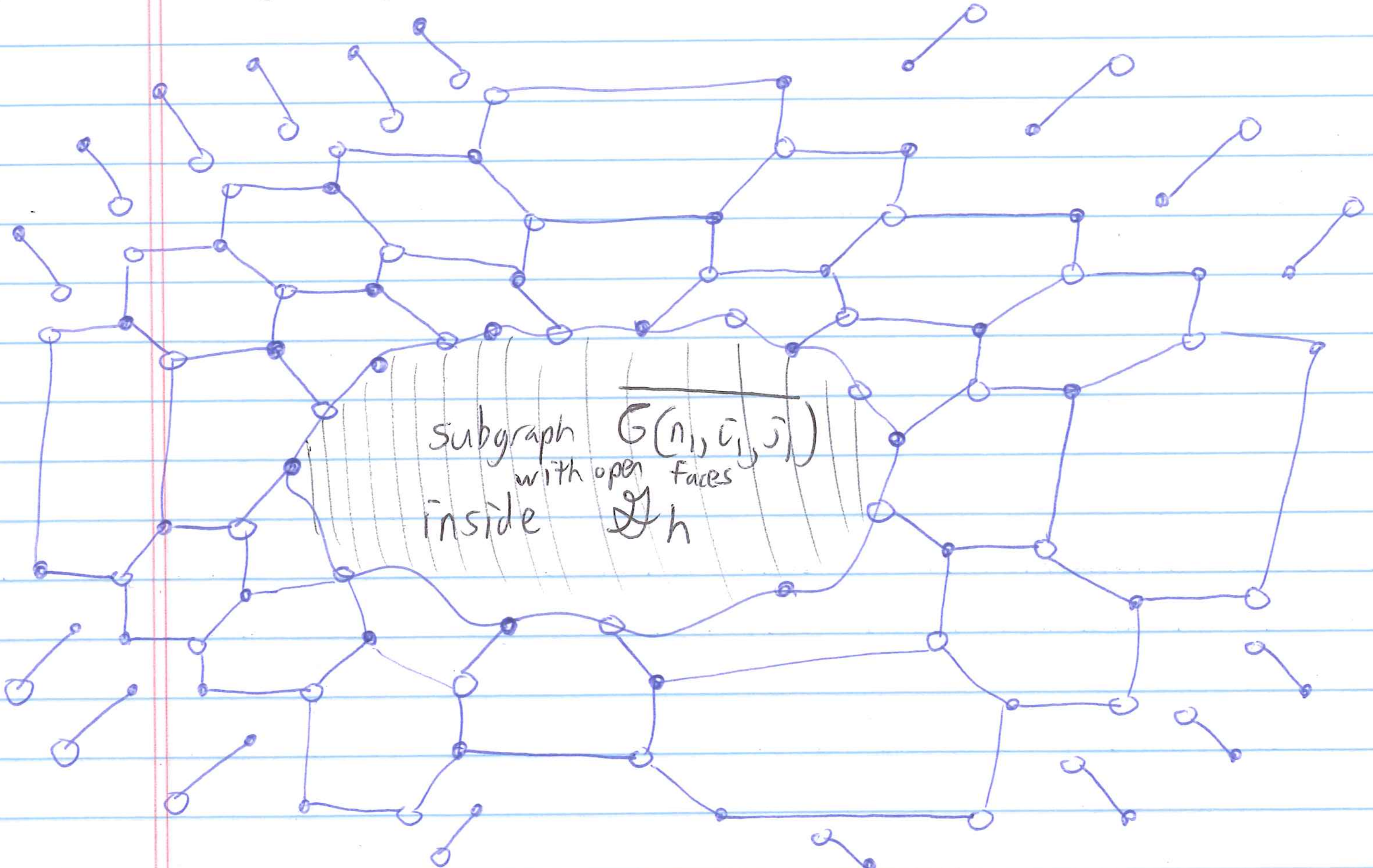
We now construct $\mathcal{G}_{\tilde{h}}$ instead of $\mathcal{G}_h$'s:

e.g. (*) From page 3



5/4/14 ⑤ In general, $\mathcal{G}_{\tilde{h}}$ looks like:



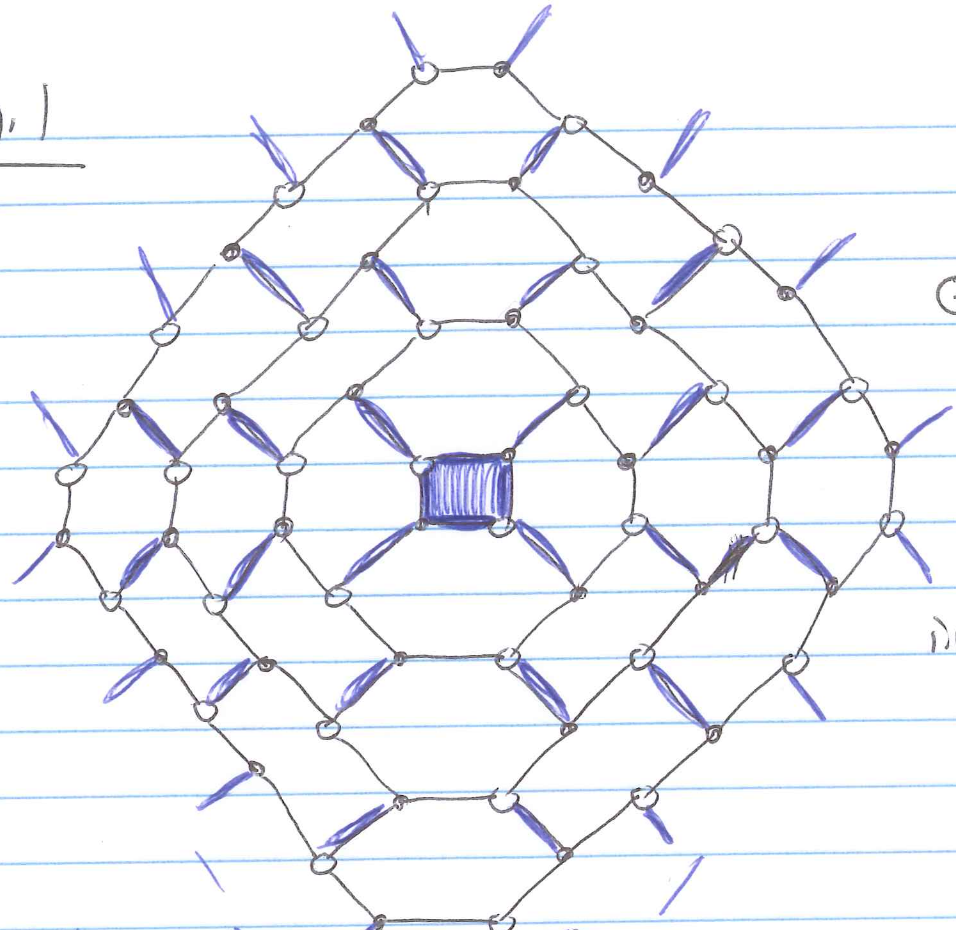

<u>Definition</u>: An <u>infinite completion</u> of a perfect matching $M$ of $G(n,i,j)$ to an <u>"infinite matching" $\tilde{M}$</u> of $\mathcal{G}_{\tilde{h}}$ so that $\tilde{M}$ uses the diagonal/wrench edges on a co-finite region of $\mathcal{G}_{\tilde{h}}$, i.e. outside $G(n,i,j)$.

<u>Claim</u>: Face weight $w(M) = w(\tilde{M})$ since the exterior of additional hexagons have 2 edges in $\tilde{M}$ on each face. Treat all faces of $\mathcal{G}_{\tilde{h}}$ as "closed faces".

closed faces of $G(n,i,j)$ have weights as expected.

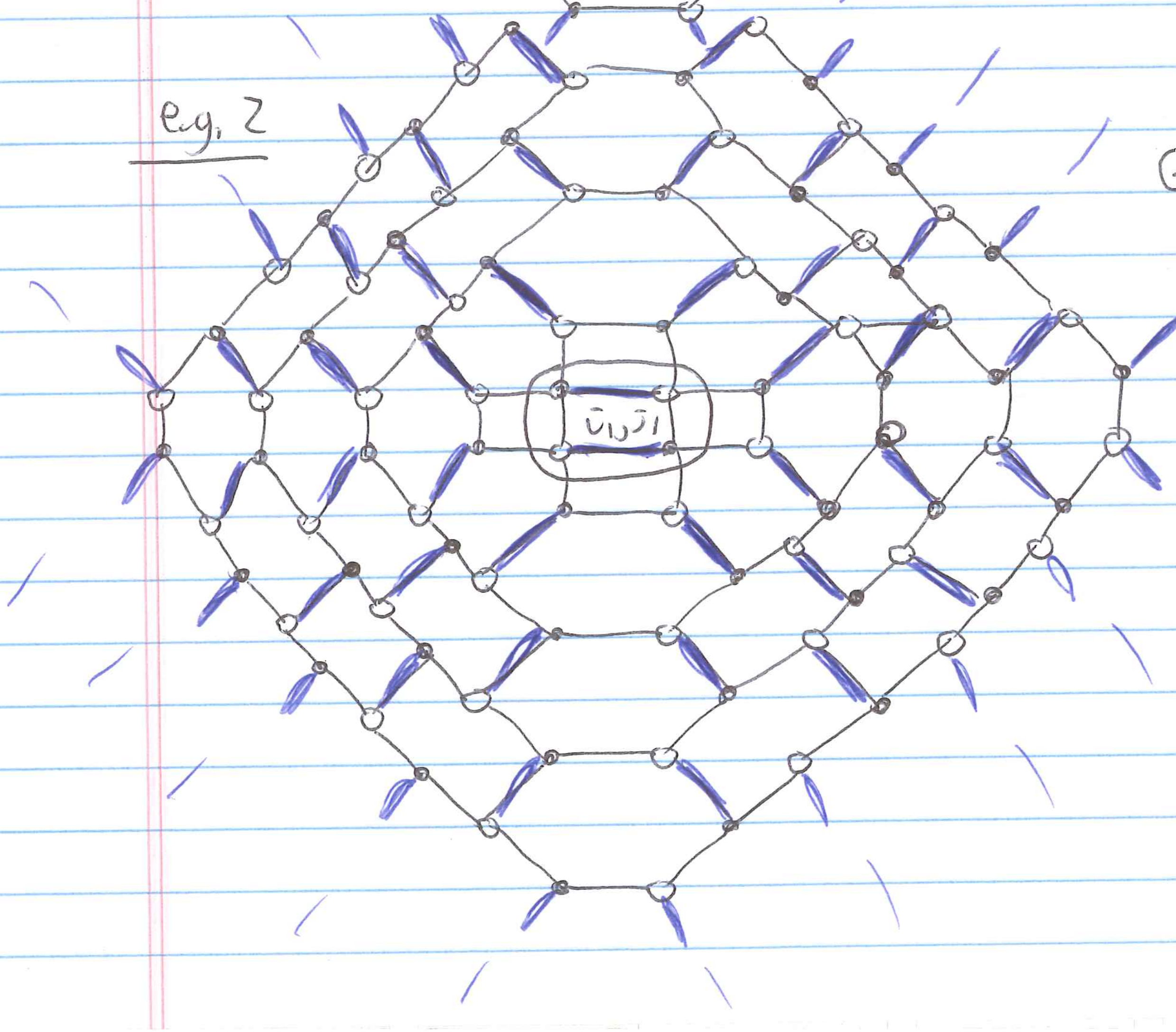
open faces of $\overline{G(n,i,j)}$ bordering hexagons have weights <u>off-by-one</u> as in Speyer's formula.

5/4/14 (6) e.g. 1



Case where $G(n_1, i_1, j_1) = \emptyset$, i.e. "just an open face", since $n_1 = h(i_1, j_1)$, i.e. $(n_1, i_1, j_1) \in \mathcal{C}\ell$

e.g. 2

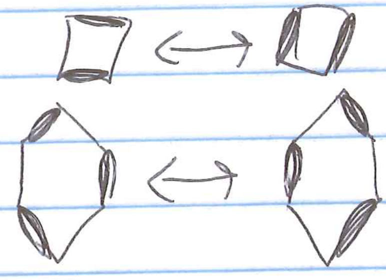



Case where $G(n_1, i_1, j_1)$ contains a single closed face, i.e. $(n_1 - 2, i_1, j_1) \in \mathcal{C}\ell$.

5/4/15 (7) In other words,

**Lemma:** The set of perfect matchings of $\mathcal{G}_{\tilde{h}}$ reachable from $\tilde{M} :=$ arbitrary matching of $G(n_1, i_1, j_1) \subset \mathcal{G}_{\tilde{h}}$ completed via including diagonal/wrench edges outside of $G(n_1, i_1, j_1)$

by twisting



or , etc.

is in bijection with perfect matchings of $G(n_1, i_1, j_1)$ s.t. weights agree.

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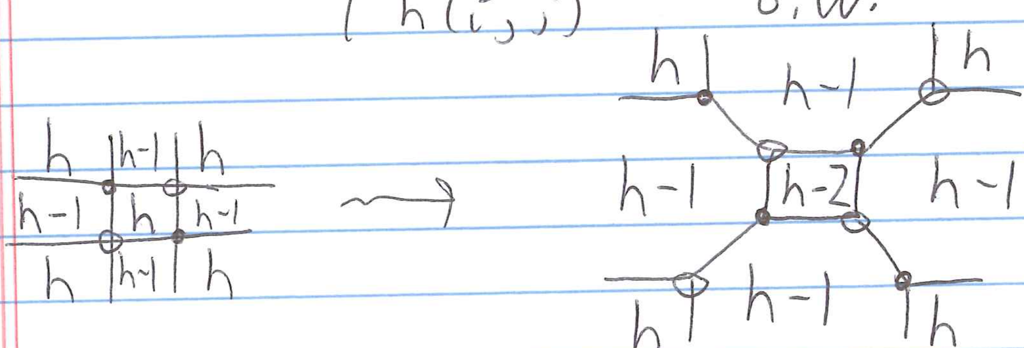
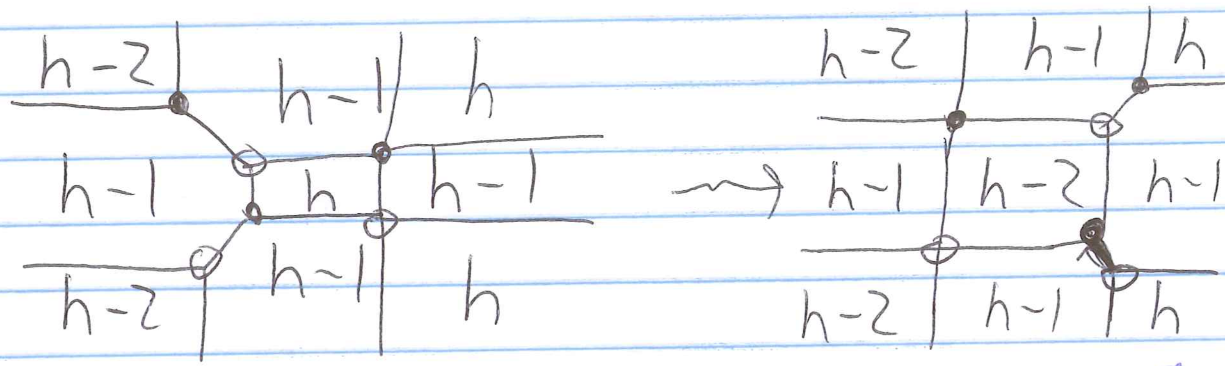
To prove the Main Theorem, it thus suffices to compare "reachable" perfect matchings of $\mathcal{G}_{\tilde{h}}$ to $\mathcal{G}_{\tilde{h}'}$ where

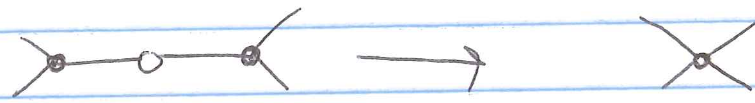
$$h'(i,j) := \begin{cases} h(i_1, j_1) + 2 & \text{if } i = i_1, j = j_1 \\ h(i,j) & \text{o.w.} \end{cases}$$



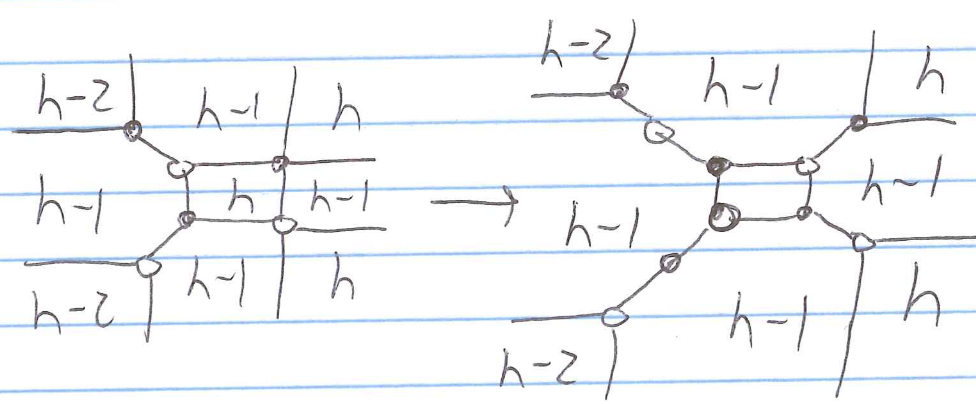


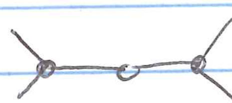

etc.

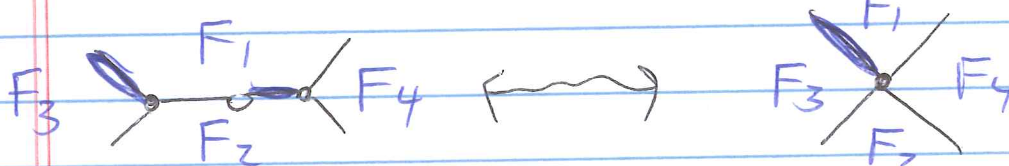
Called "urban renewal", possibly plus



e.g.

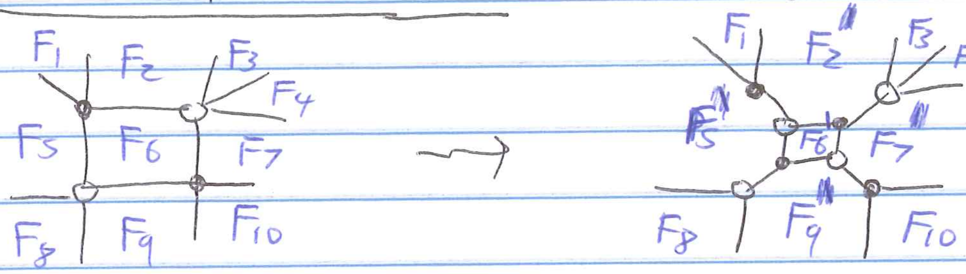



**Claim:** Changing graph by  to  does not affect (face)-weighted enumeration of perfect matchings.




Faces $F_1, F_2$ have two fewer edges, one fewer edge in $M$
$F_3, F_4$ same lengths and same edges of $M$ incident.

5/4/15 (9) More complicated Claim: Changing graph from




also does not affect (face) weighted enumeration of perfect matchings (with relation $x_6 x_6' = x_2 x_9 + x_5 x_7$)

Rem: Notice in this representative example, faces $F_1, F_3, F_4, F_6, F_8, F_{10}$ have same number of edges before & after.

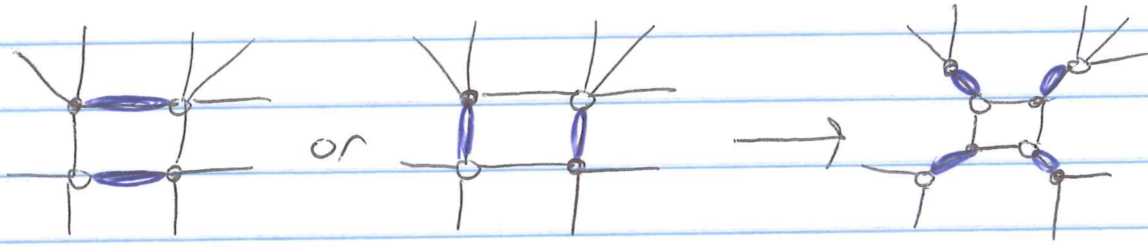
Faces $F_2, F_5, F_7, F_9$ have two more edges each (and it is possible to immediately use ≻—o—≺ → ✕ transf.)

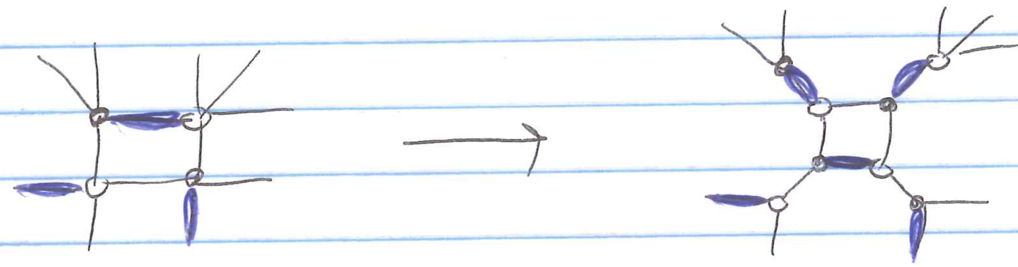
Pf of Claim: Three families of perfect matchings

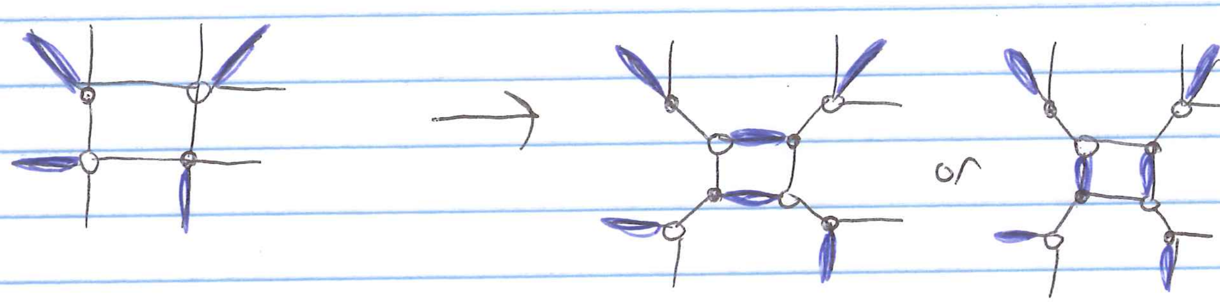
| | |
|---|---|
| $M_0$ has no perfect matching edges on center face ($F_6$) | $M_0'$ " " no edge |
| $M_1$ " " one edges | $M_1'$ " " one edge |
| $M_2$ " two edges | $M_2'$ " " two edges |

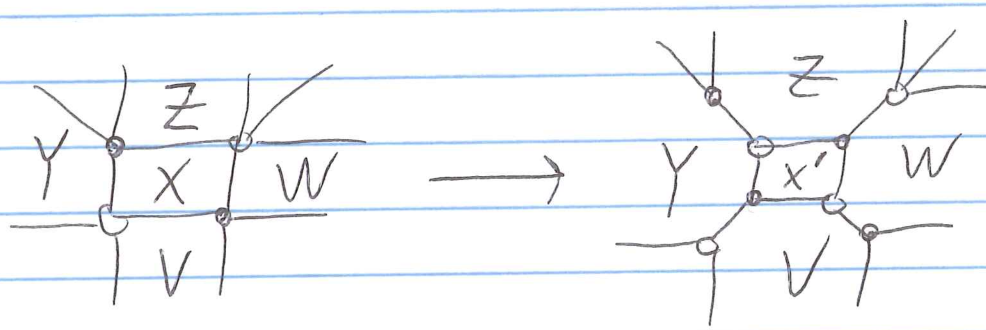
weighted 2-to-1 map $M_2 \to M_0'$
weighted 1-to-1 map $M_1 \to M_1'$
weighted 1-to-2 map $M_0 \to M_2'$

5/4/15 (10)





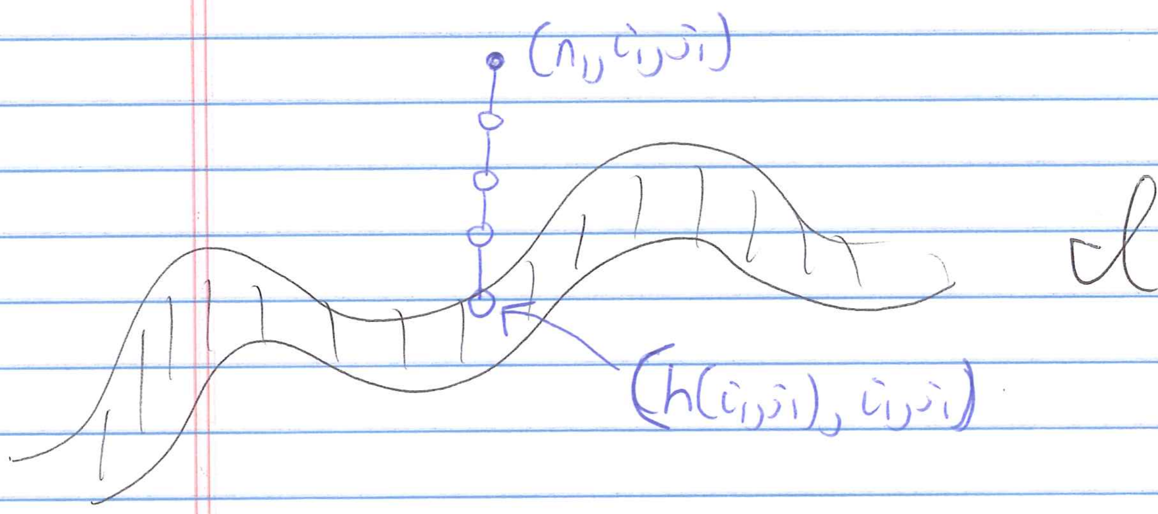

if $X'X = WY + VZ$, then

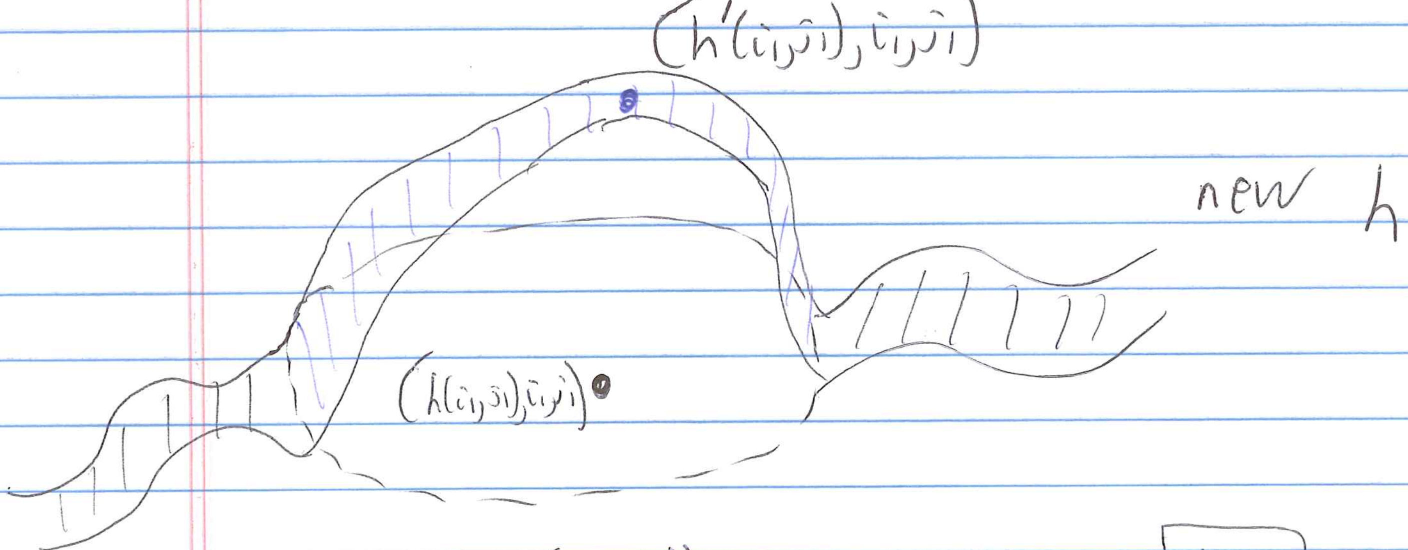
sum of weighted perfect matchings on LHS = sum of weighted perfect matchings on RHS

5/4/14 (11)

Thus applying the relation $XX' = WY + VZ$ inductively, we can "mutate" height function until it is in $c\ell$ and $f(n_1^{(k)}, i_1, j_1) = x_{i_1 j_1}$

then undoing the sequence of height changes, we find $f(n_1, i_1, j_1)$ when $n_1 > h(i_1, j_1)$




need to "mutate" $\mathcal{G}_{\tilde{h}'}$ until get $\boxed{\mathcal{G}_{\tilde{h}}}$ with original $c\ell$.

5/4/15 (12) <u>Approach via DiFrancesco-Kedem</u>

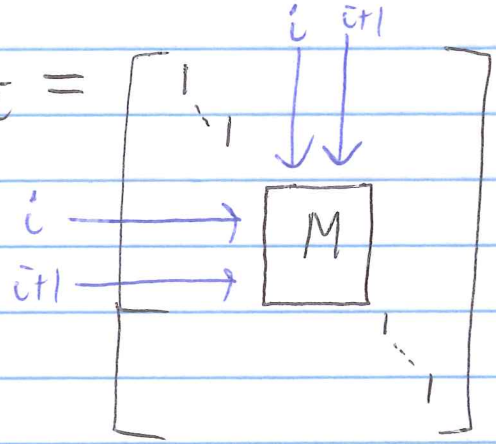
Let $U(a,b,c) = \begin{pmatrix} 1 & 0 \\ \frac{c}{b} & \frac{a}{b} \end{pmatrix}$, $V(a,b,c) = \begin{pmatrix} \frac{b}{c} & \frac{a}{c} \\ 0 & 1 \end{pmatrix}$

<u>Claim:</u> $U_i(a,b,c)\, V_{i+1}(b,c,d) = V_{i+1}(a,c,d)\, U_i(a,b,d)$

$V_i(a,b,c)\, U_{i+1}(d,e,f) = U_{i+1}(d,e,f)\, V_i(a,b,c)$

& $U(a,b,u)\, V(v,b,c) = V(v,a,b')\, U(b',c,u)$

$\iff \underline{bb' = uv + ac}$

Here $M_i =$ 


<u>T-system ($A_\infty$ unrestricted)</u> defined as

$$T_{i,j,k+1}\, T_{i,j,k-1} = T_{i,j+1,k}\, T_{i,j-1,k} + T_{i+1,j,k}\, T_{i-1,j,k}$$

Technically, get two independent systems depending on parity of $i+j+k \mod 2$, so enough to focus on one such class like Speyer did.

5/4/15 (13) <u>Rem:</u> There is also an $A_1$ case

$$T_{j,k+1}\,T_{j,k-1} = T_{j+1,k}\,T_{j-1,k} + 1,$$

an $A_r$ case: (same as $A_\infty$ case except $i \in \{1,2,\dots,r\}$ with boundary conditions $T_{0,j,k} = T_{r+1,j,k} = 1$),

and cases for other Dynkin Diagrams.

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Thm 3.6 of "T-systems, Networks, and Dimers" by Di Francesco arxiv: 1307.0095

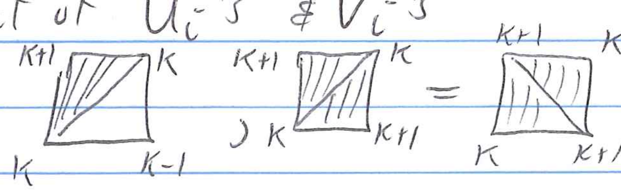
For the $A_\infty$ case with initial conditions $T_{i,j,k_{ij}} = t_{ij}$ $((i,j) \in \mathbb{Z}^2)$ for some <u>stepped surface</u> $\mathbf{k}$,

$$T_{ijk} = \frac{\left|\det M_{\mathcal{D}^0}\right|^{i-\ell+1,\dots,i-1,i}_{i-\ell+1,\dots,i-1,i} \prod_{b=1}^{\ell} t_{i'_b,j'_b}}{\prod_{a=1}^{\ell-1} t_{i_a,j_a}}$$

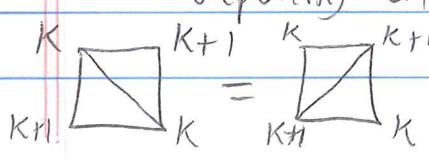
where <u>shadow $\mathcal{D}$</u> $\longleftrightarrow$ cone $C(n,i,j)$

$\mathcal{D}^0$ $\longleftrightarrow$ open cone $\mathring{C}(n,i,j)$

stepped surface $\underline{\mathbf{k}}$ $\longleftrightarrow$ $d$ defined by height function $h$

Matrix $M$ is a product of $U_i$'s & $V_i$'s depending on heights 





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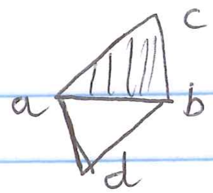
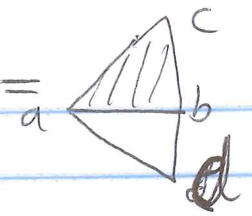
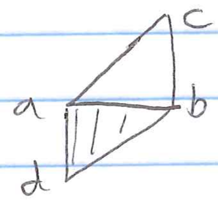
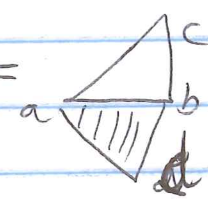



$= U(a,b,d) =$

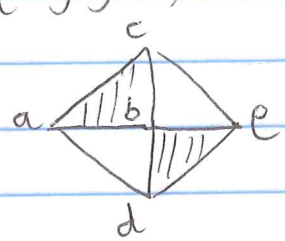


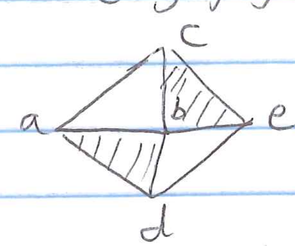




$= V(c,a,b) =$




$U(a,b,d)\, V(c,b,e) = V(c,a,b)\, U(b,e,d)$

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$\det M_{\mathcal{D}^o} = \overset{\text{weighted \#}}{\text{non-intersecting lattice paths}}$