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Lecture 6: Cluster Complexes, Generalized Associahedra, and ~~sketch of~~ towards the proof of the finite type classification

See Section 4 of [Fomin-Reading] or Chapters 5-6 of [Marsh] for surveys,

"Y-systems and Generalized Associahedra" or

"Polytopal realizations of Generalized Associahedra" for original research articles

Consider a cluster algebra $\mathcal{A}_Q$ associated to a (valued) quiver Q which is an orientation of a Dynkin Diagram of finite type.

<u>Rem</u>: By mutating at sources/sinks only, can obtain $Q' \sim Q$ which is bipartite (i.e. alternating).

e.g.
$$\begin{tikzcd} & & & & & 6 \arrow[dl] \\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 & 5 \arrow[l] \arrow[dr] & \\ & & & & & 7 \end{tikzcd} \xrightarrow[\underline{\mu}]{} \begin{tikzcd} & & & & & 6 \\ 1 \arrow[r] & 2 & 3 \arrow[l] \arrow[r] & 4 & 5 \arrow[l] \arrow[ur] \arrow[dr] & \\ & & & & & 7 \end{tikzcd}$$

by mutation sequence $\underline{\mu} = \mu_2 \mu_1 \mu_6$ (reading right to left)

<u>We assume Q is bipartite for now on.</u>

Let vertices of Q (i.e. Q_0) decompose as

$$Q_0 = \underset{\text{(sources)}}{I_+} \sqcup \underset{\text{(sinks)}}{I_-} \qquad \left[\begin{array}{l} \text{e.g. } I_+ = \{1,3,5\} \\ \phantom{\text{e.g. }} I_- = \{2,4,6,7\} \end{array} \right]$$

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Let $t_+ = \prod_{i \in I_+} s_i$, $t_- = \prod_{i \in I_-} s_i$

and $t = t_- t_+$.

t is an element of the corresponding reflection group w/ the property that it is the product of all simple reflections exactly once

Such a t is called a Coxeter elt of W.

If we take the product in a different order, would get a different Coxeter element, but

- all Coxeter elts are conjugate in W
  $\Longrightarrow$ all have same order
  Called h = Coxeter number of W.

Examples of Coxeter numbers

Type	h
A_n	$n+1$
B_n/C_n	$2n$
D_n	$2n-2$
E_6	12
E_7	18
E_8	30

Type	h
F_4	12
G_2	6
H_3	10
H_4	30
$I_2(m)$	m

2/9/15 (3) We let $\Phi = \Phi_W$ denote the finite root system associated to W, and by transitivity, to quiver Q.

Each simple reflection s_i acts linearly on Φ.

We now define a piecewise-linear variant:

Let L = root lattice $\mathbb{Z}^n$ if Φ of dim n w/ basis $\{\alpha_1, \ldots, \alpha_n\}$ of simple pos. roots

Def: For $i \in Q_0$, we define $\sigma_i : L \to L$ by

$$\sigma_i(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_j \text{ (for } j \neq i) \\ s_i(\alpha) & \text{otherwise} \end{cases}$$

Rem: Letting $[\alpha : \alpha_i]$ denote the coefficient c_i in $\alpha = c_1\alpha_1 + \ldots + c_n\alpha_n$ for positive root α and simpleroot α_i

we can rewrite actions of s_i and σ_i as

$$[s_i\alpha : \alpha_k] = \begin{cases} [\alpha : \alpha_k] & \text{if } k \neq i \\ -[\alpha : \alpha_i] - \sum_{j \neq i} \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}[\alpha : \alpha_j] & \text{if } k = i \end{cases}$$

$$[\sigma_i\alpha : \alpha_k] = \begin{cases} [\alpha : \alpha_k] & \text{if } k \neq i \\ -[\alpha : \alpha_i] - \sum_{j \neq i} \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_j \rangle} \cdot \max([\alpha : \alpha_j], 0) & \text{if } k = i \end{cases}$$

Note: $\sigma_i(-\alpha_i) = s_i(-\alpha_i) = +\alpha_i$

but $\sigma_i(-\alpha_k) = -\alpha_k$ for any $k \neq i$.

2/9/15 ④ <u>Properties</u>: • Each σ_i is an involution.

• σ_i & σ_j commute if vertices i & j are not adjacent in Q

• Each σ_i sends $\Phi_{\geq -1}$ to $\Phi_{\geq -1}$

where $\Phi_{\geq -1} = \overset{\{\text{positive roots}\}}{\Phi_+} \sqcup \{-\alpha_1, -\alpha_2, \dots, -\alpha_n\}$

Let $\tau_+ = \prod_{i \in I_+} \sigma_i$, τ_-, and τ be defined similarly.

$\boxed{\tau_+ \text{ \& } \tau_- \text{ are involutions since all constituent } \sigma_i\text{'s commute}}$

<u>Example</u> (A_2) $\quad \underset{1}{\circ} \longrightarrow \underset{2}{\circ} \quad \tau = \tau_- \tau_+ = (\sigma_2)(\sigma_1)$

$$\begin{tikzcd}
-\alpha_1 \arrow[loop below, "\sigma_2"] \arrow[r, leftrightarrow, "\sigma_1"] & \alpha_1 \arrow[r, leftrightarrow, "\sigma_2"] & \alpha_1+\alpha_2 \arrow[r, leftrightarrow, "\sigma_1"] & \alpha_2 \arrow[r, leftrightarrow, "\sigma_2"] & -\alpha_2 \arrow[loop below, "\sigma_1"]
\end{tikzcd}$$

$\tau(-\alpha_1) = \alpha_1+\alpha_2$, $\tau(\alpha_1+\alpha_2) = -\alpha_2$, $\tau(-\alpha_2) = \alpha_2$

$\tau(\alpha_2) = \alpha_1$, $\tau(\alpha_1) = -\alpha_1$.

<u>Example</u> (A_3) $\quad \overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longleftarrow \overset{3}{\circ} \quad \tau_+ = \sigma_1\sigma_3$, $\tau_- = \sigma_2$, $\tau = \sigma_2\sigma_1\sigma_3$
$= \sigma_2\sigma_3\sigma_1$

$$\begin{tikzcd}
-\alpha_1 \arrow[loop below, "\tau_-"] \arrow[r, leftrightarrow, "\tau_+"'] & \alpha_1 \arrow[r, leftrightarrow, "\tau_-"'] & \alpha_1+\alpha_2 \arrow[r, leftrightarrow, "\tau_+"'] & \alpha_2+\alpha_3 \arrow[r, leftrightarrow, "\tau_-"'] & \alpha_3 \arrow[r, leftrightarrow, "\tau_+"'] & -\alpha_3 \arrow[loop below, "\tau_-"]
\end{tikzcd}$$

$$\begin{tikzcd}
-\alpha_2 \arrow[loop below, "\tau_+"] \arrow[r, leftrightarrow, "\tau_-"'] & \alpha_2 \arrow[r, leftrightarrow, "\tau_+"'] & \alpha_1+\alpha_2+\alpha_3 \arrow[loop below, "\tau_-"]
\end{tikzcd}$$

2/9/15 (5) Example (C_2) $\underset{1}{\circ} \Rightarrow \underset{2}{\circ}$

$$\overset{\tau_-}{\circlearrowleft}\,-\alpha_1 \xleftrightarrow{\tau_+} \alpha_1 \xleftrightarrow{\tau_-} \alpha_1+\alpha_2\,\overset{\tau_+}{\circlearrowleft} \quad \Big| \quad \overset{\tau_+}{\circlearrowleft}\,-\alpha_2 \xleftrightarrow{\tau_-} \alpha_2 \xleftrightarrow{\tau_+} 2\alpha_1+\alpha_2\,\overset{\tau_-}{\circlearrowleft}$$

Thm (Cor 1.9 of [CFZ]) Let $F : \{-\alpha_1, -\alpha_2, \ldots, -\alpha_n\} \to \mathbb{R}$ satisfy

- $F(w_0(\alpha_i)) = F(-\alpha_i)$ for longest word w_0
- $\sum_{i \in Q_0} \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_j \rangle} F(-\alpha_i) > 0 \quad \forall j \in Q_0$

(i.e. F generic enough) We also get F invariant under τ_+ or τ_-.

Then we can use scalars $F(-\alpha_1), \ldots, F(-\alpha_n)$ to define a simple convex polytope P_Φ whose normal fan is $\Delta(\Phi)$, a generalized associahedron, which has

$\Phi_{\geq -1}$ as vertices and simplices are subsets of mutually compatible elts in $\Phi_{\geq -1}$

(i.e. maximal simplices correspond to clusters under $\underset{d_1\alpha_1+\cdots+d_n\alpha_n}{\overset{\shortparallel}{\alpha}} \leftrightarrow X_\alpha = \frac{*}{x_1^{d_1}\cdots x_n^{d_n}}$)

P_Φ defined by piece-wise linear inequalities

$$\max\left([B_1,\alpha_1]z_1 + [B_1,\alpha_2]z_2 + \cdots + [B_1,\alpha_n]z_n,\ [B_2,\alpha_1]z_1 + \cdots + [B_2,\alpha_n]z_n,\ \ldots,\ [B_K,\alpha_1]z_1 + \cdots + [B_K,\alpha_n]z_n\right) \leq F(-\alpha_m)$$

one for each $\langle \tau_+, \tau_- \rangle$-orbit $\{B_1, \ldots, B_K\} \ni -\alpha_m$.

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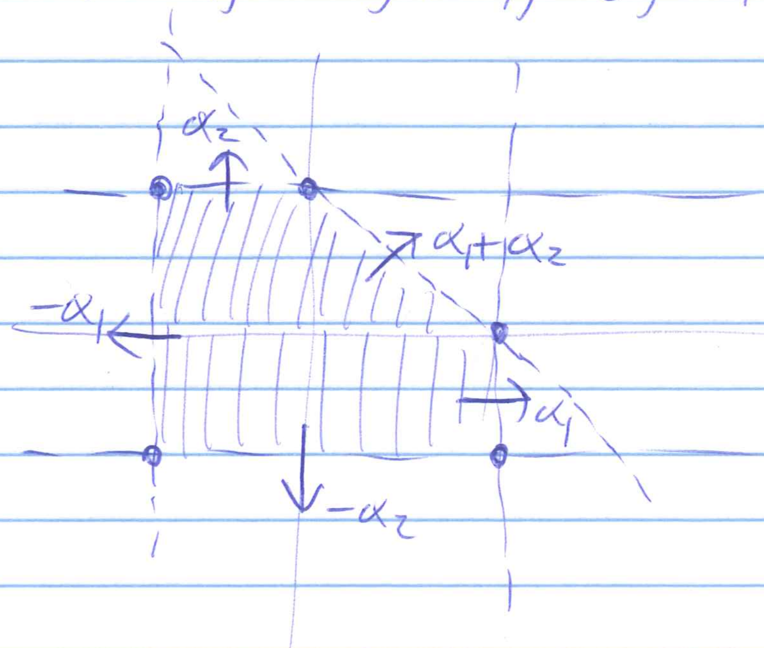
E.g.'s again

A_2 : one $\langle \tau_+, \tau_- \rangle$-orbit. Let $F(-\alpha_1) = F(-\alpha_2) = c > 0$.

one piece-wise linear inequality :

$$\max(-z_1, -z_2, +z_1, z_2, z_1 + z_2) \le c$$



pentagon

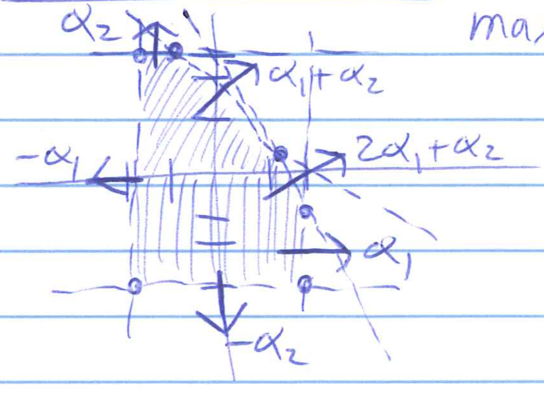
C_2 : two $\langle \tau_+, \tau_- \rangle$-orbits. Let $F(-\alpha_1) = c_1$, $F(-\alpha_2) = c_2$

$0 \le c_1 < c_2 < 2c_1$.

two PL inequalities: $\max(-z_1, z_1, z_1 + z_2) \le c_1$

$\max(-z_2, z_2, 2z_1 + z_2) \le c_2$

$\begin{pmatrix} \text{e.g.} \\ c_1 = 2 \\ c_2 = 3 \end{pmatrix}$



hexagon

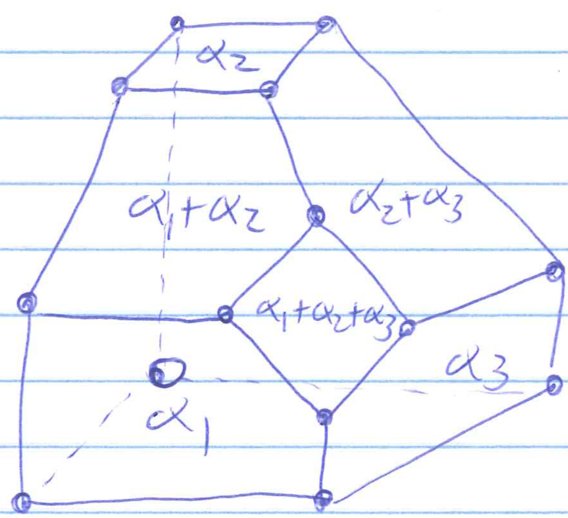
2/9/15 7 A_3: two $\langle \tau_+, \tau_- \rangle$-orbits. Let $F(-\alpha_1) = F(-\alpha_3) = c_1$

$\underline{0 < c_1 < c_2 < 2c_1}$ $\qquad F(-\alpha_2) = c_2$

$$\max(-z_1, -z_3, z_1, z_3, z_1 + z_2, z_2 + z_3) \le c_1$$

$$\max(-z_2, z_2, z_1 + z_2 + z_3) \le c_2$$

See Figure 4.7 of [FR04] w/ $c_1 = \frac{3}{2}$, $c_2 = 2$.



$-\alpha_1, -\alpha_2, -\alpha_3$ correspond to the three faces incident to 0 and not shown.

More benefits to using PL maps τ_+ and τ_-:

We define the compatibility degree $\Phi_{\geq -1} \times \Phi_{\geq -1} \to \mathbb{Z}_{\geq 0}$
$(\alpha, \beta) \mapsto (\alpha \| \beta)$

by $(-\alpha_i \| \beta) = \max([\beta : \alpha_i], 0)$

and extend by $(\tau_\pm \alpha \| \tau_\pm \beta) = (\alpha \| \beta)$.

Let $\alpha \leftrightarrow x_\alpha$, $\beta \leftrightarrow x_\beta$. Then x_α & x_β are in same cluster together $\Leftrightarrow$ $(\alpha \| \beta) = 0$.

Further, if $\alpha, \beta \in \Phi(J)_{\geq -1}$ for $J \subsetneq Q_0$, comp. degree same w.r.t. smaller quiver on J.