Brane Tilings and Cluster Algebras

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Outline.

1. Introduction to Cluster Algebras.
2. What is a Brane Tiling
3. The Del Pezzo 3 Quiver and Lattice
4. Gale-Robinson Sequences
5. Further Examples: Aztec Castles (work of Leoni-Neel-Turner)

http//math.umn.edu/∼musiker/Brane.pdf

Introduction to Cluster Algebras

In the late 1990's: Fomin and Zelevinsky were studying total positivity and canonical bases of algebraic groups. They noticed recurring combinatorial and algebraic structures.

Let them to define cluster algebras, which have now been linked to quiver representations, Poisson geometry Teichmüller theory, tilting theory, mathematical physics, discrete integrable systems, string theory, and many other topics.

Cluster algebras are a certain class of commutative rings which have a distinguished set of generators that are grouped into overlapping subsets, called clusters, each having the same cardinality.

What is a Cluster Algebra?

Definition (Sergey Fomin and Andrei Zelevinsky 2001) A cluster algebra $\mathcal{A}$ (of geometric type) is a subalgebra of $k(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ constructed cluster by cluster by certain exchange relations.

Generators:

Specify an initial finite set of them, a Cluster, $\{x_1, x_2, \ldots, x_{n+m}\}$.

Construct the rest via Binomial Exchange Relations:

$$x_\alpha x'_\alpha = \prod x_{\gamma_i}^{d_i^+} + \prod x_{\gamma_i}^{d_i^-}.$$

The set of all such generators are known as Cluster Variables, and the initial pattern of exchange relations (described as a *valued* quiver, i.e. a directed graph) determines the Seed.

Relations:

Induced by the Binomial Exchange Relations.

Example: Rank 2 Cluster Algebras

Let $B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$, $b, c \in \mathbb{Z}_{>0}$. $(\{x_1, x_2\}, B)$ is a seed for a cluster algebra $\mathcal{A}(b, c)$ of rank 2. (E.g. when $b = c$, $B = B(Q)$ where Q is a 2-vertex quiver with b arrows from $v_1 \to v_2$.)

$$\mu_1(B) = \mu_2(B) = -B \text{ and } x_1 x_1' = x_2^c + 1, \quad x_2 x_2' = 1 + x_1^b.$$

Thus the cluster variables in this case are

$$\{x_n : n \in \mathbb{Z}\} \text{ satisfying } x_n x_{n-2} = \begin{cases} x_{n-1}^b + 1 \text{ if } n \text{ is odd} \\ x_{n-1}^c + 1 \text{ if } n \text{ is even} \end{cases}.$$

Example ($b = c = 1$): (Finite Type, of Type A_2)

$$x_3 = \frac{x_2 + 1}{x_1}. \quad x_4 = \frac{x_3 + 1}{x_2} = \frac{\frac{x_2+1}{x_1} + 1}{x_2} = \frac{x_1 + x_2 + 1}{x_1 x_2}.$$

$$x_5 = \frac{x_4 + 1}{x_3} = \frac{\frac{x_1+x_2+1}{x_1 x_2} + 1}{(x_2 + 1)/x_1} = \frac{x_1(x_1 + x_2 + 1 + x_1 x_2)}{x_1 x_2 (x_2 + 1)} = \frac{x_1 + 1}{x_2}. \quad x_6 = x_1.$$

Example: Rank 2 Cluster Algebras

Example ($b = c = 2$): (Affine Type, of Type $\widetilde{A}_1$)

$$x_3 = \frac{x_2^2 + 1}{x_1}. \quad x_4 = \frac{x_3^2 + 1}{x_2} = \frac{x_2^4 + 2x_2^2 + 1 + x_1^2}{x_1^2 x_2}.$$

$$x_5 = \frac{x_4^2 + 1}{x_3} = \frac{x_2^6 + 3x_2^4 + 3x_2^2 + 1 + x_1^4 + 2x_1^2 + 2x_1^2 x_2^2}{x_1^3 x_2^2}, \quad \dots$$

If we let $x_1 = x_2 = 1$, we obtain $\{x_3, x_4, x_5, x_6\} = \{2, 5, 13, 34\}$.

The next number in the sequence is $x_7 = \frac{34^2+1}{13} = \frac{1157}{13} = 89$, an integer!

What is a Brane Tiling (in Physics & Algebraic Geometry)

In physics, **Brane Tilings** are combinatorial models that are used to

Decribe the world volume of both D_3 and M_2 branes, and describe certain $(3+1)$-dimensional superconformal field theories arising in string theory (Type II B).

In Algebraic Geometry, they are used to

Probe certain toric Calabi-Yau singularities, and relate to non-commutative crepant resolutions and the 3-dimensional McKay correspondence.

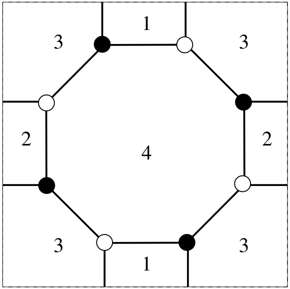
Certain examples of path algebras with relations (Jacobian Algebras) can be constructed by a quiver and potential coming from a brane tiling.

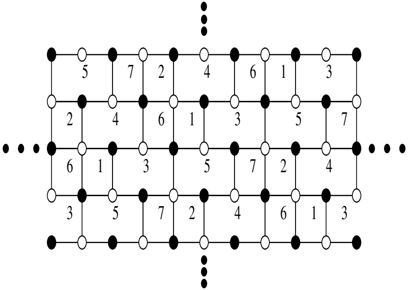
What is a Brane Tiling (Combinatorially)

However, this is a combinatorics talk, not a physics talk, so I will henceforth focus on combinatorial motivation instead.

Most simply stated, a Brane Tiling is a **Bipartite graph on a torus**.

We view such a tiling as a doubly-periodic tiling of its universal cover, the Euclidean plane.

Examples: 

, 

Brane Tilings from a Quiver Q with Potential W

A **Brane Tiling** can be associated to a pair (Q, W), where Q is a quiver and W is a potential (called a superpotential in the physics literature).

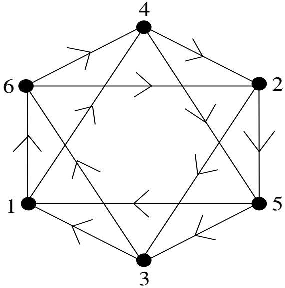
A quiver Q is a directed graph where each edge is referred to as an arrow, and multiple edges are allowed.

A potential W is a linear combination of cyclic paths in Q (possibly an infinite linear combination).

For combinatorial purposes, we assume other conditions on (Q, W), such as

• Each arrow of Q appears in one term of W with a positive sign, and one term with a negative sign.

• The number of terms of W with a positive sign equals the number with a negative sign. All coefficients in W are ± 1.

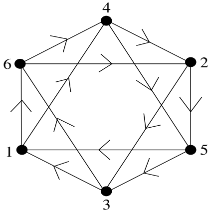
Example (The dP_3 Quiver): $Q_{dP_3} = Q =$ 

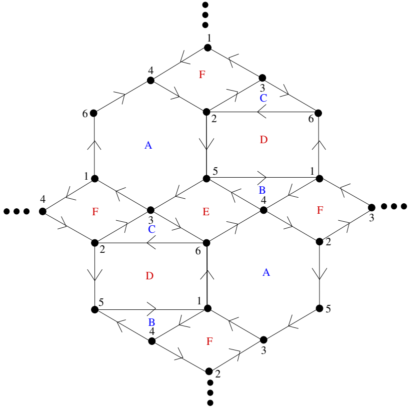
,

$$W = A_{16}A_{64}A_{42}A_{25}A_{53}A_{31} + A_{14}A_{45}A_{51} + A_{23}A_{36}A_{62} - A_{16}A_{62}A_{25}A_{51} - A_{36}A_{64}A_{45}A_{53} - A_{14}A_{42}A_{23}A_{31}.$$

We now unfold Q onto the plane, letting the three positive (resp. negative) terms in W depict clockwise (resp. counter-clockwise) cycles on $\widetilde{Q}$.

Example (continued):

$Q =$ 

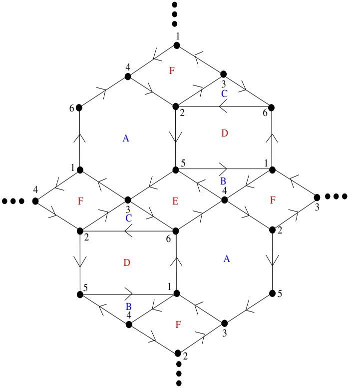
unfolds to $\widetilde{Q} =$ 

$$W = A_{16}A_{64}A_{42}A_{25}A_{53}A_{31}(A) + A_{14}A_{45}A_{51}\ (B) + A_{23}A_{36}A_{62}\ (C)$$
$$- A_{16}A_{62}A_{25}A_{51}\ (D) - A_{36}A_{64}A_{45}A_{53}\ (E) - A_{14}A_{42}A_{23}A_{31}(F).$$

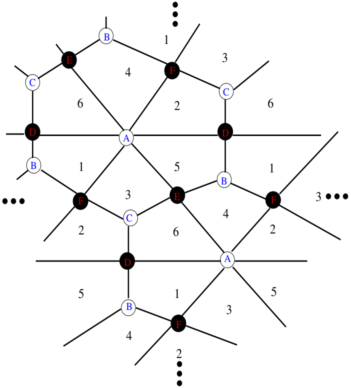
Locally, the configurations around vertices of Q and $\widetilde{Q}$ are identical.

Brane Tilings from a Quiver Q with Potential W

Taking the planar dual yields a bipartite graph on a torus (**Brane Tiling**):



$$\widetilde{Q} \longrightarrow \mathcal{T}_Q =$$

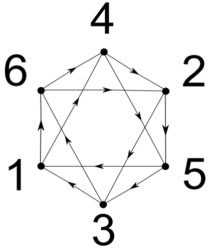


Negative Term in $W \longleftrightarrow$ Counter-Clockwise cycle in $\widetilde{Q} \longleftrightarrow \bullet$ in $\mathcal{T}_Q$

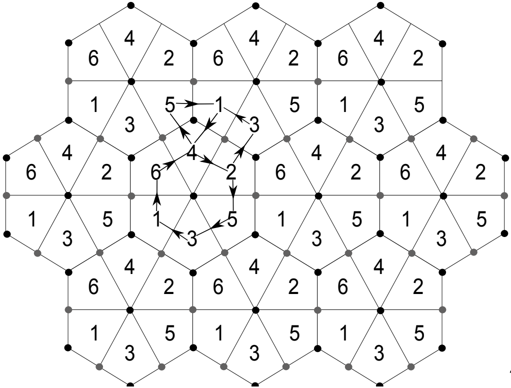
Positive Term in $W \longleftrightarrow$ Clockwise cycle in $\widetilde{Q} \longleftrightarrow \circ$ in $\mathcal{T}_Q$

(To obtain $\widetilde{Q}$ from $\mathcal{T}_Q$, we dualize edges so that white is on the right.)

Summarizing the dP_3 Example:



Q



$\mathcal{T}_Q$

Negative Term in $W \longleftrightarrow$ Counter-Clockwise cycle in $\widetilde{Q} \longleftrightarrow \bullet$ in $\mathcal{T}_Q$

Positive Term in $W \longleftrightarrow$ Clockwise cycle in $\widetilde{Q} \longleftrightarrow \circ$ in $\mathcal{T}_Q$

(To obtain $\widetilde{Q}$ from $\mathcal{T}_Q$, we dualize edges so that white is on the right.)

Brane Tilings in Physics

Face $\longleftrightarrow$ $U(N)$ Gauge Group

Edge $\longleftrightarrow$ Bifundamental Chiral Fields (Representations)

Vertex $\longleftrightarrow$ Gauge-invariant operator (Term in the Superpotential)

Together, this data yields a **quiver gauge theory**. One can apply Seiberg duality to get a different quiver gauge theory.

Combinatorial connection:
Seiberg duality corresponds to mutation in cluster algebra theory.

Description of Seiberg Duality (from physics)

From **"Brane Dimers and Quiver Gauges Theories (2005)** by Franco, Hanany, Kennaway, Wegh, and Wecht:

After picking a node to dualize at: "Reverse the direction of all arrows entering or exiting the dualized node. This is because Seiberg duality requires that the dual quarks transform in the conjugate flavor representations to the originals. ...

Next, draw in ... bifundamentals which correspond to composite (mesonic) operators. ... the Seiberg mesons are promoted to the fields in the bifundamental representation of the gauge group. ...

It is possible that this will make some fields massive, in which case the appropriate fields should then be integrated out."

Description of Seiberg Duality (rephrased combinatorially)

Pick a vertex j of the quiver Q (equiv. face of the brane tiling $\mathcal{T}_Q$) at which to mutate. Then, reverse the direction of all arrows incident to j, i.e. $A_{ij} \rightarrow A^*_{ji}$. Next, for every two-path $i \rightarrow j \rightarrow k$, *"meson"*, in Q draw in a new arrow $i \rightarrow k$, *"the Seiberg mesons are promoted to the fields"*. Let Q' denote this new quiver.

We similarly alter the superpotential W to get W'. For every 2-path $i \rightarrow j \rightarrow k$ in Q, we replace any appearance of the product $A_{ij}A_{jk}$ in W with the singleton A_{ik}, and add or subtract a new degree 3-term, $A_{ik}A^*_{kj}A^*_{ji}$.

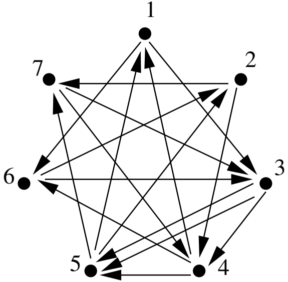
It is possible, that this will make some of the terms of W' of degree two, *"massive"*, in which case there should be an associated 2-cycle in the mutated quiver Q' that can be deleted, *"the appropriate fields should then be integrated out"*.

This is in fact Mutation of Quivers with potential from cluster algebras (as defined by Derksen-Weyman-Zelevinsky)!

Description of Seiberg Duality (on the Brane Tiling)

In the special case, that we are mutating at a vertex with two arrows in and out, a **toric vertex**, this corresponds to a Urban Renewal of a square face in the brane tiling.

Example ($Q_7^{(2,3)}$):

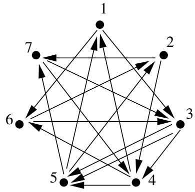


with potential

$$W = A_{13}A_{34}A_{41} + A_{16}A_{63}A_{35}A_{51} + A_{35}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62}$$
$$- A_{16}A_{62}A_{24}A_{41} - A_{34}A_{46}A_{63} - A_{13}A_{35}A_{51} - A_{27}A_{73}A_{35}A_{52} - A_{45}A_{57}A_{74}.$$

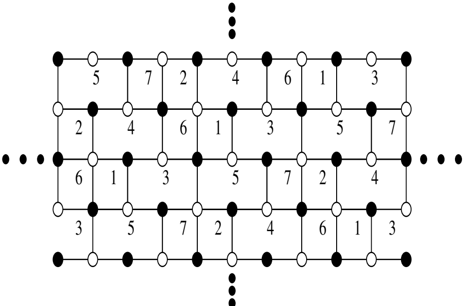
Consider the corresponding Brane Tiling $\mathcal{T}_7^{(2,3)}$ and mutation of (Q, W) at the toric vertex labeled 1. **(Associated to Gale-Robinson Sequence)**

Description of Seiberg Duality (on the Brane Tiling)

Example ($Q_7^{(2,3)}$): 

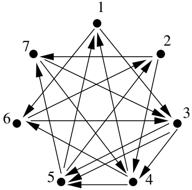
with potential

$$
\begin{aligned}
W &= A_{13}A_{34}A_{41} + A_{16}A_{63}A_{35}^{(V)}A_{51} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\
&- A_{16}A_{62}A_{24}A_{41} - A_{34}A_{46}A_{63} - A_{13}A_{35}^{(H)}A_{51} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74}.
\end{aligned}
$$



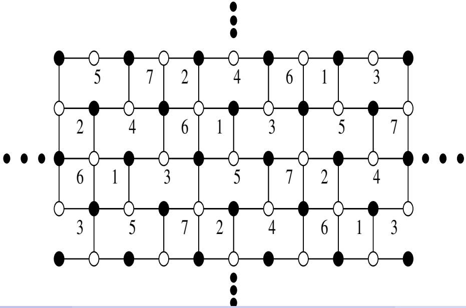
Description of Seiberg Duality (on the Brane Tiling)

Example ($Q_7^{(2,3)}$):

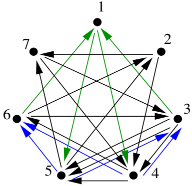


Rotate potential terms containing 1

$$\begin{aligned} W \quad = \quad & {\color{blue}A_{41}A_{13}}A_{34} + {\color{blue}A_{51}A_{16}}A_{63}A_{35}^{(V)} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\ - \quad & {\color{blue}A_{41}A_{16}}A_{62}A_{24} - A_{34}A_{46}A_{63} - {\color{blue}A_{51}A_{13}}A_{35}^{(H)} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74}. \end{aligned}$$

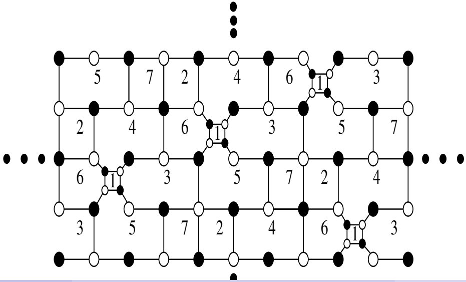


Example ($Q_7^{(2,3)}$):

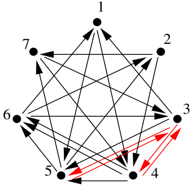


Mutating at 1 yields

$$\begin{aligned} W' &= A_{43}A_{34} + A_{56}A_{63}A_{35}^{(V)} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\ &- A_{46}^{(D)}A_{62}A_{24} - A_{34}A_{46}A_{63} - A_{53}^{(H)}A_{35}^{(H)} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74} \\ &+ A_{14}^{*}A_{46}^{(D)}A_{61}^{*} + A_{15}^{*}A_{53}^{(H)}A_{31}^{*} - A_{14}^{*}A_{43}A_{31}^{*} - A_{15}^{*}A_{56}A_{61}^{*}. \end{aligned}$$

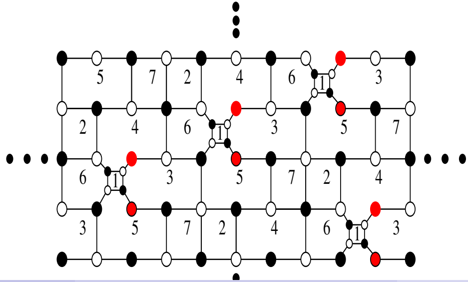


Example ($Q_7^{(2,3)}$):



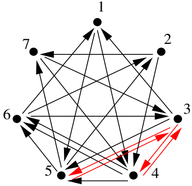
Highlighting Massive terms

$$
\begin{aligned}
W' \quad = \quad & {\color{red}A_{43}A_{34}} + A_{56}A_{63}A_{35}^{(V)} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\
- \quad & A_{46}^{(D)}A_{62}A_{24} - A_{34}A_{46}A_{63} - {\color{red}A_{53}^{(H)}A_{35}^{(H)}} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74} \\
+ \quad & A_{14}^{*}A_{46}^{(D)}A_{61}^{*} + A_{15}^{*}A_{53}^{(H)}A_{31}^{*} - A_{14}^{*}A_{43}A_{31}^{*} - A_{15}^{*}A_{56}A_{61}^{*}.
\end{aligned}
$$



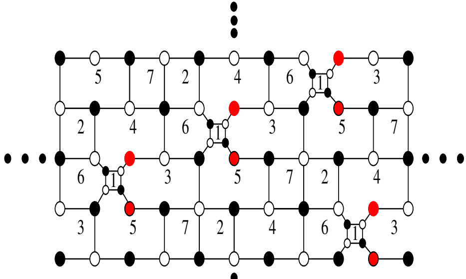
Description of Seiberg Duality (on the Brane Tiling)

Example ($Q_7^{(2,3)}$):

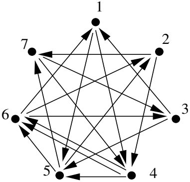


Highlighting complementary terms

$$\begin{aligned} W' &= A_{43}A_{34} + A_{56}A_{63}A_{35}^{(V)} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\ &- A_{46}^{(D)}A_{62}A_{24} - A_{34}A_{46}A_{63} - A_{53}^{(H)}A_{35}^{(H)} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74} \\ &+ A_{14}^{*}A_{46}^{(D)}A_{61}^{*} + A_{53}^{(H)}A_{31}^{*}A_{15}^{*} - A_{43}A_{31}^{*}A_{14}^{*} - A_{15}^{*}A_{56}A_{61}^{*}. \end{aligned}$$

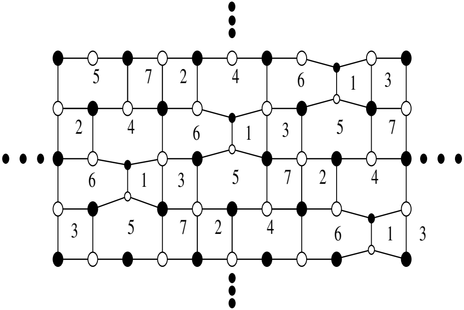


Example ($Q_7^{(2,3)}$):

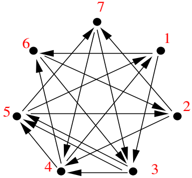


Reduces the potential to

$$
\begin{aligned}
W'' &= A_{56}A_{63}A_{35}^{(V)} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} - A_{46}^{(D)}A_{62}A_{24} - A_{27}A_{73}A_{35}^{(V)}A_{52} \\
&- A_{45}A_{57}A_{74} + A_{14}^{*}A_{46}^{(D)}A_{61}^{*} - A_{15}^{*}A_{56}A_{61}^{*} - A_{46}A_{63}A_{31}^{*}A_{14}^{*} + A_{31}^{*}A_{15}^{*}A_{57}A_{73}.
\end{aligned}
$$

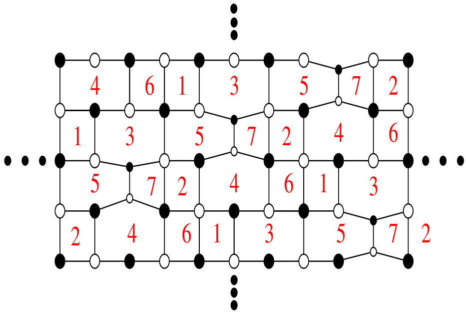


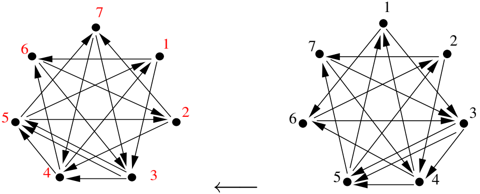
Description of Seiberg Duality (on the Brane Tiling)

Example ($Q_7^{(2,3)}$): 

If we cyclically permute vertices

$$\begin{aligned} W'' &= A_{45}A_{52}A_{24}^{(V)} + A_{13}A_{34}A_{41} + A_{16}A_{63}A_{35}A_{51} - A_{35}^{(D)}A_{51}A_{13} - A_{16}A_{62}A_{24}^{(V)}A_{41} \\ &- A_{34}A_{46}A_{63} + A_{73}^{*}A_{35}^{(D)}A_{57}^{*} - A_{74}^{*}A_{45}A_{57}^{*} - A_{35}A_{52}A_{27}^{*}A_{73}^{*} + A_{27}^{*}A_{74}^{*}A_{46}A_{62}. \end{aligned}$$

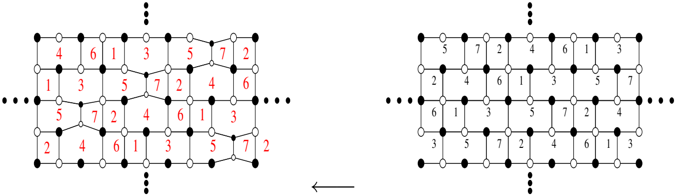




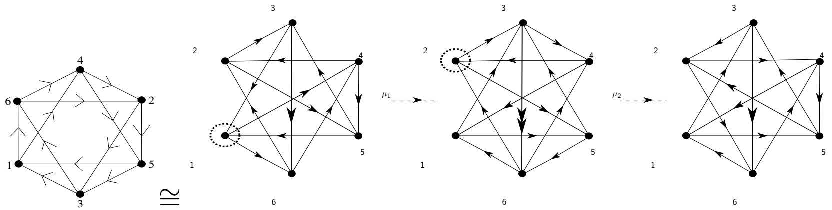
Example ($Q_7^{(2,3)}$):

The cyclic permutation yields the **original** Brane Tiling and (Q, W)!

$$
\begin{aligned}
W'' &= A_{45}A_{52}A_{24}^{(V)} + A_{13}A_{34}A_{41} + A_{16}A_{63}A_{35}A_{51} - A_{35}^{(D)}A_{51}A_{13} - A_{16}A_{62}A_{24}^{(V)}A_{41} \\
&- A_{34}A_{46}A_{63} + A_{73}^{*}A_{35}^{(D)}A_{57}^{*} - A_{74}^{*}A_{45}A_{57}^{*} - A_{35}A_{52}A_{27}^{*}A_{73}^{*} + A_{27}^{*}A_{74}^{*}A_{46}A_{62} \\
W &= A_{13}A_{34}A_{41} + A_{16}A_{63}A_{35}^{(V)}A_{51} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\
&- A_{16}A_{62}A_{24}A_{41} - A_{34}A_{46}A_{63} - A_{13}A_{35}^{(H)}A_{51} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74}.
\end{aligned}
$$



The quiver Q_{dP_3} has a similar **periodicity** property.



If we mutate Q_{dP_3} by $1, 2, 3, 4, 5, 6, 1, 2, \ldots$, after the first two mutations, we obtain same quiver back up to cyclically permuting the vertex labels.

Point: Mutating once in the $Q_N^{(r,s)}$ case, or twice in the Q_{dP_3} case, yields a quiver with potential that is equivalent up to cyclic rotation.

Such quivers are called periodic in the Fordy-Marsh sense.

Cluster Variable Mutation

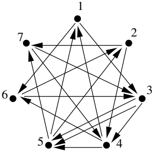
In addition to the mutation of quivers, there is also a complementary **cluster mutation** that can be defined.

Cluster mutation yields a sequence of Laurent polynomials in $\mathbb{Q}(x_1, x_2, \ldots, x_n)$ known as **cluster variables**.

Given a quiver Q (the potential is irrelevant here) and an initial cluster $\{x_1, \ldots, x_N\}$, then mutating at vertex 1 yields a new cluster variable x_{N+1}

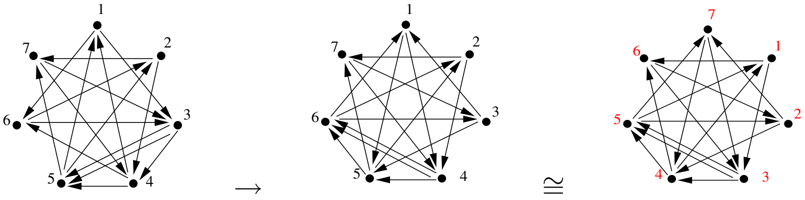
$$\text{defined by} \qquad x_{N+1} = \left(\prod_{1 \to i \in Q} x_i + \prod_{i \to 1 \in Q} x_i \right) \Big/ x_1.$$

Example ($Q_N^{(r,s)}$): In Q, $1 \to r+1,\ N-r+1$ and $1 \leftarrow s+1,\ N-s+1$.



For $r = 2, s = 3, N = 7$, we get $x_8 = (x_3 x_6 + x_4 x_5)/x_1$.

Example ($Q_N^{(r,s)}$): (e.g. $r = 2,\ s = 3,\ N = 7$)



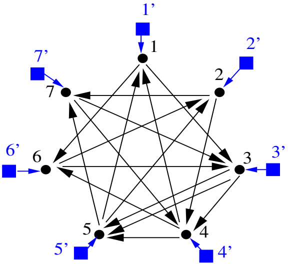
Mutating at $1, 2, 3, \ldots, N, 1, 2, \ldots$ yields the same quiver, up to cyclic permutation, at each step, hence we obtain the infinite sequence of $x_{N+1}, x_{N+2}, \ldots$ satsifying

$$x_n = \left(x_{n-r}x_{n-N+r} + x_{n-s}x_{n-N+s}\right)/x_{n-N} \quad \text{for} \quad n > N.$$

Known as the **Gale-Robinson Sequence** of Laurent polynomials.

The Gale-Robinson Sequence (with coefficients)

Example ($Q_N^{(r,s)}$): (e.g. $r = 2,\ s = 3,\ N = 7$)



We add N frozen vertices to $Q_N^{(r,s)}$ with incoming arrows. Let y_i denote the **cluster variable** corresponding to vertex i'.

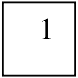
Mutating again at $1, 2, 3, \ldots, N, 1, 2, \ldots$ (never at frozen vertices) yields a infinite sequence of cluster variables with a more complicated recurrence:

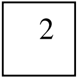
$$x_n x_{n-N} = x_{n-r} x_{n-N+r} + \prod_{i=1}^{n} y_i^{d(N-n-i,s,n-s)}\ x_{n-s} x_{n-N+s} \quad \text{for} \quad n > N.$$

where $d(M, s, s') = \#$ ways to write M as $A \cdot s + B \cdot s'$ with $A, B \in \mathbb{Z}_{\geq 0}$

Gale-Robinson Sequence Example

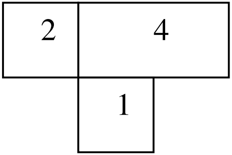
For $Q_7^{(2,3)}$, $x_8 = \frac{x_4x_5{\color{red}y_1}+x_3x_6}{x_1}$, $x_9 = \frac{x_5x_6{\color{red}y_2}+x_4x_7}{x_2}$, $x_{10} = \frac{x_1x_6x_7{\color{red}y_1y_3}+x_4x_5^2{\color{red}y_1}+x_3x_5x_6}{x_1x_3}$,
$x_{11} = \frac{x_2x_4x_5x_7{\color{red}y_1y_2y_4}+x_2x_3x_6x_7{\color{red}y_2y_4}+x_1x_5x_6^2{\color{red}y_2}+x_1x_4x_6x_7}{x_1x_2x_4}$, ...

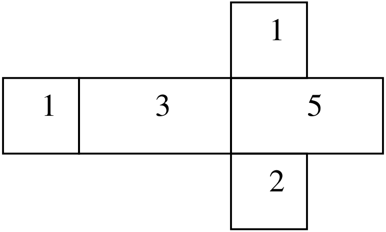
$x_8 \leftrightarrow$ 

, $x_9 \leftrightarrow$ 

, $x_{10} \leftrightarrow$ 

,

$x_{11} \leftrightarrow$ 

, $x_{12} \leftrightarrow$ 

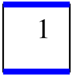
, ...

Gale-Robinson Sequence Example (continued)

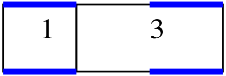
With **Minimal Matchings** Highlighted:

For $Q_7^{(2,3)}$, $x_8 = \frac{x_4x_5y_1 + x_3x_6}{x_1}$, $x_9 = \frac{x_5x_6y_2 + x_4x_7}{x_2}$, $x_{10} = \frac{x_1x_6x_7y_1y_3 + x_4x_5^2y_1 + x_3x_5x_6}{x_1x_3}$,

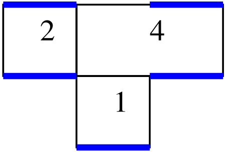
$x_{11} = \frac{x_2x_4x_5x_7y_1y_2y_4 + x_2x_3x_6x_7y_2y_4 + x_1x_5x_6^2y_2 + x_1x_4x_6x_7}{x_1x_2x_4}$, ...

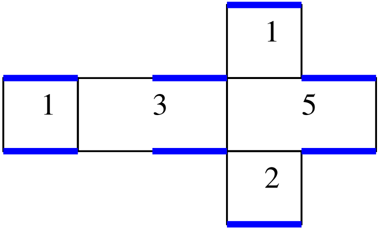
$x_8 \leftrightarrow$ 

, $x_9 \leftrightarrow$ 

, $x_{10} \leftrightarrow$ 

,

$x_{11} \leftrightarrow$ 

, $x_{12} \leftrightarrow$ 

, ...

Main Theorem (Jeong-M-Zhang)

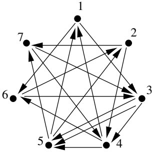
For certain periodic quivers Q, which include the **Gale-Robison** quiver family, the dP_3 quiver, and some other 2-periodic quivers, we can use the Brane Tiling $\mathcal{T}_Q$ to obtain combinatorial formulas for an infinite sequence of cluster variables in $\mathcal{A}_Q$.

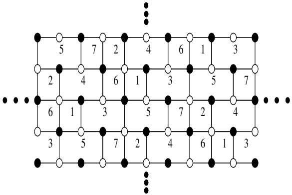
$$\text{For } n > N, \quad x_n = cm(G_n) \sum_{M = \text{ perfect matching of } G_n} x(M)y(M), \text{ where}$$

$\{G_n : n > N\}$'s are a collection of subgraphs of $\mathcal{T}_Q$, $x(M) = \prod_{\text{edge } e \in M} \frac{1}{x_i x_j}$ (for edge e straddling faces i and j), $y(M) =$ height of M (recording what faces need to be twisted to obtain matching M starting from the minimal matching, and $cm(G_n) =$ the covering monomial of the graph G_n (which records what face labels are contained in G_n and along its boundary).

Remark: This weighting scheme is a reformulation of schemes appearing in works of Speyer (**"Octahedron Recurrence"**) and Goncharov-Kenyon.

Gale-Robinson Example ($Q_7^{(2,3)}$, Mutating $1, 2, \ldots, 7, \ldots$)

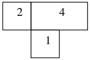
$Q =$ 

$\mathcal{T}_Q =$ 

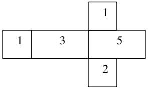
$x_8 \leftrightarrow$ 

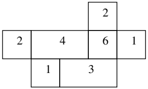
, $x_9 \leftrightarrow$ 

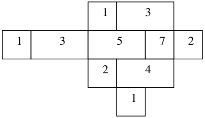
, $x_{10} \leftrightarrow$ 

, $x_{11} \leftrightarrow$ 

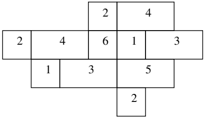
,

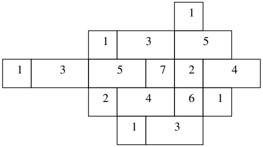
$x_{12} \leftrightarrow$ 

, $x_{13} \leftrightarrow$ 

, $x_{14} \leftrightarrow$ 

,

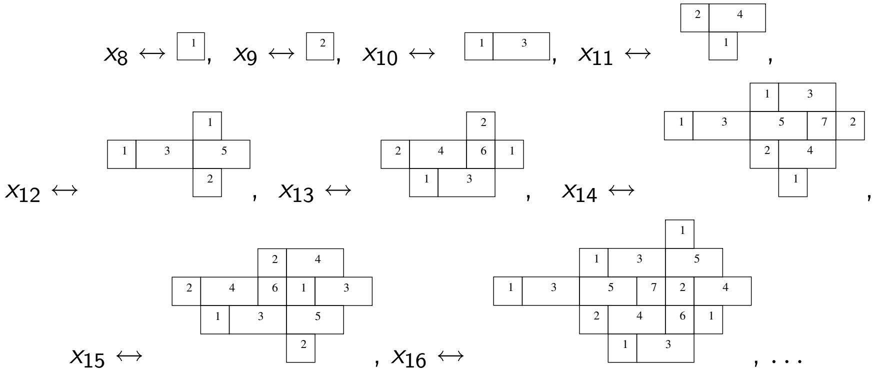
$x_{15} \leftrightarrow$ 

, $x_{16} \leftrightarrow$ 

, ...

Obtain **pinecone graphs** from Bousquet-Mélou, Propp, and West in terms of Brane Tilings Terminology.

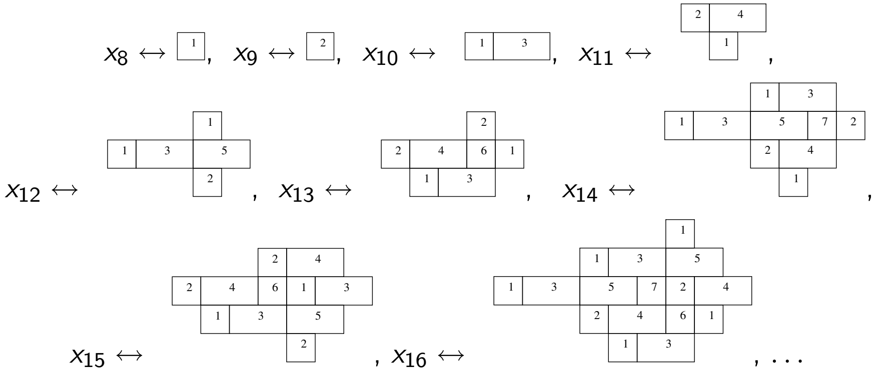
Furthermore, to get cluster variable formulas with coefficients, need only use weights (Goncharov-Kenyon, Speyer) and heights (Kenyon-Propp-...)



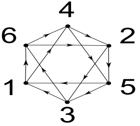
Gale-Robinson Example ($Q_7^{(2,3)}$, Mutating $1, 2, \ldots, 7, \ldots$)

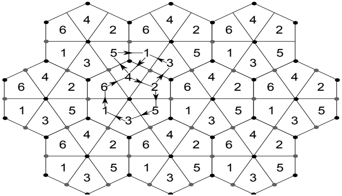
Similar connections (without principal coefficients) also observed in **"Brane tilings and non-commutative geometry"** by Richard Eager.

Eager uses physics terminology where he looks at $Y^{p,q}$ and $L^{a,b,c}$ quiver gauge theories, and their periodic Seiberg duality (i.e. quiver mutations).

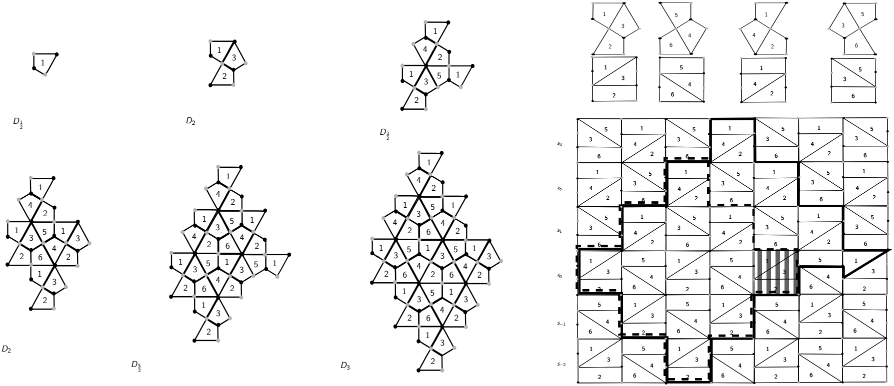


dP_3 Example (Mutating $1, 2, 3, 4, 5, 6, 1, 2, \ldots$)





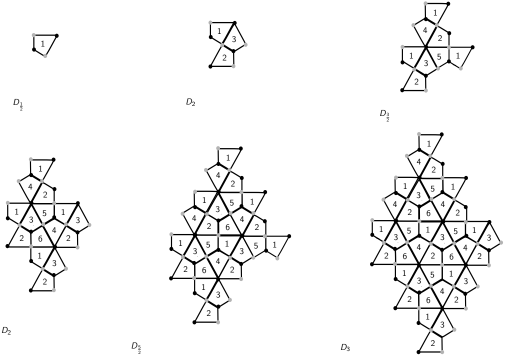
$Q \longrightarrow \mathcal{T}_Q$:

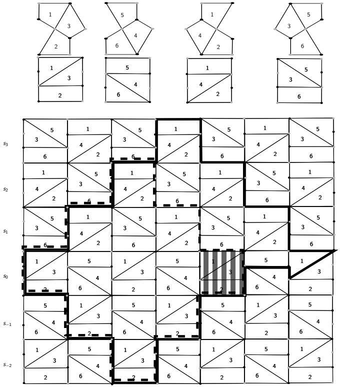


dP_3 Example (Mutating $1, 2, 3, 4, 5, 6, 1, 2, \ldots$)

These subgraphs appear in work by Cottrell-Young and a subsequence of them appear in M. Ciucu's work **"Perfect matchings and perfect powers"**, where they are called **Aztec Dragons**.

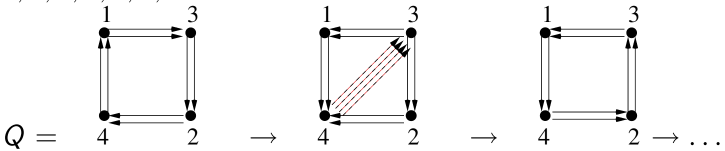
More on Aztec Dragons and the dP_3 lattice shortly.

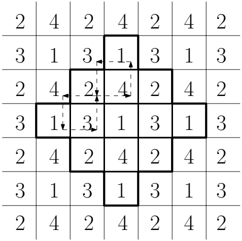


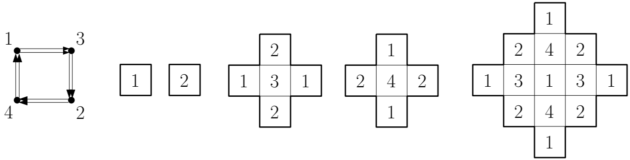


Hirzebruch Quiver F_0: **Aztec Diamonds** and Fortresses

The quiver Q below is 2-periodic, as illustrated by mutating in order $1, 2, 3, 4, 1, 2, \ldots$

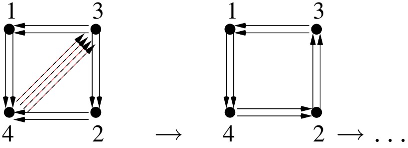


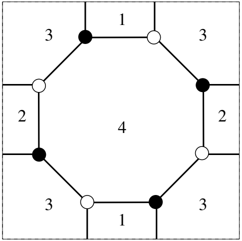




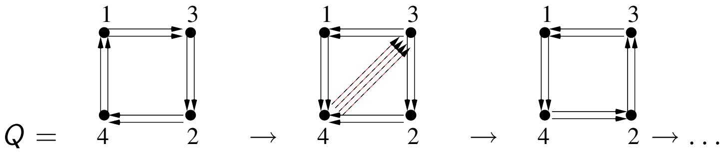
Hirzebruch Quiver F_0: Aztec Diamonds and **Fortresses**

The quiver Q' below is 2-periodic, as illustrated by mutating in order $1, 2, 3, 4, 1, 2, \ldots$

$Q' =$ 

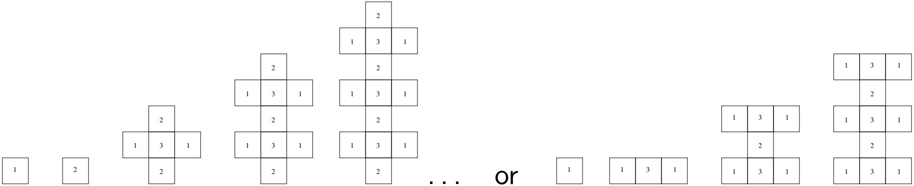


Fortresses from M. Ciucu's work **"Perfect matchings and perfect powers"**.



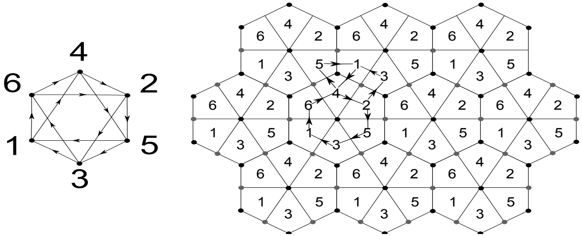
If we instead mutate by $1, 2, 3, 2, 3, \dots$ or $1, 3, 2, 3, 2, \dots$, we obtain quivers where the number of arrows grows without bound.

Nonetheless, a combinatorial interpretation for the cluster variables is



. . . .

Non-periodic mutation sequences in the dP_3 Lattice



Mutating at a vertex of dP_3 and its antipode commute so a mutation such as $1, 2, 1, 2$ can be reordered to $1, 1, 2, 2$, and hence is the identity.

Letting $\tau_1 = \mu_1 \circ \mu_2$, $\tau_2 = \mu_3 \circ \mu_4$, $\tau_3 = \mu_5 \circ \mu_6$, the above can be written as $\tau_1^2 = \tau_2^2 = \tau_3^2 = 1$.

As discussed with Pylyavskyy, we see the further relations $(\tau_1\tau_2)^3 = (\tau_1\tau_3)^3 = (\tau_2\tau_3)^3 = 1$, and it can be shown that there are no other relations. Thus $\langle \tau_1, \tau_2, \tau_3 \rangle \cong \tilde{A}_2$.

τ-Mutation-Sequences

We call a mutation sequence built out of a concatenation of τ_1, τ_2, and τ_3 a τ-**mutation sequence**.

As an example, the periodic sequences $1, 2, 3, 4, 5, 6, 1, 2, \ldots$ yielding the Aztec Dragons, were examples of τ-mutation sequences given by $\tau_1, \tau_2, \tau_3, \tau_1, \ldots$.

Up to the $\tilde{A}_2$ relations and relabeling vertices, other τ-mutation sequences do not necessarily give new cluster variables.

First non-trivial τ-mutation sequence: $\tau_1, \tau_2, \tau_1, \tau_3 = 1, 2, 3, 4, 1, 2, 5, 6$ yields the following as the last cluster variable:

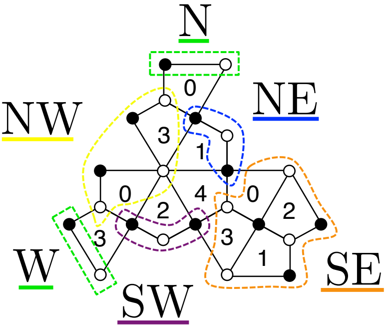
$$\begin{aligned} & (x_1x_2^2x_3^3x_5^4 + x_2^3x_3^2x_4x_5^4 + 2x_1^2x_2x_3^3x_5^3x_6 + 4x_1x_2^2x_3^2x_4x_5^3x_6 + 2x_2^3x_3x_4^2x_5^3x_6 + x_1^3x_3^3x_5^2x_6^2 \\ + \quad & 5x_1^2x_2x_3^2x_4x_5^2x_6^2 + 5x_1x_2^2x_3x_4^2x_5^2x_6^2 + x_2^3x_4^3x_5^2x_6^2 + 2x_1^3x_3^2x_4x_5x_6^3 + 4x_1^2x_2x_3x_4^2x_5x_6^3 \\ + \quad & 2x_1x_2^2x_4^3x_5x_6^3 + x_1^3x_3x_4^2x_6^4 + x_1^2x_2x_4^3x_6^4)/x_1^2x_2^2x_3^2x_4^2x_6 \end{aligned}$$

What is a combinatorial interpretation of this Laurent polynomial?

Aztec Castles

UMN 2013 REU Students Megan Leoni, Seth Neel, and Paxton Turner investigated these non-periodic mutation sequences in the dP_3 Lattice.

They developed a two-parameter family of graphs, **Aztec Castles**, to encode cluster variables arising from τ-mutation sequences.

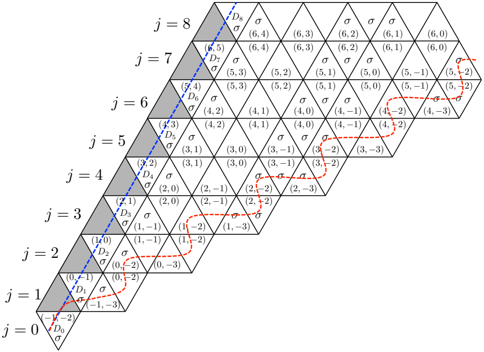


Example: $x_{\tau_1,\tau_2,\tau_1,\tau_3}$ corresponds to .
(Up to a shift in indexing on faces by one.)

Aztec Castles (cont.)

For a general τ-mutation sequence, due to $\tilde{A}_2$ symmetry, the last cluster variable can be associated to an alcove in the $\tilde{A}_2$ Coxeter Lattice, i.e. indexed by a pair (k, ℓ).

More precisely, they cut the lattice into twelve "cones" and by symmetry it suffices to describe the families of graphs for two adjacent cones.



Aztec Castles (cont.)

For a general τ-mutation sequence, due to $\tilde{A}_2$ symmetry, the last cluster variable can be associated to an alcove in the $\tilde{A}_2$ Coxeter Lattice, i.e. indexed by a pair (k, ℓ).

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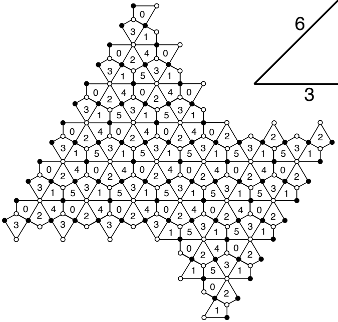

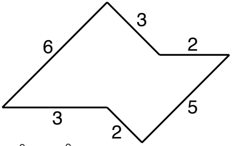

Example $(k = 2, \ell = -2)$**:**

Open Problems and Work in Progress

• τ-mutation sequences and the periodic mutation sequences discussed are examples of **toric** mutation sequences. Mutation only performed at vertices with at most two incoming and two outgoing arrows.

"Colored BPS Pyramid Partition Functions, Quivers, and Cluster Transformations" by Richard Eager and Sebastian Franco gives a recipe for getting combinatorial interpretations for toric mutation sequences but there is a gap for examples like $\tau_1\tau_2\tau_1\tau_3$.

Current work in progress with S. Franco to resolve this issue using their formulation involving Jacobian algebras to find a general working recipe.

• Extend definition of heights to other cases to obtain cluster variables with principal coefficients.

• Obtain more combinatorial interpretations for non-toric mutation sequences (like the F_0 case).

Thank You For Listening

In-Jee Jeong, *Bipartite Graphs, Quivers, and Cluster Variables*, REU Report, http://www.math.umn.edu/∼reiner/REU/Jeong2011.pdf

Sicong Zhang, *Cluster Variables and Perfect Matchings of Subgraphs of the dP_3 Lattice*, REU Report, http://www.math.umn.edu/∼reiner/REU/Zhang2012.pdf

I. Jeong, G. Musiker, and S. Zhang. Gale-Robinson Sequences and Brane Tilings. *Discrete Mathematics and Theoretical Computer Science Proc.* **AS** (2013), 737-748. (Longer version in preparation.)

Megan Leoni, Seth Neel, and Paxton Turner, *Aztec Castles and the dP3 Quiver*, arXiv:1308.3926.

Slides Available at http//math.umn.edu/∼musiker/Brane.pdf