Applications of New F-polynomial Formulas in terms of C-Vectors

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http://math.umn.edu/~musiker/Fpoly19.pdf

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arXiv:1812.01910 and forthcoming work
Given a quiver $Q$ (i.e. a directed graph) with $n$ vertices, we build an $n$-by-$n$ skew-symmetric matrix $B_Q = [b_{ij}]_{i=1, j=1}^n$ whose entries are

\[ b_{ij} = (\# \text{arrows from } i \text{ to } j) - (\# \text{arrows from } j \text{ to } i). \]

**Note:** More generally, we can let $B_Q$ be skew-symmetrizable, meaning there exists a diagonal matrix $D$ with positive integer entries such that $DB_Q$ is skew-symmetric, i.e. satisfies $(DB_Q)^T = -DB_Q$. However, for this talk we will focus on the quiver, i.e. the skew-symmetric, case.

We build the corresponding $2n$-by-$n$ exchange matrix with principal coefficients via $\tilde{B}_Q = \begin{bmatrix} B_Q & I_n \end{bmatrix}$, where $I_n$ denotes the $n$-by-$n$ identity matrix.

Equivalently, $\tilde{B}_Q$ corresponds to the exchange matrix of the framed quiver $\tilde{Q} = Q \cup \{1', 2', \ldots, n'\}$ with a single arrow from $i' \to i$ for each $1 \leq i \leq n$. 
If $Q = 1 \rightarrow 2$, then $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\tilde{Q} = 1' \ 2'$ and $\tilde{B_Q} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

If $Q = 1 \Rightarrow 2$, then $B_Q = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, $\tilde{Q} = 1' \ 2'$ and $\tilde{B_Q} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

If $Q = 1 \Rightarrow 2 \Leftarrow 3 \Leftarrow 4$, then $\tilde{Q} = 1' \ 2' \ 3' \ 4'$ and $\tilde{B_Q} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. 

$B_Q = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$.

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Quiver Mutation

Given a quiver $Q$ and its vertex $j$, we can define $Q' = \mu_j Q$, the **mutation of $Q$ at $j$**, by a 3 step process:

1) For any 2-path $i \to j \to k$, add a new arrow $i \toarrow j \toarrow k$.

2) Reverse the direction of all arrows incident to $j$.

3) Delete any 2-cycle $i \toarrow j \toarrow k$ created from the above two steps.

**Examples:** If $Q = 1 \to 2 \leftarrow 3 \leftarrow 4$, then

\[
\mu_1 Q = 1 \leftarrowarrow 2 \leftarrowarrow 3 \leftarrowarrow 4, \quad \mu_2 Q = 1 \leftarrowarrow 2 \rightarrowarrow 3 \rightarrowarrow 4
\]

\[
\mu_3 Q = 1 \toarrow 2 \toarrow 3 \toarrow 4, \quad \mu_4 Q = 1 \toarrow 2 \leftarrowarrow 3 \leftarrowarrow 4
\]

**Note:** Mutation is an **involution**, meaning that $\mu_j^2 Q = Q$ for any vertex $j$. 
Exchange Matrix Mutation

Quiver mutation induces an analogous dynamic on exchange matrices $B_Q$. We define $[b'_{ij}] = B'_Q = \mu_k B_Q$, the mutation of $B_Q = [b_{ij}]$ at $k$, by

$$b'_{ij} = \begin{cases} 
-b_{ij} & \text{if } i = k \text{ or } j = k \\
b_{ij} + [b_{ik}]_+ [b_{kj}]_+ - [-b_{ik}]_+ [-b_{kj}]_+ & \text{otherwise}
\end{cases}$$

using $[\alpha]_+ = \max(\alpha, 0)$.

Examples: If $Q = 1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4$, $B_Q = \begin{bmatrix} 0 & 2 & 0 & 0 \\
-2 & 0 & -1 & 1 \\
0 & 1 & 0 & -1 \\
0 & -1 & 1 & 0 \end{bmatrix}$, then

$$\mu_1 Q = 1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4, \quad \mu_1 B_Q = \begin{bmatrix} 0 & -2 & 0 & 0 \\
2 & 0 & -1 & 1 \\
0 & 1 & 0 & -1 \\
0 & -1 & 1 & 0 \end{bmatrix}. $$
Exchange Matrix Mutation

Quiver mutation induces an analogous dynamic on exchange matrices $B_Q$. We define $[b'_{ij}] = B'_Q = \mu_k B_Q$, the **mutation of** $B_Q = [b_{ij}]$ **at** $k$, by

$$b'_{ij} = \begin{cases} 
-b_{ij} & \text{if } i = k \text{ or } j = k \\
 b_{ij} + [b_{ik}]_+ [b_{kj}]_+ - [-b_{ik}]_+ [-b_{kj}]_+ & \text{otherwise}
\end{cases}$$

using $[\alpha]_+ = \max(\alpha, 0)$.

**Examples:** If $Q = 1 \rightarrow 2 \leftrightarrow 3 \leftrightarrow 4$, $B_Q = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$, then

$$\mu_2 Q = 1 \leftrightarrow 2 \rightarrow 3 \leftrightarrow 4,$$

$$\mu_2 B_Q = \begin{bmatrix} 0 & -2 & 0 & 2 \\ 2 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ -2 & 1 & 0 & 0 \end{bmatrix}.$$
Exchange Matrix Mutation

Quiver mutation induces an analogous dynamic on exchange matrices $B_Q$. We define $[b'_{ij}] = B'_Q = \mu_k B_Q$, the mutation of $B_Q = [b_{ij}]$ at $k$, by

$$b'_{ij} = \begin{cases} 
-b_{ij} & \text{if } i = k \text{ or } j = k \\
 b_{ij} + [b_{ik}] + [b_{kj}] - [-b_{ik}] - [-b_{kj}] & \text{otherwise}
\end{cases}$$

using $[\alpha]_+ = \max(\alpha, 0)$.

**Examples:** If $Q = 1 \Rightarrow 2 \Leftarrow 3 \Leftarrow 4$, $B_Q =$

$$\begin{bmatrix}
0 & 2 & 0 & 0 \\
-2 & 0 & -1 & 1 \\
0 & 1 & 0 & -1 \\
0 & -1 & 1 & 0
\end{bmatrix}, \text{ then}

\mu_3 Q = 1 \Rightarrow 2 \rightarrow 3 \rightarrow 4, \quad \mu_3 B_Q =$

$$\begin{bmatrix}
0 & 2 & 0 & 0 \\
-2 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix}.$$
Exchange Matrix Mutation

Quiver mutation induces an analogous dynamic on exchange matrices $B_Q$. We define $[b'_{ij}] = B'_Q = \mu_k B_Q$, the **mutation of $B_Q = [b_{ij}]$ at $k$**, by

$$
b'_{ij} = \begin{cases} 
-b_{ij} & \text{if } i = k \text{ or } j = k \\
b_{ij} + [b_{ik}] + [b_{kj}] - [-b_{ik}] + [-b_{kj}] & \text{otherwise} 
\end{cases}
$$

using $[\alpha]_+ = \max(\alpha, 0)$.

**Examples:** If $Q = 1 \Rightarrow 2 \Leftarrow 3 \Leftarrow 4$, $B_Q = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$, then

$$\mu_4 Q = 1 \Rightarrow 2 \Leftarrow 3 \Rightarrow 4$$

$$\mu_4 B_Q = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}.$$
Examples of mutation with principal coefficients

As framed quivers (for the case of a type $A_2$ quiver):

\[
\begin{array}{ccccccccccc}
1' & 2' & \rightarrow^{\mu_1} & 1' & 2' & \rightarrow^{\mu_2} & 1' & 2' & \rightarrow^{\mu_1} & 1' & 2' & \rightarrow^{\mu_2} & 1' & 2' \\
\downarrow & \downarrow & & \uparrow & \uparrow & & \uparrow & \uparrow & & \uparrow & \uparrow & & \uparrow & \uparrow \\
1 & \rightarrow & 2 & & & & & & & & & & \leftarrow & 2
\end{array}
\]

As $2n$-by-$n$ exchange matrices:

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\rightarrow^{\mu_1}
\begin{pmatrix}
0 & 1 \\
1 & 0 \\
-1 & 1 \\
0 & 1
\end{pmatrix}
\rightarrow^{\mu_2}
\begin{pmatrix}
0 & -1 \\
-1 & 0 \\
0 & -1 \\
1 & -1
\end{pmatrix}
\rightarrow^{\mu_1}
\begin{pmatrix}
0 & -1 \\
1 & 0 \\
0 & -1 \\
-1 & 0
\end{pmatrix}
\rightarrow^{\mu_2}
\begin{pmatrix}
0 & 1 \\
-1 & 0 \\
0 & 1 \\
-1 & 0
\end{pmatrix}
\rightarrow^{\mu_1}
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
1 & 0
\end{pmatrix}.
\]
Examples of mutation with principal coefficients

Starting with the framed quiver for the case of the Kronecker quiver

\[
\begin{array}{c}
1' \\
\downarrow \\
1 \Rightarrow 2
\end{array}
\]

As \(2n\)-by-\(n\) exchange matrices:

\[
\begin{bmatrix}
0 & 2 \\
-2 & 0 \\
1 & 0 \\
0 & 1
\end{bmatrix} \twoheadrightarrow_{\mu_1} \begin{bmatrix}
0 & -2 \\
2 & 0 \\
-1 & 2 \\
0 & 1
\end{bmatrix} \twoheadrightarrow_{\mu_2} \begin{bmatrix}
0 & 2 \\
-2 & 0 \\
3 & -2 \\
2 & -1
\end{bmatrix} \twoheadrightarrow_{\mu_1} \begin{bmatrix}
0 & -2 \\
2 & 0 \\
-3 & 4 \\
-2 & 3
\end{bmatrix} \twoheadrightarrow_{\mu_2} \begin{bmatrix}
0 & 2 \\
-2 & 0 \\
7 & -6 \\
6 & -5
\end{bmatrix} \twoheadrightarrow_{\mu_1} \begin{bmatrix}
0 & 2 \\
-2 & 0 \\
-7 & 8 \\
-6 & 7
\end{bmatrix} \twoheadrightarrow \ldots
\]
Cluster seeds and their mutation

A seed for a cluster algebra is defined as a choice of a quiver (equivalently an exchange matrix) on $N$ vertices and a choice of a cluster $\{x_1, x_2, \ldots, x_N\}$ where the $x_i$ are formal variables, called cluster variables.

We define cluster mutation alongside quiver mutation yielding (a priori) rational functions in $\mathbb{Q}(x_1, x_2, \ldots, x_N)$ defined by

$$\{x_1, \ldots, x_N\} \rightarrow^{\mu_k} \{x_1, \ldots, x_N\} \cup \{x'_k\} \setminus \{x_k\} \text{ where}$$

$$x'_k = \frac{\prod_{i=1}^{n} x_i^{[b_{ik}]_+} + \prod_{k=1}^{n} x_i^{-[b_{ik}]_+}}{x_k} = \prod_{i \rightarrow k} x_i + \prod_{k \rightarrow i} x_i$$

using the exchange matrix $B_Q$, or equivalently the arrows in the quiver $Q$. 

Theorem (Fomin-Zelevinsky 2001)
The Laurent Phenomenon holds for all cluster variables, namely the rational functions resulting from iterating cluster mutation are in fact Laurent polynomials, i.e.

$$P(x_1, \ldots, x_N) = \prod_{i=1}^{n} x_i^{d_1} \cdots x_i^{d_n}$$

where $P$ is a polynomial with integer coefficients and each $d_i$ is a nonnegative integer.
Cluster seeds and their mutation

A seed for a cluster algebra is defined as a choice of a quiver (equivalently an exchange matrix) on \( N \) vertices and a choice of a cluster \( \{x_1, x_2, \ldots, x_N\} \) where the \( x_i \) are formal variables, called cluster variables.

We define cluster mutation alongside quiver mutation yielding (a priori) rational functions in \( \mathbb{Q}(x_1, x_2, \ldots, x_N) \) defined by

\[
\{x_1, \ldots, x_N\} \rightarrow^{\mu_k} \{x_1, \ldots, x_N\} \cup \{x'_k\} \setminus \{x_k\}
\]

where

\[
x'_k = \frac{\prod_{i=1}^n x_i^{b_{ik}} + \prod_{k=1}^n x_i^{-b_{ik}}}{x_k} = \prod_{i \rightarrow k} x_i + \prod_{k \rightarrow i} x_i
\]

using the exchange matrix \( B_Q \), or equivalently the arrows in the quiver \( Q \).

**Theorem (Fomin-Zelevinsky 2001)** The Laurent Phenomenon holds for all cluster variables, namely the rational functions resulting from iterating cluster mutation are in fact Laurent polynomials, i.e.

\[
\frac{P(x_1, \ldots, x_N)}{x_1^{d_1} \cdots x_n^{d_n}}
\]

where \( P \) is a polynomial with integer coefficients and each \( d_i \) is a nonnegative integer.
F-polynomials

If we start with a framed quiver $\tilde{Q} = Q \cup \{1', 2', \ldots, n'\}$ and the initial cluster $\{x_1, \ldots, x_N\} = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, we iterate cluster mutation with the extra restriction disallowing mutation at vertices $i'$. Consequently, the binomial exchange relation for cluster mutation

$$x'_k = \frac{\prod_{i=1}^{n} x_i^{[b_{ik}]+}}{x_k} + \frac{\prod_{k=1}^{n} x_i^{-b_{ik}+}}{x_k} = \prod_{i \rightarrow k} x_i + \prod_{k \rightarrow i} x_i$$

will involve $y_1, y_2, \ldots, y_n$ in the numerator, but never in the denominator.

By letting $x_1 = x_2 = \cdots = x_n = 1$, and iterating cluster mutation, we replace cluster variables (which are Laurent polynomials in $x_i$’s and $y_i$’s) with polynomials in $y_1, y_2, \ldots, y_n$, which are called **F-polynomials**.
F-polynomials

\[ x'_k = \frac{\prod_{i=1}^{n} x_i^{[b_{ik}]+} + \prod_{k=1}^{n} x_i^{-b_{ik}+}}{x_k} \]

By letting \( x_1 = x_2 = \cdots = x_n = 1 \), and iterating cluster mutation, we replace cluster variables (which are Laurent polynomials in \( x_i \)'s and \( y_i \)'s) with polynomials in \( y_1, y_2, \ldots, y_n \), which are called **F-polynomials**.

**Example:**

\[
\begin{align*}
1' & \rightarrow_{\mu_1} 1' \quad 2' \rightarrow_{\mu_2} 1' \quad 2' \rightarrow_{\mu_1} 1' \quad 2' \rightarrow_{\mu_2} 1' \quad 2' \rightarrow_{\mu_1} 1' \quad 2' \\
1 \rightarrow 2 & \quad 1 \leftarrow 2 \quad 1 \rightarrow 2 \quad 1 \leftarrow 2 \quad 1 \rightarrow 2 \quad 1 \leftarrow 2
\end{align*}
\]

\[
\{F_1, F_2\} = \{1, 1\} \rightarrow_{\mu_1} \{y_1 + 1, 1\} \rightarrow_{\mu_2} \{y_1 + 1, y_1y_2 + y_1 + 1\}
\]

\[
\rightarrow_{\mu_1} \{y_2 + 1, y_1y_2 + y_1 + 1\} \rightarrow_{\mu_2} \{y_2 + 1, 1\} \rightarrow_{\mu_1} \{1, 1\}
\]
c-vectors

Given a framed quiver $\tilde{Q}$ and its images under a sequence of mutations, we define the $c$-vectors associated to the seed $t$ by

$$c_{j,t} = [c_{1j}, c_{2j}, \ldots, c_{nj}]^T$$

where $c_{ij} = \#\text{arrows from } i' \to j$. Equivalently, $c_{j,t}$ is the $j$th column of the bottom half of the $2n$-by-$n$ exchange matrix associated to seed $t$.

In particular, the initial $c$-vectors, for seed $t_0$, equal unit vectors

$$c_{1,t_0} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad c_{2,t_0} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad c_{n,t_0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

and then recursively $c$-vectors mutate alongside quivers and exchange matrices. Letting $c_{j, \mu_k t} = [c'_{1j}, c'_{2j}, \ldots, c'_{nj}]^T$ for each $1 \leq j \leq n$, we have

$$c'_{ij} = \begin{cases} -c_{ij} = -c_{ik} & \text{if } j = k \\ c_{ij} + [c_{ik}]_+ + [b_{kj}]_+ - [-c_{ik}]_+ - [-b_{kj}]_+ & \text{otherwise} \end{cases}.$$
c-vectors for $1 \to 2$

\[
\begin{align*}
1' &\rightarrow_{\mu_1} 2' & 1' &\rightarrow_{\mu_2} 2' & 1' &\rightarrow_{\mu_1} 2' & 1' &\rightarrow_{\mu_2} 2' & 1' &\rightarrow_{\mu_1} 1' &\leftarrow 2' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 &\leftrightarrow 2 & 1 &\leftrightarrow 2 & 1 &\leftrightarrow 2 & 1 &\leftrightarrow 2 & 1 &\leftrightarrow 2 & 1 &\leftrightarrow 2
\end{align*}
\]

\[
\begin{align*}
t_0 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow_{\mu_1} t_1 &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow_{\mu_2} t_2 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix} \\
\rightarrow_{\mu_1} t_3 &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow_{\mu_2} t_4 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \rightarrow_{\mu_1} t_5 &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
c_{1,t_0} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & c_{2,t_0} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & c_{1,t_1} &= \begin{bmatrix} -1 \\ 0 \end{bmatrix}, & c_{2,t_1} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, & c_{1,t_2} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & c_{2,t_2} &= \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\
c_{1,t_3} &= \begin{bmatrix} 0 \\ -1 \end{bmatrix}, & c_{2,t_3} &= \begin{bmatrix} -1 \\ 0 \end{bmatrix}, & c_{1,t_4} &= \begin{bmatrix} 0 \\ -1 \end{bmatrix}, & c_{2,t_4} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & c_{1,t_5} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & c_{2,t_5} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{align*}
\]
c-vectors for $1 \Rightarrow 2$

\[
t_0 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \mu_1 \quad t_1 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \rightarrow \mu_2 \quad t_2 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 3 & -2 \\ 2 & -1 \end{bmatrix}
\]

\[
\rightarrow \mu_1 \quad t_3 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \\ -3 & 4 \\ -2 & 3 \end{bmatrix} \rightarrow \mu_2 \quad t_4 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \\ 5 & -4 \\ 4 & -3 \end{bmatrix} \rightarrow \mu_1 \quad t_5 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \\ -5 & 6 \\ -4 & 5 \end{bmatrix} \rightarrow \ldots
\]

\[
c_{1,t_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad c_{2,t_2} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \quad c_{1,t_3} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}, \quad c_{2,t_4} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}, \quad c_{1,t_5} = \begin{bmatrix} -5 \\ -4 \end{bmatrix}, \ldots
\]
\textbf{c-vector Sign Coherence}

For $1 \rightarrow 2$ and $\mu_1 \mu_2 \mu_1 \mu_2 \mu_1$, 

\[ c_{1,t_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad c_{2,t_2} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad c_{1,t_3} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad c_{2,t_4} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c_{1,t_5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

For $1 \Rightarrow 2$ and $\mu_1 \mu_2 \mu_1 \mu_2 \mu_1 \cdots$, 

\[ c_{1,t_1} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad c_{2,t_2} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \quad c_{1,t_3} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}, \quad c_{2,t_4} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}, \quad c_{1,t_5} = \begin{bmatrix} -5 \\ -4 \end{bmatrix}, \cdots \]

\textbf{Theorem (Derksen-Weyman-Zelevinsky 2010)} Each $c$-vector consists exclusively of nonnegative entries or exclusively of nonpositive entries.

Sign Coherence implies we can assign a sign $\epsilon_{j,t_r} \in \{ \pm 1 \}$ to each $c_{j,t_r}$.

Three definitions of $g$-vectors

1) For a framed quiver $\tilde{Q}$ with exchange matrix $\begin{bmatrix} B_Q \\ I_n \end{bmatrix}$, define a $\mathbb{Z}^n$-grading by $\deg(x_i) = e_i$ and $\deg(y_j) = -b_j$, where $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ is the initial cluster, $e_i$ is the $i$th unit vector and $b_j$ is the $j$th column of $B_Q$. 

Then for any cluster variable $x'$ written as a Laurent polynomial in $Q\left[x_i^{\pm 1}, x_1, \ldots, x_n, y_1, \ldots, y_n\right]$, the $\mathbb{Z}^n$-grading of each such Laurent monomial of $x'$ coincide. This common multidegree is defined to be the $g$-vector attached to $x'$. (See Section 6 of Cluster Algebras IV.)

2) As a consequence of sign coherence, any $F$-polynomial has a constant term of 1. Utilizing this, the $g$-vector of $x'$ agrees with the exponent vector, in $x_i$'s, of the unique Laurent monomial of $x'$ containing no $y_j$'s.

3) Let $C_t$ (resp. $G_t$) denote the matrices whose columns are the $c$-vectors (resp. $g$-vectors) associated to seed $t$.

Theorem 4.1 of Nakanishi 2011: 

As another consequence of sign coherence, $G_t = (C_T^t)^{-1}$. 

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F-polynomials and C-Vectors 

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Three definitions of \(g\)-vectors

1) For a framed quiver \(\tilde{Q}\) with exchange matrix \(\begin{bmatrix} B_Q \\ I_n \end{bmatrix}\), define a \(\mathbb{Z}^n\)-grading by \(\deg(x_i) = e_i\) and \(\deg(y_j) = -b_j\), where \(\{x_1, \ldots, x_n, y_1, \ldots, y_n\}\) is the initial cluster, \(e_i\) is the \(i\)th unit vector and \(b_j\) is the \(j\)th column of \(B_Q\).

Then for any cluster variable \(x'\) written as a Laurent polynomial in \(\mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, y_2, \ldots, y_n]\), the \(\mathbb{Z}^n\)-grading of each such Laurent monomial of \(x'\) coincide. This common multidegree is defined to be the \(g\)-vector attached to \(x'\). (See Section 6 of Cluster Algebras IV.)
Three definitions of $g$-vectors

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Then for any cluster variable $x'$ written as a Laurent polynomial in $\mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, y_2, \ldots, y_n]$, the $\mathbb{Z}^n$-grading of each such Laurent monomial of $x'$ coincide. This common multidegree is defined to be the $g$-vector attached to $x'$. (See Section 6 of Cluster Algebras IV.)

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Three definitions of $g$-vectors

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Then for any cluster variable $x'$ written as a Laurent polynomial in $\mathbb{Q}[x_1^{\pm1}, x_2^{\pm1}, \ldots, x_n^{\pm1}, y_1, y_2, \ldots, y_n]$, the $\mathbb{Z}^n$-grading of each such Laurent monomial of $x'$ coincide. This common multidegree is defined to be the $g$-vector attached to $x'$. (See Section 6 of Cluster Algebras IV.)

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3) Let $C_t$ (resp. $G_t$) denote the matrices whose columns are the $c$-vectors (resp. $g$-vectors) associated to seed $t$. Theorem 4.1 of Nakanishi 2011:

As another consequence of sign coherence, $G_t = (C_t^T)^{-1}$. 
Theorem (Gupta ’18) as will be re-expressed in (Gupta-M ’19+) : Given a framed quiver \( \tilde{Q} \) and a mutation sequence \( \bar{\mu} = \mu_i_1 \mu_i_2 \cdots \mu_i_\ell \), consider the sequence of cluster seeds \( t_0 \xrightarrow{\mu_i_1} t_1 \xrightarrow{\mu_i_2} \cdots t_{\ell-1} \xrightarrow{\mu_i_\ell} t_\ell \).

Then the F-polynomial resulting from the final mutation, i.e. \( F_{i_\ell; t_\ell} \), is expressible as a product of recursively defined formulas, dependent only on c-vectors and g-vectors, followed by a monomial specialization:
Theorem (Gupta ’18) as will be re-expressed in (Gupta-M ’19+): Given a framed quiver \( \tilde{Q} \) and a mutation sequence \( \bar{\mu} = \mu_1 \mu_2 \cdots \mu_{i_\ell} \), consider the sequence of cluster seeds \( t_0 \to \mu_1 \ t_1 \to \mu_2 \cdots t_{\ell-1} \to \mu_{i_\ell} \ t_\ell \).

Then the F-polynomial resulting from the final mutation, i.e. \( F_{i_\ell};t_\ell \), is expressible as a product of recursively defined formulas, dependent only on \( c \)-vectors and \( g \)-vectors, followed by a monomial specialization:

Let \( L_1 = 1 + z_1 \) and \( L_k = 1 + z_k L_1^{c_1 \cdot B_Q|c_k|} L_2^{c_2 \cdot B_Q|c_k|} \cdots L_{k-1}^{c_{k-1} \cdot B_Q|c_k|} \) for \( k \geq 2 \).

Then \( F_{i_\ell};t_\ell = \prod_{j=1}^{\ell} L_j^{c_j \cdot g_{i_\ell}} \big|_{z_1 = y|c_1|, \ldots, z_\ell = y|c_\ell|} \). Also see [Nagao10] and [Keller12].

**Note:** Before the monomial specialization, the \( L_j \)'s and \( F_{i_\ell};t_\ell \)'s may be rational functions in the \( z_i \)'s.

Here, \( c_p \) (resp. \( |c_p| \) or \( g_p \)) denotes the \( p \)th \( c \)-vector (resp. the normalized \( c \)-vector \( \epsilon_p c_p \) or the \( g \)-vector) along the mutation sequence \( \bar{\mu} \), \( B_Q \) denotes the exchange matrix associated to \( Q \) before any mutations, \( a \cdot b \) denotes ordinary dot product, and \( y^{(d_1,d_2,\ldots,d_n)} \) is shorthand for \( y_1^{d_1} y_2^{d_2} \cdots y_n^{d_n} \).
Type $A_2$ Quiver Example

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{c_1} B_Q |c_k| L_2^{c_2} B_Q |c_k| \cdots L_{k-1}^{c_{k-1}} B_Q |c_k|$ for $k \geq 2$.

Suppose $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\bar{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \mu_1$. 
Type $A_2$ Quiver Example

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{c_1} B_Q | c_k | L_2^{c_2} B_Q | c_k | \ldots L_{k-1}^{c_{k-1}} B_Q | c_k |$ for $k \geq 2$.

Suppose $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\bar{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \mu_1$. Then

$$c_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, c_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, c_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, c_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, c_5 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_Q | c_2 | = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, B_Q | c_3 | = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_Q | c_4 | = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, B_Q | c_5 | = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$L_1 = 1 + z_1$, $L_2 = 1 + z_2 L_1^{-1} = 1 + z_2 (1 + z_1)^{-1} = \frac{1 + z_1 + z_2}{1 + z_1}$

$L_3 = 1 + z_3 L_1^{-1} L_2^{-1} = 1 + \frac{z_3}{1 + z_1} \frac{1 + z_1}{1 + z_1 + z_2} = \frac{1 + z_1 + z_2 + z_3}{1 + z_1 + z_2}$
Type $A_2$ Quiver Example (continued)

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k L_1^{c_1 B_Q | c_k} L_2^{c_2 B_Q | c_k} \cdots L_{k-1}^{c_{k-1} B_Q | c_k}$ for $k \geq 2$.

Suppose $B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\bar{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \mu_1$. Then

$$c_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad c_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c_5 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_Q | c_2 | = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad B_Q | c_3 | = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_Q | c_4 | = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad B_Q | c_5 | = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$L_4 = 1 + z_4 L_1^0 L_2^1 L_3^1 = 1 + z_4 \frac{1 + z_1 + z_2}{1 + z_1} \frac{1 + z_1 + z_2 + z_3}{1 + z_1 + z_2} = \frac{1 + z_1 + z_4 (1 + z_1 + z_2 + z_3)}{1 + z_1}$$

$$L_5 = 1 + z_5 L_1^{-1} L_2^{-1} L_3^0 L_4^1 = 1 + \frac{z_5}{1 + z_1} \frac{1 + z_1}{1 + z_1 + z_2} \frac{1 + z_1 + z_4 (1 + z_1 + z_2 + z_3)}{1 + z_1}$$

$$= \frac{(1 + z_1)(1 + z_1 + z_2) + z_5 + z_1 z_5 + z_4 z_5 (1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1)}.$$
**Type $A_2$ Quiver Example (continued)**

\[ B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \bar{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \mu_1. \]

\[ F_{i\ell} |_{z_1 = |c_1|, \ldots, z_\ell = |c_\ell|} = \prod_{j=1}^{\ell} \mathcal{L}_j^{c_j \cdot g_\ell} |_{z_1 = y|c_1|, \ldots, z_\ell = y|c_\ell|} \]

\[ L_1 = 1 + z_1, \quad L_2 = \frac{1 + z_1 + z_2}{1 + z_1}, \quad L_3 = \frac{1 + z_1 + z_2 + z_3}{1 + z_1 + z_2}, \quad L_4 = \frac{1 + z_1 + z_4(1 + z_1 + z_2 + z_3)}{1 + z_1}, \]

\[ L_5 = \frac{(1 + z_1)(1 + z_1 + z_2) + z_5 + z_1 z_5 + z_4 z_5 (1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1)}, \]

\[ c_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad c_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c_5 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]

\[ g_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad g_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad g_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad g_5 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]

\[ F_1 = L_1 = 1 + z_1, \quad F_2 = L_1 L_2 = 1 + z_1 + z_2, \]

\[ F_3 = L_2 L_3 = \frac{1 + z_1 + z_2 + z_3}{1 + z_1}, \]

\[ F_4 = L_1^{-1} L_2^{-1} L_4 = \frac{1 + z_1 + z_4(1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1)}, \]

\[ F_5 = L_2^{-1} L_3^{-1} L_5 = \frac{(1 + z_1)(1 + z_1 + z_2) + z_5 + z_1 z_5 + z_4 z_5 (1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1 + z_2 + z_3)}. \]
Type $A_2$ Quiver Example (continued)

\[ B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \overline{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \mu_1. \]

\[ F_{i\ell}; t_\ell = \prod_{j=1}^{\ell} L_j^{c_j \cdot g_\ell} \big|_{z_1 = y | c_1 |, \ldots, z_\ell = y | c_\ell |} \]

\[
F_1 = L_1 = 1 + z_1, \quad F_2 = L_1 L_2 = 1 + z_1 + z_2, \\
F_3 = L_2 L_3 = \frac{1 + z_1 + z_2 + z_3}{1 + z_1}, \\
F_4 = L_1^{-1} L_2^{-1} L_4 = \frac{1 + z_1 + z_4(1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1)}, \\
F_5 = L_2^{-1} L_3^{-1} L_5 = \frac{(1 + z_1)(1 + z_1 + z_2) + z_5 + z_1 z_5 + z_4 z_5(1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1 + z_2 + z_3)}
\]

\[ c_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad c_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c_5 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

Based on $\epsilon_3 = -1$, $\epsilon_4 = +1$, $\epsilon_5 = +1$, and $B_Q$ as above, we get

\[ F_3 F_1 = F_2 + z_3, \quad F_4 F_2 = z_4 F_3 + 1, \quad F_5 F_3 = z_5 F_4 + 1, \]

and these recurrences are valid for these expressions as rational functions.
Type $A_2$ Quiver Example (continued)

$B_Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\mu = \mu_1\mu_2\mu_1\mu_2\mu_1$.  

$F_{i\ell; t_{\ell}} = \prod_{j=1}^{\ell} L_j^{c_j \cdot g_{\ell}} |_{z_1 = y|c_1|, \ldots, z_\ell = y|c_\ell|}$

$F_1 = L_1 = 1 + z_1$,  
$F_2 = L_1L_2 = 1 + z_1 + z_2$,  
$F_3 = L_2L_3 = \frac{1 + z_1 + z_2 + z_3}{1 + z_1}$,  
$F_4 = L_1^{-1}L_2^{-1}L_4 = \frac{1 + z_1 + z_4(1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1)}$,  
$F_5 = L_2^{-1}L_3^{-1}L_5 = \frac{(1 + z_1)(1 + z_1 + z_2) + z_5 + z_1z_5 + z_4z_5(1 + z_1 + z_2 + z_3)}{(1 + z_1 + z_2)(1 + z_1 + z_2 + z_3)}$

$c_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $c_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $c_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $c_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $c_5 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Letting $z_1 = y_1$, $z_2 = y_1y_2$, $z_3 = y_2$, $z_4 = y_1$, $z_5 = y_2$, we get polynomials

$F_1 = y_1 + 1$,  
$F_2 = y_1y_2 + y_1 + 1$,  
$F_3 = y_2 + 1$,  
$F_4 = 1$,  
$F_5 = 1$.  

Theorem (Gupta ’18) as will be re-expressed in (Gupta-M ’19+) : Given a framed quiver $\tilde{Q}$ and a mutation sequence $\mu = \mu_{i_1}\mu_{i_2} \cdots \mu_{i_\ell}$, consider the sequence of cluster seeds $t_0 \rightarrow^{\mu_{i_1}} t_1 \rightarrow^{\mu_{i_2}} \ldots t_{\ell-1} \rightarrow^{\mu_{i_\ell}} t_\ell$.

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k \ell_1^{c_1 \cdot B_Q|c_k|} \ell_2^{c_2 \cdot B_Q|c_k|} \cdots \ell_{k-1}^{c_{k-1} \cdot B_Q|c_k|}$ for $k \geq 2$

and $F_{i_\ell; t_\ell} = \prod_{j=1}^{\ell} \ell_j^{c_j \cdot g_{i_\ell}} |z_1 = y| c_1, \ldots, z_\ell = y| c_\ell|.$
Theorem (Gupta ’18) as will be re-expressed in (Gupta-M ’19+) :
Given a framed quiver $\tilde{Q}$ and a mutation sequence $\vec{\mu} = \mu_{i_1}\mu_{i_2} \cdots \mu_{i_\ell}$, consider the sequence of cluster seeds $t_0 \rightarrow^{\mu_{i_1}} t_1 \rightarrow^{\mu_{i_2}} \cdots t_{\ell-1} \rightarrow^{\mu_{i_\ell}} t_\ell$.

Let $L_1 = 1 + z_1$ and $L_k = 1 + z_k \prod_{j=1}^{\ell} L_{c_j}^{c_j \cdot B_Q|c_k|} L_{c_{j+1}}^{c_{j+1} \cdot B_Q|c_k|} \cdots L_{c_{k-1}}^{c_{k-1} \cdot B_Q|c_k|}$ for $k \geq 2$

and $F_{i_\ell};t_\ell = \prod_{j=1}^{\ell} L_{c_j}^{c_j \cdot g_j|c_k|} \big|_{z_1 = y|c_1|, \ldots, z_\ell = y|c_\ell|}$.

Then the F-polynomial resulting from the final mutation, i.e. $F_{i_\ell};t_\ell$, can also be expressed as a sum of a product of binomial coefficients:

$$F_{i_\ell};t_\ell = \sum_{(m_1, \ldots, m_\ell) \in \mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \left( c_j \cdot \left(G_{\ell} + \sum_{k=j+1}^{\ell} m_k B_Q|c_k| \right) \right) y \sum_{j=1}^{\ell} m_j |c_j|.$$

**Note:** This expression as a power series leaves the polynomiality (finiteness of the sum) and positivity of the coefficients as surprising consequences.
Kronecker Quiver Example

\[ F_{i_\ell; t_\ell} = \sum_{(m_1, \ldots, m_\ell) \in \mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \left( c_j \cdot \left( g_\ell + \sum_{k=j+1}^{\ell} m_k B_Q |c_k| \right) \right) y^{\sum_{j=1}^{\ell} m_j |c_j|}. \]

Suppose \( B_Q = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \) and \( \bar{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \cdots \mu_{i_\ell}. \)
Kronecker Quiver Example

\[ F_{i_\ell; t_\ell} = \sum_{(m_1, \ldots, m_\ell) \in \mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \left( c_j \cdot \left( g_\ell + \sum_{k=j+1}^{\ell} m_k B_Q |c_k| \right) \right) \ y \sum_{j=1}^{\ell} m_j |c_j| . \]

Suppose \( B_Q = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \) and \( \bar{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \cdots \mu_{i_\ell} \). Then

\[ c_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} , \ c_2 = \begin{bmatrix} -2 \\ -1 \end{bmatrix} , \ c_3 = \begin{bmatrix} -3 \\ -2 \end{bmatrix} , \ldots , \ c_p = \begin{bmatrix} -p \\ -p+1 \end{bmatrix} , \ |c_p| = \begin{bmatrix} p \\ p+1 \end{bmatrix} , \]

and \( g_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} , \ g_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix} , \ g_3 = \begin{bmatrix} -3 \\ 4 \end{bmatrix} , \ldots , \ g_q = \begin{bmatrix} -q \\ q+1 \end{bmatrix} . \]
Kronecker Quiver Example

\[ F_{i\ell; t\ell} = \sum_{(m_1, \ldots, m_{\ell}) \in \mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \left( c_j \cdot \left( g_\ell + \sum_{k=j+1}^{\ell} m_k B_Q |c_k| \right) \right) y^\sum_{j=1}^{\ell} m_j |c_j| . \]

Suppose \( B_Q = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \) and \( \bar{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \cdots \mu_i \ell \). Then

\[
\begin{align*}
c_1 &= \begin{bmatrix} -1 \\ 0 \end{bmatrix}, & c_2 &= \begin{bmatrix} -2 \\ -1 \end{bmatrix}, & c_3 &= \begin{bmatrix} -3 \\ -2 \end{bmatrix}, & \ldots, & c_p &= \begin{bmatrix} -p \\ -p+1 \end{bmatrix}, & |c_p| &= \begin{bmatrix} p \\ p+1 \end{bmatrix}, \\
g_1 &= \begin{bmatrix} -1 \\ 2 \end{bmatrix}, & g_2 &= \begin{bmatrix} -2 \\ 3 \end{bmatrix}, & g_3 &= \begin{bmatrix} -3 \\ 4 \end{bmatrix}, & \ldots, & g_q &= \begin{bmatrix} -q \\ q+1 \end{bmatrix}.
\end{align*}
\]

and

\[
\begin{align*}
c_j \cdot g_\ell &= \begin{bmatrix} -j \\ -j+1 \end{bmatrix} \cdot \begin{bmatrix} -\ell \\ -\ell+1 \end{bmatrix} = \ell - j + 1, & c_j \cdot B_Q |c_k| &= \begin{bmatrix} -j \\ -j+1 \end{bmatrix} \cdot \begin{bmatrix} -2k+2 \\ -2k \end{bmatrix} = 2(j - k).
\end{align*}
\]
Kronecker Quiver Example

\[
F_{i; t} = \sum_{(m_1, \ldots, m_{\ell}) \in \mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \left( c_j \cdot \left( g_{\ell} + \sum_{k=j+1}^{\ell} m_k B_Q |c_k| \right) \right) y^{\sum_{j=1}^{\ell} m_j |c_j|}.
\]

Suppose \( B_Q = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \) and \( \bar{\mu} = \mu_1 \mu_2 \mu_1 \mu_2 \cdots \mu_i \). Then

\[
c_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \quad c_3 = \begin{bmatrix} -3 \\ -2 \end{bmatrix}, \ldots, \quad c_p = \begin{bmatrix} -p \\ -p+1 \end{bmatrix}, \quad |c_p| = \begin{bmatrix} p \\ p+1 \end{bmatrix},
\]

and \( g_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad g_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \quad g_3 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}, \ldots, \quad g_q = \begin{bmatrix} -q \\ q+1 \end{bmatrix}. \)

Hence

\[
c_j \cdot g_{\ell} = \begin{bmatrix} -j \\ -j+1 \end{bmatrix} \cdot \begin{bmatrix} -\ell \\ \ell+1 \end{bmatrix} = \ell - j + 1, \quad c_j \cdot B_Q |c_k| = \begin{bmatrix} -j \\ -j+1 \end{bmatrix} \cdot \begin{bmatrix} -2k+2 \\ -2k \end{bmatrix} = 2(j - k).
\]

Consequently, we simplify the formula in the Kronecker case to

\[
F_{i; t} = \sum_{(m_1, \ldots, m_{\ell}) \in \mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \left( \ell - i + 1 - 2 \sum_{j=i+1}^{\ell} (j - i) m_j \right) y^{\sum_{i=1}^{\ell} i m_i} y^{\sum_{i=1}^{\ell} (i-1) m_i}.
\]
Kronecker Quiver Example (continued)

\[ F_{i\ell; t\ell} = \sum_{(m_1, \ldots, m_\ell) \in \mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \left( \ell - i + 1 - 2 \sum_{j=i+1}^{\ell} (j - i) m_j \right) y_1^{\sum_{i=1}^{\ell} im_i} y_2^{\sum_{i=1}^{\ell} (i-1) m_i}. \]

\[ F_{1; t_1} = \sum_{m_1=0}^{\infty} \binom{1}{m_1} y_1^{m_1} = 1 + y_1 \]

\[ F_{2; t_2} = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \binom{2 - 2m_2}{m_1} \binom{1}{m_2} y_1^{m_1+2m_2} y_2^{m_2} = 1 + 2y_1 + y_1^2 + y_1^2 y_2. \]

\[ F_{1; t_3} = \sum_{m_1, m_2, m_3 \in \mathbb{Z}_{\geq 0}} \binom{3 - 2m_2 - 4m_3}{m_1} \binom{2 - 2m_3}{m_2} \binom{1}{m_3} y_1^{m_1+2m_2+3m_3} y_2^{m_2+2m_3} = 1 + 3y_1 + 3y_1^2 + y_1^3 + 2y_1^2 y_2 + 2y_1^3 y_2 + y_1^3 y_2^2. \]
Kronecker Quiver Example (continued)

\[ F_{i_\ell; t_\ell} = \sum_{(m_1, \ldots, m_\ell) \in \mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \left( \ell - i + 1 - 2 \sum_{j=i+1}^{\ell} (j - i) m_j \right) y_1^{\sum_{i=1}^{\ell} i m_i} y_2^{\sum_{i=1}^{\ell} (i-1) m_i}. \]

\[ F_{1; t_1} = \sum_{m_1 = 0}^{\infty} \binom{1}{m_1} y_1^{m_1} = 1 + y_1 \]

These two terms correspond to \( m_1 = 0 \) and \( m_1 = 1 \), respectively. There are no contributions for \( m_1 \geq 2 \).
Kronecker Quiver Example (continued)

\[ F_{i; t} = \sum_{(m_1, \ldots, m_\ell) \in \mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \left( \ell - i + 1 - 2 \sum_{j=i+1}^{\ell} (j - i) m_j \right) y_1^{\sum_{i=1}^{\ell} i m_i} y_2^{\sum_{i=1}^{\ell} (i-1) m_i}. \]

\[ F_{1; 1} = \sum_{m_1=0}^{\infty} \binom{1}{m_1} y_1^{m_1} = 1 + y_1 \]

These two terms correspond to \( m_1 = 0 \) and \( m_1 = 1 \), respectively. There are no contributions for \( m_1 \geq 2 \).

\[ F_{2; 2} = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{m_1} \left( 2 - 2m_2 \right) \binom{1}{m_1} y_1^{m_1+2m_2} y_2^{m_2} = 1 + 2y_1 + y_1^2 + y_1^2 y_2. \]

The two underlined contributions correspond to \( m_2 = 0 \) and \( m_2 = 1 \), respectively. Analogously, there are no contributions for \( m_2 \geq 2 \).
Kronecker Quiver Example (continued)

\[ F_{i; t} = \sum_{(m_1, \ldots, m_\ell) \in \mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \left( \ell - i + 1 - 2 \sum_{j=i+1}^{\ell} (j - i) m_j \right) y_1^{\sum_{i=1}^{\ell} i m_i} y_2^{\sum_{i=1}^{\ell} (i-1) m_i} . \]

\[ F_{1; t} = \sum_{m_1=0}^{\infty} \left( \begin{array}{c} 1 \\ m_1 \end{array} \right) y_1^{m_1} = 1 + y_1 \]

These two terms correspond to \( m_1 = 0 \) and \( m_1 = 1 \), respectively. There are no contributions for \( m_1 \geq 2 \).

\[ F_{2; t} = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \left( \begin{array}{c} 2 - 2 m_2 \\ m_1 \end{array} \right) \left( \begin{array}{c} 1 \\ m_2 \end{array} \right) y_1^{m_1+2m_2} y_2^{m_2} = 1 + 2y_1 + y_1^2 + y_1^2 y_2. \]

The two underlined contributions correspond to \( m_2 = 0 \) and \( m_2 = 1 \), respectively. Analogously, there are no contributions for \( m_2 \geq 2 \).

The first three terms correspond to \( m_1 = 0, m_1 = 1, m_1 = 2 \), respectively, and there are no contributions for \( m_1 \geq 2 \).
Kronecker Quiver Example (continued)

\[ F_{1; t_3} = \sum_{m_1, m_2, m_3 \in \mathbb{Z}_{\geq 0}} \binom{3 - 2m_2 - 4m_3}{m_1} \binom{2 - 2m_3}{m_2} \binom{1}{m_3} y_1^{m_1+2m_2+3m_3} y_2^{m_2+2m_3} = \]

\[
1 + 3y_1 + 3y_1^2 + y_1^3 + 2y_1^2 y_2 + 2y_1^3 y_2 + y_1^3 y_2^2.
\]

The two underlined contributions correspond to \( m_3 = 0 \) and \( m_3 = 1 \), respectively. Again, there are no contributions for \( m_3 \geq 2 \).
Kronecker Quiver Example (continued)

\[ F_{1;3} = \sum_{m_1,m_2,m_3 \in \mathbb{Z}_{\geq 0}} \left( \begin{array}{c} 3 - 2m_2 - 4m_3 \\ m_1 \end{array} \right) \left( \begin{array}{c} 2 - 2m_3 \\ m_2 \end{array} \right) \left( \begin{array}{c} 1 \\ m_3 \end{array} \right) y_1^{m_1+2m_2+3m_3} y_2^{m_2+2m_3} = \]

\[ 1 + 3y_1 + 3y_1^2 + y_1^3 + 2y_2 + 2y_1^3 y_2 + y_1^3 y_2^2. \]

The two underlined contributions correspond to \( m_3 = 0 \) and \( m_3 = 1 \), respectively. Again, there are no contributions for \( m_3 \geq 2 \).

Further refinement of this sum by tracking \( m_2 = 0 \) and \( m_2 = 1 \), respectively, under the assumption \( m_3 = 0 \) yields

\[ 1 + 3y_1 + 3y_1^2 + y_1^3 + 2y_1^3 y_2 + y_1^3 y_2^2. \]
Kronecker Quiver Example (continued)

\[ F_{1;\, t_3} = \sum_{m_1, m_2, m_3 \in \mathbb{Z}_{\geq 0}} \left( \frac{3 - 2m_2 - 4m_3}{m_1} \right) \left( \frac{2 - 2m_3}{m_2} \right) \left( \frac{1}{m_3} \right) y_1^{m_1+2m_2+3m_3} y_2^{m_2+2m_3} = \]

\[ \frac{1 + 3y_1 + 3y_1^2 + y_1^3 + 2y_1^2 y_2 + 2y_1^3 y_2 + y_1^3 y_2^2}{1 + 3y_1 + 3y_1^2 + y_1^3 + 2y_1^2 y_2 + 2y_1^3 y_2 + y_1^3 y_2^2}. \]

The two underlined contributions correspond to \( m_3 = 0 \) and \( m_3 = 1 \), respectively. Again, there are no contributions for \( m_3 \geq 2 \).

Further refinement of this sum by tracking \( m_2 = 0 \) and \( m_2 = 1 \), respectively, under the assumption \( m_3 = 0 \) yields

\[ \frac{1 + 3y_1 + 3y_1^2 + y_1^3 + 2y_1^2 y_2 + 2y_1^3 y_2 + y_1^3 y_2^2}{1 + 3y_1 + 3y_1^2 + y_1^3 + 2y_1^2 y_2 + 2y_1^3 y_2 + y_1^3 y_2^2}. \]

However, in addition we get an **infinite** number of contributions

\[ \sum_{m_1=0}^{\infty} \left( \frac{-1}{m_1} \right) y_1^{m_1+4} y_2^2 + \sum_{m_1=0}^{\infty} \left( \frac{-1}{m_1} \right) y_1^{m_1+3} y_2^2; \text{ recall } \left( \frac{-1}{m_1} \right) = (-1)^{m_1} \]

arising when \( m_2 = 2, m_3 = 0 \) or \( m_2 = 0, m_3 = 1 \).
Kronecker Quiver Example (continued)

\[ F_{1; t_3} = \sum_{m_1, m_2, m_3 \in \mathbb{Z}_{\geq 0}} \left( \frac{3 - 2m_2 - 4m_3}{m_1} \right) \left( \frac{2 - 2m_3}{m_2} \right) \left( \frac{1}{m_3} \right) y_1^{m_1+2m_2+3m_3} y_2^{m_2+2m_3} = \]

\[ 1 + 3y_1 + 3y_1^2 + y_1^3 + 2y_1^2y_2 + 2y_1^3y_2 + y_1^3y_2^2. \]

The two underlined contributions correspond to \( m_3 = 0 \) and \( m_3 = 1 \), respectively. Again, there are no contributions for \( m_3 \geq 2 \).

Further refinement of this sum by tracking \( m_2 = 0 \) and \( m_2 = 1 \), respectively, under the assumption \( m_3 = 0 \) yields

\[ 1 + 3y_1 + 3y_1^2 + y_1^3 + 2y_1^2y_2 + 2y_1^3y_2 + y_1^3y_2^2. \]

However, in addition we get an \textbf{infinite} number of contributions

\[ \sum_{m_1=0}^{\infty} \left( -1 \right)^{m_1} y_1^{m_1+4} y_2^2 + \sum_{m_1=0}^{\infty} \left( -1 \right)^{m_1} y_1^{m_1+3} y_2^2; \]  recall \( \left( -1 \right)^{m_1} = (-1)^{m_1} \)

arising when \( m_2 = 2, m_3 = 0 \) or \( m_2 = 0, m_3 = 1 \). This telescoping infinite sum vanishes except for the term of \( y_1^3y_2^2 \) for \( m_1 = 0, m_2 = 0, m_3 = 1 \).
Kronecker Quiver Example (continued)

The formulae continue as

\[ F_{2,t_4} = \sum_{m_1,m_2,m_3,m_4 \in \mathbb{Z}_{\geq 0}} \binom{4 - 2m_2 - 4m_3 - 6m_4}{m_1} \binom{3 - 2m_3 - 4m_4}{m_2} \times \binom{2 - 2m_4}{m_3} \binom{1}{m_4} \ y_1^{m_1+2m_2+3m_3+4m_4} y_2^{m_2+2m_3+3m_4} \]

\[ F_{1,t_5} = \sum_{m_1,m_2,m_3,m_4,m_5 \in \mathbb{Z}_{\geq 0}} \binom{5 - 2m_2 - 4m_3 - 6m_4 - 8m_5}{m_1} \binom{4 - 2m_3 - 4m_4 - 6m_5}{m_2} \times \binom{3 - 2m_4 - 4m_5}{m_3} \binom{2 - 2m_5}{m_4} \binom{1}{m_5} \ y_1^{m_1+2m_2+3m_3+4m_4+5m_5} y_2^{m_2+2m_3+3m_4+4m_5} \]

\[ F_{1,t_5} \] includes terms such as \( 6y_1^5 y_2^3 - 2y_1^5 y_2^3 = 4y_1^5 y_2^3 \) in its expansion, corresponding to \((m_1, m_2, m_3, m_4, m_5) = (0, 1, 1, 0, 0)\) and \((1, 0, 0, 1, 0)\), respectively. In particular, the contributions from negative binomial coefficients yield a positive term, yet arises from a non-trivial difference.
Formula for general Rank Two, i.e. $r$-Kronecker Case

For the case of $B_Q = \begin{bmatrix} 0 & r \\ -r & 0 \end{bmatrix}$ and $\mu = \mu_1\mu_2\mu_1\mu_2 \cdots \mu_i$, where $s_{-1} = 0$, $s_0 = 1$, $s_{k+1} = rs_k - s_{k-1}$ for $k \geq 0$.

$$F_{i\ell,t\ell} = \sum_{(m_1, \ldots, m_\ell) \in \mathbb{Z}_{\geq 0}} \prod_{i=1}^{\ell} \left( s_{\ell-i} - r \sum_{j=i+1}^{\ell} s_{j-i-1} m_j \right) y_1^{\sum_{i=1}^{\ell} s_{i-1} m_i} y_2^{\sum_{i=1}^{\ell} s_{i-2} m_i}$$
Theorem (Gupta ’18) as will be re-expressed in (Gupta-M ’19+) : Given a framed quiver $\tilde{Q}$ and a mutation sequence $\overline{\mu} = \mu_1 \mu_2 \cdots \mu_\ell$, consider the sequence of cluster seeds $t_0 \to \mu_1 t_1 \to \mu_2 \cdots t_{\ell-1} \to \mu_\ell t_\ell$.

Let $\{ F_1; t_\ell, F_2; t_\ell, \ldots, F_n; t_\ell \}$ be the $F$-polynomials associated to the cluster seed after the final mutation. Let $F_{t_\ell}^{(d_1, \ldots, d_n)} = F_{1; t_\ell}^{d_1} F_{2; t_\ell}^{d_2} \cdots F_{n; t_\ell}^{d_n}$ and $g^{(d_1, d_2, \ldots, d_n)}$ be the associated $d$-weighted linear combination of $g$-vectors.
Theorem (Gupta ’18) as will be re-expressed in (Gupta-M ’19+) : Given a framed quiver $\tilde{Q}$ and a mutation sequence $\overline{\mu} = \mu_1 \mu_2 \cdots \mu_\ell$, consider the sequence of cluster seeds $t_0 \rightarrow \mu_1 t_1 \rightarrow \mu_2 \cdots t_{\ell-1} \rightarrow \mu_\ell t_\ell$.

Let $\{F_1;t_\ell, F_2;t_\ell, \ldots, F_n;t_\ell\}$ be the $F$-polynomials associated to the cluster seed after the final mutation. Let $F_{t_\ell}^{(d_1,\ldots,d_n)} = F_{1;t_\ell}^{d_1} F_{2;t_\ell}^{d_2} \cdots F_{n;t_\ell}^{d_n}$ and $g^{(d_1,d_2,\ldots,d_n)}$ be the associated $d$-weighted linear combination of $g$-vectors. Then this $F$-polynomial analogue of a cluster monomial can be expressed as a sum of products of binomial coefficients:

$$F_{t_\ell}^{(d_1,\ldots,d_n)} = \sum_{(m_1,\ldots,m_\ell) \in \mathbb{Z}_{\geq 0}} \prod_{j=1}^\ell \left( c_j \cdot \left( g^{(d_1,d_2,\ldots,d_n)} + \sum_{k=j+1}^\ell m_k B_Q |c_k| \right) \right) y^{\sum_{j=1}^\ell m_j |c_j|}.$$

Here, $c_p$ (resp. $|c_p|$) denotes the $p$th $c$-vector (resp. the normalized $c$-vector $\epsilon_p c_p$) along the mutation sequence $\overline{\mu}$, $B_Q$ denotes the exchange matrix associated to $Q$ before any mutations, $a \cdot b$ denotes ordinary dot product, and $y^{(d_1,d_2,\ldots,d_n)}$ is shorthand for $y_1^{d_1} y_2^{d_2} \cdots y_n^{d_n}$. 

G. Musiker and M. Gupta (AMS 2019)
Meghal Gupta, A formula for F-polynomials in terms of C-Vectors and Stabilization of F-polynomials, REU 2018, arXiv:1812.01910

Meghal Gupta and Gregg Musiker, Applications of F-polynomials in terms of C-Vectors, (in preparation).


