Graph Theoretical Cluster Expansion Formulas

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MIT

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http//math.mit.edu/∼ musiker/GraphTalk.pdf

Outline.

Cluster Expansions

Definition (Sergey Fomin and Andrei Zelevinsky 2001) A cluster algebra $\mathcal{A}$ is a certain subalgebra of $k(x_1, \ldots, x_m)$, the field of rational functions over $\{x_1, \ldots, x_m\}$. Generators constructed by a series of exchange relations, which in turn induce all relations satisfied by the generators.

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Theorem. (The Laurent Phenomenon FZ 2001) For any cluster algebra defined by initial seed $(\{x_1, x_2, \ldots, x_m\}, B)$, all cluster variables of $\mathcal{A}(B)$ are *Laurent polynomials* in $\{x_1, x_2, \ldots, x_m\}$
(with no coefficient $x_{n+1}, \ldots, x_m$ in the denominator).

Thus, any cluster variable $x_\alpha = \frac{P_\alpha(x_1, \ldots, x_m)}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}$ where $P_\alpha \in \mathbb{Z}[x_1, \ldots, x_n]$.

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Conjecture. (Positivity Conjecture FZ 2001) For any cluster variable x_α the polynomial $P_\alpha(x_1, \ldots, x_n)$ has nonnegative integer coefficients.

Some Prior Work on Positivity Conjecture

Work of [Carroll-Price 2002] gave expansion formulas for case of Ptolemy algeras, cluster algebras of type A_n with boundary coefficients.

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Positivity also proven for those cluster variables for an acyclic seed [Caldero-Reineke 2006],

as well as for Cluster algebras arising from unpunctured surfaces [Schiffler-Thomas 2007, Schiffler 2008], generalizing Trails model of Carroll-Price.

Cluster Algebras of Triangulated Surfaces

We follow (Fomin-Shapiro-Thurston) and have a surface (S, M).
We assume marked points $M \subset \partial S$ (no punctures).

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4. γ does not cut out a monogon or digon.

$$\text{Seed} \leftrightarrow \text{Triangulation } T = \{\tau_1, \tau_2, \ldots, \tau_n\}$$

$$\text{Cluster Variable} \leftrightarrow \text{Arc } \gamma$$

$$x_i \leftrightarrow \tau_i \in T.$$

For $\gamma \notin T$ let $e_i(T : \gamma)$ be the minimal intersection number of τ_i and γ.

A Graph Theoretic Approach

Recall from Ralf Schiffler's Talk:

Theorem. (M-Schiffler 2008) For every triangulation T (in a surface without punctures) and arc γ, we construct a snake graph $G_{\gamma,T}$ such that

$$x_\gamma = \frac{\sum_{\text{perfect matching } M \text{ of } G_{\gamma,T}} x(M)y(M)}{x_1^{e_1(T,\gamma)} x_2^{e_2(T,\gamma)} \cdots x_n^{e_n(T,\gamma)}}$$

where $e_i(T,\gamma)$ is the crossing number of τ_i and γ, and $x(M)$, $y(M)$ are each monomials. (x_γ is cluster variable with *principal coefficients*.)

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Definition. Given a simple undirected graph $G = (V, E)$, a perfect matching $M \subseteq E$ is a set of distinguished edges so that every vertex of V is covered exactly once. (Each edge has weight $x(e)$ where $x(e)$ is allowed to be 1 (unweighted) or some variable x_i.)

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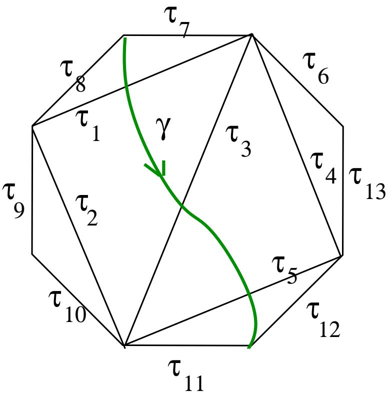
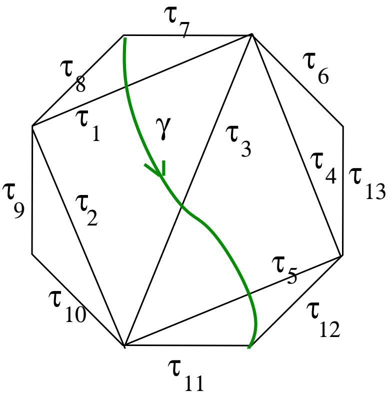
The weight of a matching M is the product of the weights of the constituent edges, i.e. $x(M) = \prod_{e \in M} x(e)$.

Example of Octagon



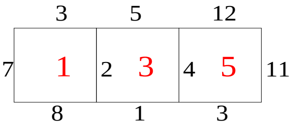


Recall that are 5 completed (T, γ)-paths of this octagon, with weights

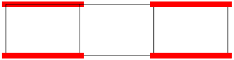
$$\frac{x_3^2 + x_3x_4 + x_2x_3 + x_2x_4 + x_1x_5}{x_1x_3x_5}.$$

Example of Octagon (continued)

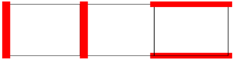
Consider the graph $G_{T_O,\gamma} =$



$G_{T_O,\gamma}$ has five perfect matchings $(x_7, x_8, \ldots, x_{13} = 1)$:

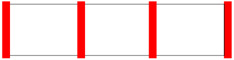


$x_3(x_8)x_3(x_{13})$,



$x_2(x_7)x_3(x_{13})$,

$x_3(x_8)x_4(x_{11})$,



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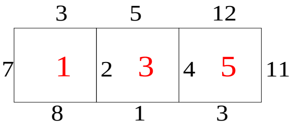

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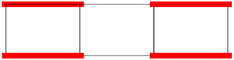

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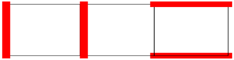

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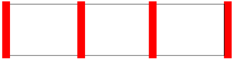

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Dividing each monomial by $x_1x_3x_5$, we obtain weights of (T,γ)-paths.

How to construct $G_{T,\gamma}$'s (unpunctured surfaces)

Definition. For $1 \leq i \leq n$ (i.e. all $\tau_i \in T$), define Tile $\overline{S_i}$ to be (weighted) triangulated quadrilateral homeomorphic to the quadrilateral bounding arc τ_i in surface S. (Diagonal $NW - SE$ and opposite sides still opposite)

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1. Now given arc γ: Pick orientation of $\gamma : s \rightarrow t$.
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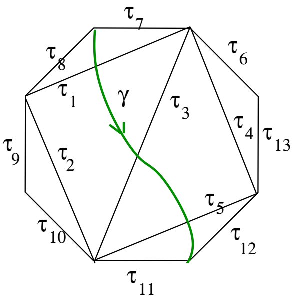
7. We define $\overline{G_{T,\gamma}}$ to be $\overline{G_{\gamma,d}}$. (Erase diagonals to obtain $G_{T,\gamma}$.)



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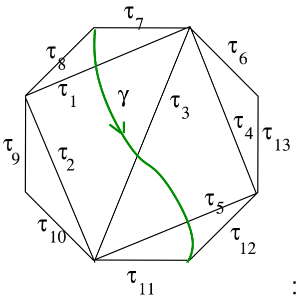
Examples of $G_{T,\gamma}$



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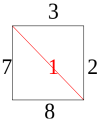

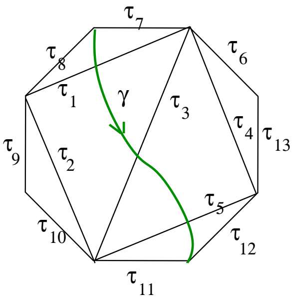

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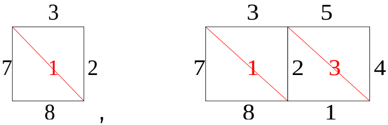

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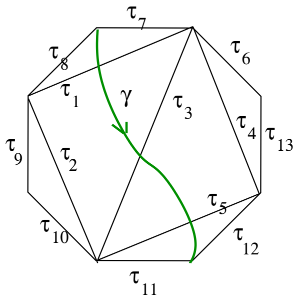

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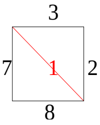

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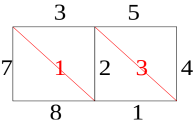

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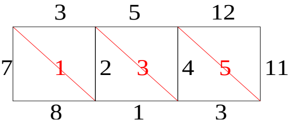
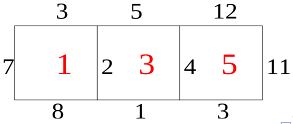


. Thus

$G_{T_O,\gamma} =$



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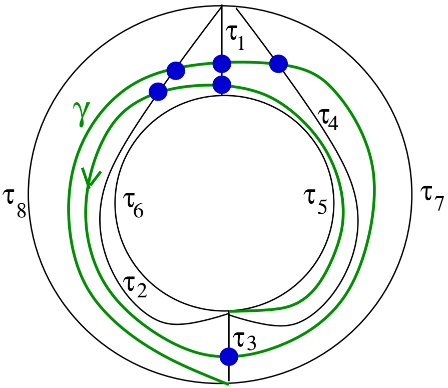
Example 2. We now construct graph $G_{T_A,\gamma}$.



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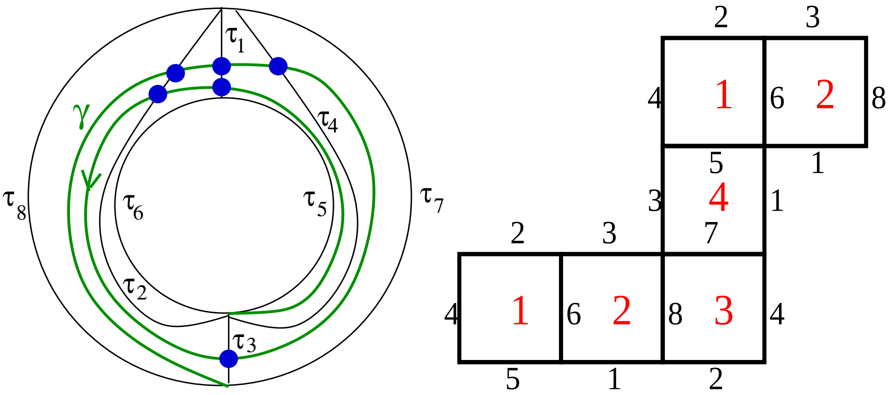

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Given a snake graph G, up to orientation, there is a choice of minimal matching (M_-) which consists of every-other edge on the boundary.

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Given a snake graph G, up to orientation, there is a choice of minimal matching (M_-) which consists of every-other edge on the boundary.

Given any other matching M, let $M \ominus M_-$ denote the symmetric difference.

The height $h_M : Faces(G) \to \mathbb{Z}_{\geq 0}$ of matching M is a function recording which faces are enclosed by $M \ominus M_-$.

Height Functions (of Perfect Matchings of *Snake* Graphs)

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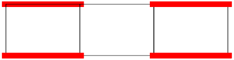
For snake graphs, $h_M(F) \in \{0, 1\}$ and we obtain the formula

$$y(M) := \prod_i y_i^{\sum_{\text{Faces Labeled } i} h_M(F)}.$$

Height Function Examples

Recall that $G_{T_o,\gamma}$ has three faces, labeled 1, 3 and 5.
$G_{T_o,\gamma}$ has five perfect matchings $(x_7, x_8, \ldots, x_{13} = 1)$:



$x_3^2 \; y_1 y_3 y_5$,

$x_2 x_3 \; y_3 y_5$,

$x_3 x_4 \; y_1 y_3$,

$x_2 x_4 \; y_3$,

$x_1 x_5 \; (1)$.

($\longleftarrow$ This matching is M_-.)

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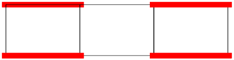
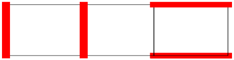


$x_3^2\ y_1 y_3 y_5$,



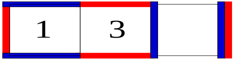
$x_2 x_3\ y_3 y_5$,

$x_3 x_4\ y_1 y_3$,

$x_2 x_4\ y_3$,

$x_1 x_5\ (1)$. ($\longleftarrow$ This matching is M_-.)

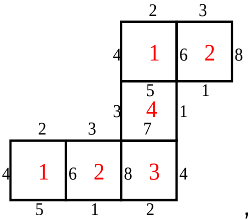
For example, we get heights $y_1 y_3$, y_3, and $y_3 y_5$ because of superpositions:

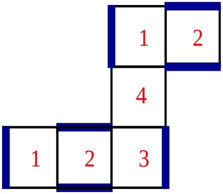


,

, and

Height Function Examples (continued)

For $G_{T_A,\gamma} =$ 

, M_- is 

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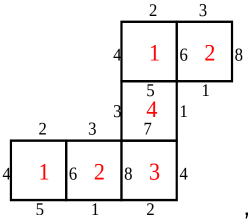
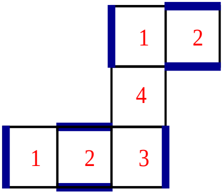
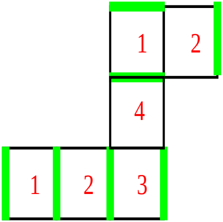
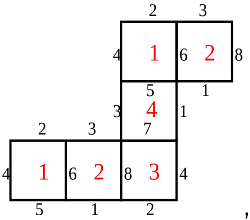
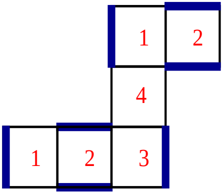
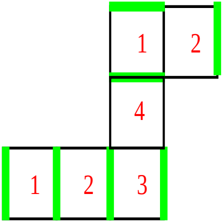
Height Function Examples (continued)

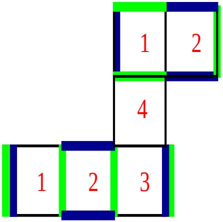
For $G_{T_{A,\gamma}} =$ , M_- is . One of the 17 matchings, M, is ,

Height Function Examples (continued)

For $G_{T_A,\gamma} =$ 

, M_- is 

. One of the 17 matchings, M, is 

, so $M \ominus M_- =$ 

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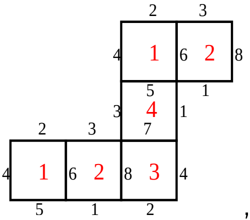
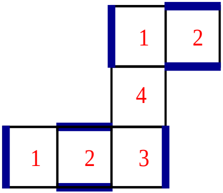
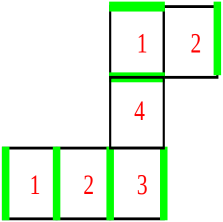
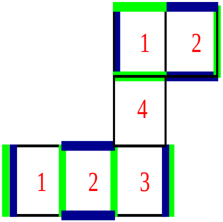
Height Function Examples (continued)

For $G_{T_A,\gamma} =$ 

, M_- is 

. One of the 17 matchings, M, is 

, so $M \ominus M_- =$ 

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which has height $y_1 y_2^2$. So one of the 17 terms in the cluster expansion of x_γ is

$$\frac{x_4^2 x_2}{x_1^2 x_2^2 x_3 x_4}(y_1 y_2^2).$$

Summary

Theorem. (M-Schiffler 2008) For every triangulation T of unpunctured surface and arc γ, we construct a snake graph $G_{\gamma,T}$ such that

$$x_\gamma = \frac{\sum_{\text{perfect matching } M \text{ of } G_{\gamma,T}} x(M)y(M)}{x_1^{e_1(T,\gamma)} x_2^{e_2(T,\gamma)} \cdots x_n^{e_n(T,\gamma)}}$$

where $e_i(T,\gamma)$ is the crossing number of τ_i and γ, $x(M)$ is the edge-weight of perfect matching M, and $y(M)$ is the height of perfect matching M. (x_γ is cluster variable with principal coefficients.)

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where $e_i(T,\gamma)$ is the crossing number of τ_i and γ, $x(M)$ is the edge-weight of perfect matching M, and $y(M)$ is the height of perfect matching M. (x_γ is cluster variable with principal coefficients.)

Corollary. The F-polynomial equals $\sum_M y(M)$, is positive, and has constant term 1.

The g-vector satisfies $\mathbf{x}^g = x(M_-)$.

Corollary. The Laurent expansion of cluster variable x_γ is positive for any cluster algebra (of geometric type) arising from a triangulated surface without punctures.

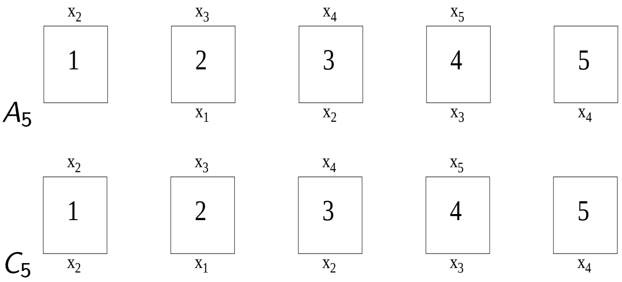
Partially Generalizes Earlier Work on Classical Types

Theorem. (M 2007) For every classical root system, let B_Φ denote the corresponding bipartite seed (without coefficients). Then there exists a family of graphs $\mathcal{G}_\Phi = \{G_\alpha\}_{\alpha \in \Phi_+}$ such that x_α, the cluster variable of $\mathcal{A}(B_\Phi)$ corresponding to $\alpha \in \Phi_+$, can be expressed as

$$x_\alpha = \frac{P_{G_\alpha}(x_1, \ldots, x_n)}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}.$$

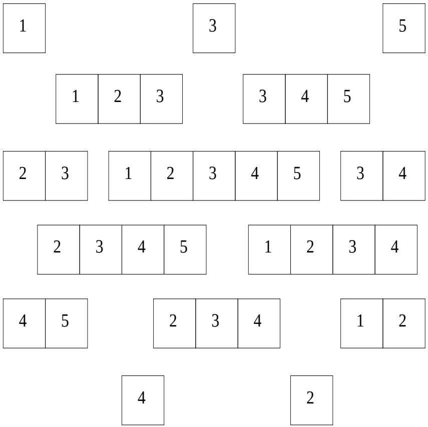
Further, we will construct the graphs in a very simple manner using the tiles T_k.

Tiles for the four classical types

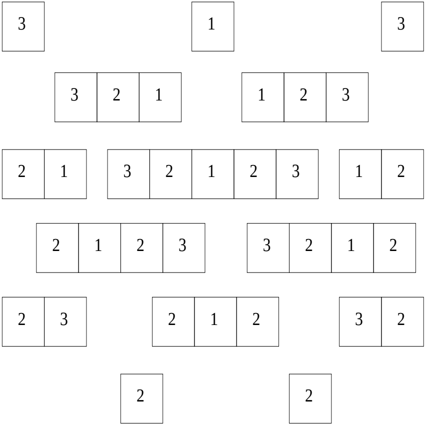


Graphs for A_n and C_n

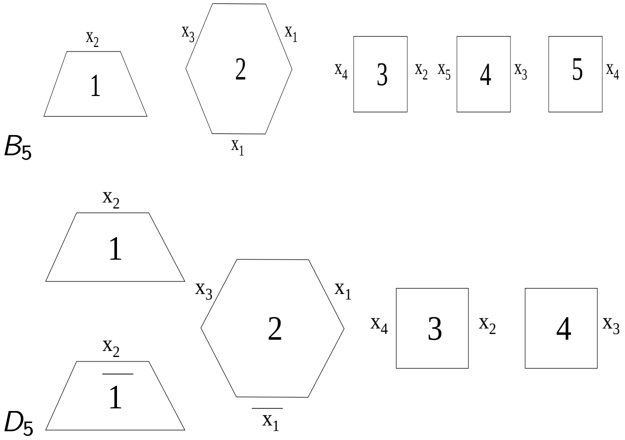
A_5



C_3 folds onto A_5 (Take right-half including middle)

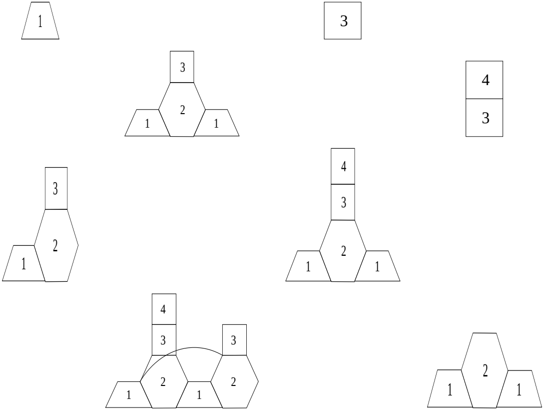


Tiles for the four classical types (cont.)

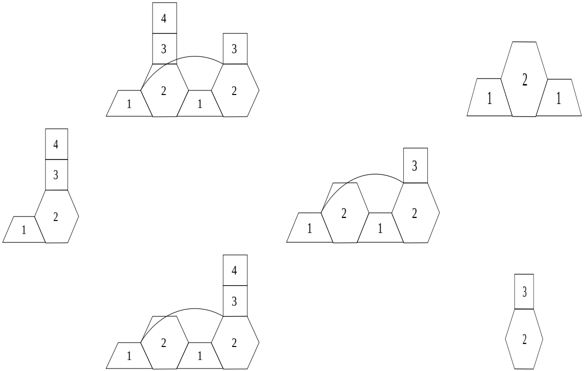


The B_n and D_n cases

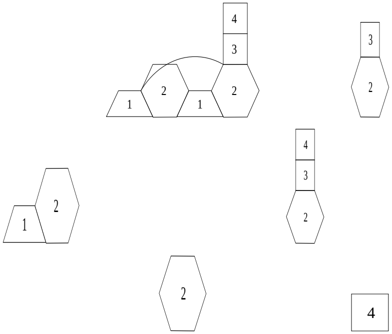
B_4 After mutating with respect to x_1 and x_3 (x_2 and x_4), we obtain



The B_n and D_n cases (cont.)

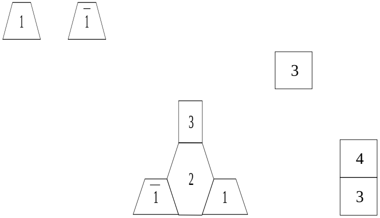


The B_n and D_n cases (cont.)



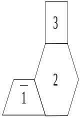
The B_n and D_n cases (cont.)

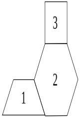
D_5

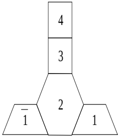


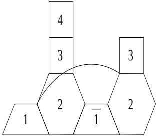
The B_n and D_n cases (cont.)

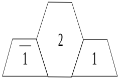
D_5 (cont.)



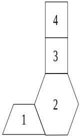


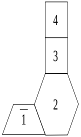


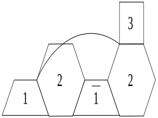


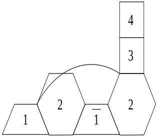


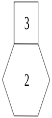
D_5 (cont.)



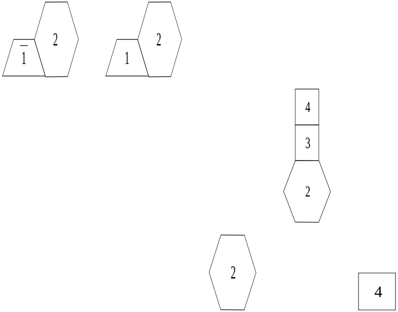








D_5 (cont.)

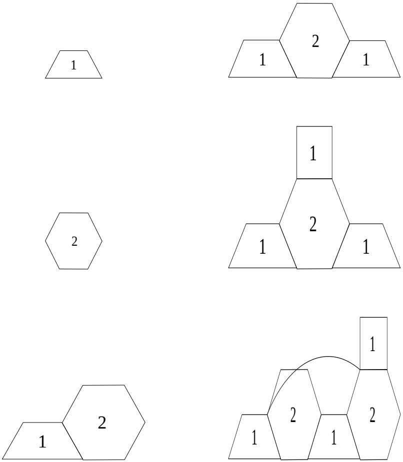


G_2

Seed matrix is $B = \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}$

Hexagon has x_1 on NW, NE, and S sides,

Trapezoid has x_2 on N side.



Affine Rank 2

Joint work with Jim Propp.

Let $B = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$.

Here we also exploit invariance of matrices B under mutation.

So we are considering (b, c)-sequence

$$x_n x_{n-2} = \begin{cases} x_{n-1}^b + 1 & \text{if } n \text{ odd} \\ x_{n-1}^c + 1 & \text{if } n \text{ even} \end{cases}$$

for $(b, c) = (2, 2)$ or $(1, 4)$.

Affine Rank 2 (cont.)

Since cluster algebra structure, (b, c) sequence consists of Laurent polynomials.

Work of Sherman and Zelevinsky verifies positive coefficients for $(1, 4)$ and $(2, 2)$ using Newton polytope, and Caldero-Zelevinsky give another proof of positivity for $(2, 2)$ case via Quiver Grassmannians.

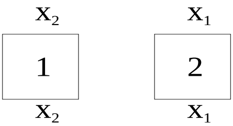
This cluster algebra also comes from an annulus with one marked point on each boundary (no punctures).

Equivalently, this is a cluster algebra of affine type $\widetilde{A}_{1,1}$.

We give proof of positivity via graph theoretical interpretation similar to above.

Affine Rank 2 (cont.)

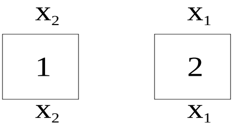
(2, 2): all cluster variables have denominators $x_1^d x_2^{d+1}$ (resp. $x_1^{d+1} x_2^d$)
We string together corresponding number of sqares



in an intertwining fashion.

Affine Rank 2 (cont.)

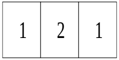
$(2,2)$: all cluster variables have denominators $x_1^d x_2^{d+1}$ (resp. $x_1^{d+1} x_2^d$)
We string together corresponding number of sqares



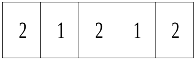
in an intertwining fashion.

Examples:

$\frac{x_2{}^4+2\,x_2{}^2+1+x_1{}^2}{x_1{}^2x_2} \leftrightarrow$

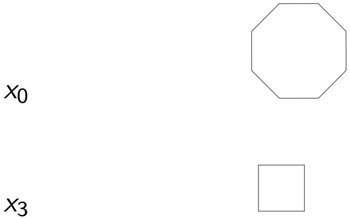


$\frac{x_1{}^6+3\,x_1{}^4+3\,x_1{}^2+2\,x_2{}^2x_1{}^2+x_2{}^4+1+2\,x_2{}^2}{x_2{}^3x_1{}^2} \leftrightarrow$



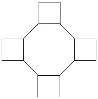
Affine Rank 2 (cont.)

$(1,4)$: Tiles are a square and an octagon:

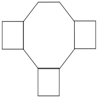


Sequnce Continues

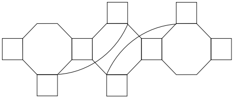
x_4 17 terms



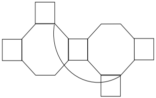
x_5 9 terms



x_6 386 terms

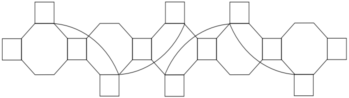


x_7 43 terms

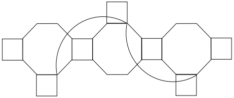


Sequnce Continues (cont.)

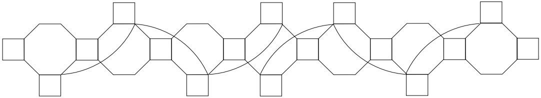
x_8 8857 terms



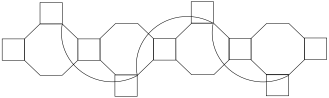
x_9 206 terms



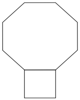
x_{10} 203321 terms

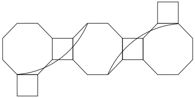


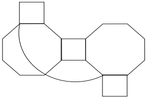
x_{11} 987 terms

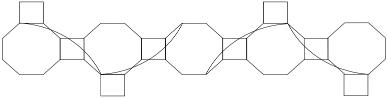


Running the $(1,4)$ sequence backwards

x_{-1} 3 terms 

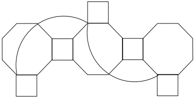
x_{-2} 41 terms 

x_{-3} 14 terms 

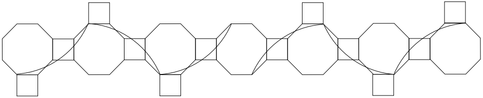
x_{-4} 937 terms 

Running the $(1, 4)$ sequence backwards (cont.)

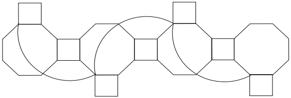
x_{-5} 67 terms



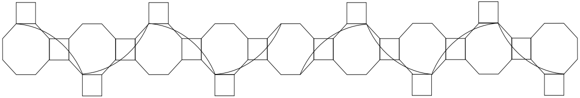
x_{-6} 21506 terms



x_{-7} 321 terms



x_{-8} 493697 terms



Markoff polynomials

Joint work by Carroll, Itsara, Le, M, Price, Thurston, and Viana under Propp in REACH program.

$$B = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix},$$ Exchange graph is free ternary tree.

B invariant under mutation. All exchanges have form $(x, y, z) \mapsto (x', y, z)$ where $xx' = y^2 + z^2$.

(Cluster algebra corresponds to once punctured torus.)

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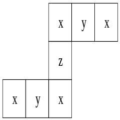
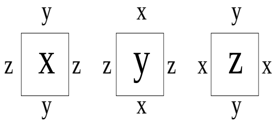
These also have graph theoretic interpretation: Snake Graphs, .e.g

$$\frac{Polynomial(x,y,z)}{x^4 y^2 z^1} \longleftrightarrow$$



with tiles



Work in Progress with Ralf Schiffler and Lauren Williams

- **Theorem.** Formulas for F-polynomials and g-vectors for types A, B, C, D with respect to any seed (not nec. acyclic).
- **In Progress.** Snake Graph Interpretations for Triangulated Surfaces (even in prescence of punctures).

Thank You For Listening

Cluster Expansion Formulas and Perfect Matchings (with Ralf Schiffler), `arXiv:math.CO/0810.3638`

A Graph Theoretic Expansion Formula for Cluster Algebras of Classical Type, `http://www-math.mit.edu/∼ musiker/Finite.pdf`, (To appear in the Annals of Combinatorics)

Combinatorial Interpretations for Rank-Two Cluster Algebras of Affine Type (with Jim Propp), Electronic Journal of Combinatorics. Vol. 14 (R15), 2007.

The Combinatorics of Frieze Patterns and Markoff Numbers (by Jim Propp), `arXiv:math.CO/0511633`

Slides Available at `http//math.mit.edu/∼ musiker/GraphTalk.pdf`