Higher Cluster Categories and QFT Dualities

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Combinatorics Seminar

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Based on Joint Work with Seba Franco

Motivation and History

There has been a fruitful dialogue between string theorists and mathematicians since the 1990's:

Seiberg duality (1995) $\longleftrightarrow$ Quiver Mutation (2001)
(Seiberg) (Fomin-Zelevinsky)

Zamolodchikov Periodicity (1991) $\longleftrightarrow$ Y-system Periodicity (2003)
(Zamolodchikov) (Fomin-Zelevinsky)

Superpotentials & Moduli Spaces (2002) $\longleftrightarrow$ Quivers with Potentials (2007)
(Berenstein-Douglas) (Derksen-Weyman-Zelevinsky)

Amplituhedron (2013) $\longleftrightarrow$ Positive Grassmannian (2006)
(Arkani-Hamed-Trnka) (Postnikov)

Brane Tilings & Gauge Theories (2005) $\longleftrightarrow$ Cluster Integrable Systems (2011)
(Franco-Hanany-Kennaway-Vegh-Wecht) (Goncharov-Kenyon)

This Talk:
Brane Bricks & Hyperbricks (2015-2016) $\longleftrightarrow$??????
(Franco-Lee-Seong-Vafa)

In the late 1990's: Fomin and Zelevinsky were studying total positivity and canonical bases of algebraic groups. They noticed recurring combinatorial and algebraic structures.

Let them to define cluster algebras, which have now been linked to quiver representations, Poisson geometry Teichmüller theory, tilting theory, mathematical physics, discrete integrable systems, string theory, and many other topics.

Cluster algebras are a certain class of commutative rings which have a distinguished set of generators that are grouped into overlapping subsets, called clusters, each having the same cardinality.

What is a Cluster Algebra?

Definition (Sergey Fomin and Andrei Zelevinsky 2001) A cluster algebra $\mathcal{A}$ (of geometric type) is a subalgebra of $k(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ constructed cluster by cluster by certain exchange relations.

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Specify an initial finite set of them, a Cluster, $\{x_1, x_2, \ldots, x_{n+m}\}$.

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$$x_\alpha x'_\alpha = \prod x_{\gamma_i}^{d_i^+} + \prod x_{\gamma_i}^{d_i^-}.$$

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The set of all such generators are known as Cluster Variables, and the initial pattern of exchange relations (described as a *valued* quiver, i.e. a directed graph) determines the Seed.

Relations:

Induced by the Binomial Exchange Relations.

Quiver Mutation (Fomin-Zelevinsky 2001)

Given a quiver Q, we mutate at vertex j by:

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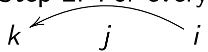

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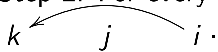

Step 3: Delete any 2-cycles created by Steps 1 and 2.

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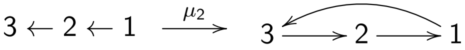

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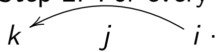

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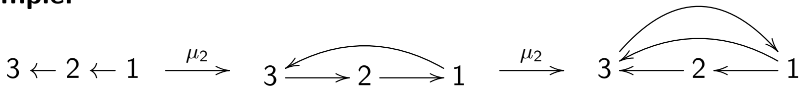

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Given a quiver Q, we mutate at vertex j by:

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Step 2: For every 2-path $k \leftarrow j \leftarrow i$ in Q, add a new arrow

$$\begin{tikzcd} k & j & i \arrow[ll, bend right] \end{tikzcd}.$$

Step 3: Delete any 2-cycles created by Steps 1 and 2.

Example:

$$3 \leftarrow 2 \leftarrow 1 \quad \xrightarrow{\mu_2} \quad \begin{tikzcd} 3 \arrow[r] & 2 \arrow[r] & 1 \arrow[ll, bend right] \end{tikzcd} \quad \xrightarrow{\mu_2} \quad \begin{tikzcd} 3 \arrow[rr, bend left=60] & 2 \arrow[l] & 1 \arrow[l] \arrow[ll, bend right] \end{tikzcd}$$

$$= \; 3 \longleftarrow 2 \longleftarrow 1\,.$$

Cluster Variable Mutation (Fomin-Zelevinsky 2001)

In addition to the mutation of quivers, there is also a complementary **cluster mutation** that can be defined.

Cluster mutation yields a sequence of Laurent polynomials in $\mathbb{Q}(x_1, x_2, \ldots, x_n)$ known as **cluster variables**.

Given a quiver Q and an initial cluster $\{x_1, \ldots, x_n\}$, then mutating at vertex j yields a new cluster variable x_j'

$$\text{defined by} \qquad x_j' = \left(\prod_{k \leftarrow j \in Q} x_k + \prod_{j \leftarrow i \in Q} x_i \right) \Big/ x_j.$$

Example: $Q = 3 \rightarrow 2 \leftarrow 1$

$$x_1 x_1' = x_2 + 1$$

$$x_2 x_2' = 1 + x_1 x_3$$

$$x_3 x_3' = x_2 + 1$$

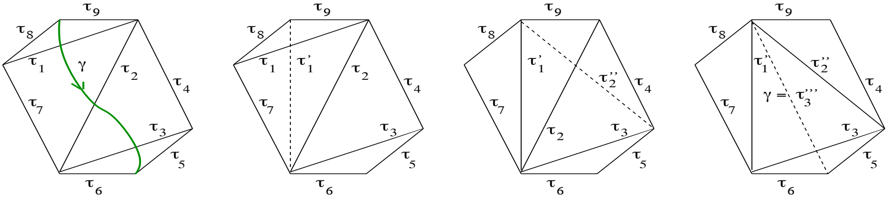
Cluster Algebras from Surfaces

Theorem (Fomin-Shapiro-Thurston 2006, based on earlier work of Fock-Goncharov and Gekhtman-Shapiro-Vainshtein): Given a Riemann surface with marked points (S, M), they define a cluster algebra $\mathcal{A}(S, M)$.

$$\text{Seed} \leftrightarrow \text{Triangulation } T = \{\tau_1, \tau_2, \ldots, \tau_n\}$$

$$\text{Cluster Variable} \leftrightarrow \text{Arc } \gamma \ (x_i \leftrightarrow \tau_i \in T)$$

Cluster Mutation (Binomial Exchange Relations) $\leftrightarrow$ Flipping Diagonals.



$$x_\gamma = \frac{x_2^2 + 2x_2 + 1 + x_1x_3}{x_1x_2x_3}, \text{ via } x_1x_1' = x_2+1, \ x_2x_2'' = x_3+x_1', \ x_3x_3''' = x_2''+x_1'.$$

Cluster Algebras from Surfaces

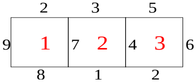
Theorem. (M-Schiffler-Williams 2009) Given a cluster algebra arising from a surface, $\mathcal{A}(S, M)$ with initial seed Σ, the Laurent expansion of every cluster variable with respect to the seed Σ has non-negative coefficients.

Proof via explicit combinatorial formulas in terms of graphs.

Cluster Algebras from Surfaces

Theorem. (M-Schiffler-Williams 2009) Given a cluster algebra arising from a surface, $\mathcal{A}(S, M)$ with initial seed Σ, the Laurent expansion of every cluster variable with respect to the seed Σ has non-negative coefficients.

Proof via explicit combinatorial formulas in terms of graphs. Example:

The graph $G_{\Sigma,\gamma} =$ 

has five perfect matchings:

 $(x_9)x_1x_3(x_6)$,  $(x_9x_7x_4x_6)$,

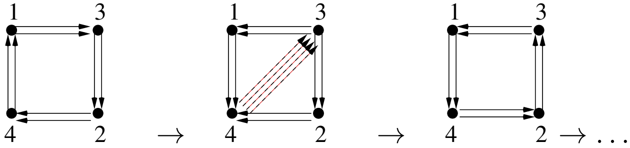
 $x_2(x_8)(x_4x_6)$,  $(x_9x_7)x_2(x_5)$,

 $x_2(x_8)x_2(x_5)$. $x_\gamma = \frac{x_1x_3+1+2x_2+x_2^2}{x_1x_2x_3}$ (with $x_4 = \cdots = x_9 = 1$)

A perfect matching is a subset of edges covering every vertex exactly once. The weight of a matching is the product of the weights of the constituent edges. The denominator corresponds to the labels of $G_{\Sigma,\gamma}$'s tiles.

Cluster Algebras and Aztec Diamonds

Consider the quiver Q (on the left below). Instead of all cluster variables, we focus on those obtained by mutating $1, 2, 3, 4, 1, 2, \ldots$ periodically:



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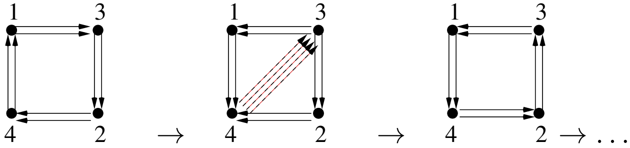


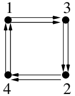
Yields a sequence of cluster variables, with initial cluster variables x_1, x_2, x_3, x_4, with x_{n+4} denoting the nth new cluster variable obtained by this mutation sequence $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, \ldots\}$.

Because of the periodicity, it follows that the x_n's satisfy the recurrences

$$x_n x_{n-4} = \begin{cases} x_{n-1}^2 + x_{n-2}^2 & \text{when } n \text{ is odd, and} \\ x_{n-2}^2 + x_{n-3}^2 & \text{when } n \text{ is even.} \end{cases}$$

For example, $x_5 = \frac{x_3^2+x_4^2}{x_1}$, $x_6 = \frac{x_3^2+x_4^2}{x_2}$, $x_7 = \frac{x_5^2+x_6^2}{x_3}$, and $x_8 = \frac{x_5^2+x_6^2}{x_4}$.

Cluster Algebras and Aztec Diamonds

Let $Q =$ 

, and mutate periodically at $1, 2, 3, 4, 1, 2, 3, 4, \ldots$.

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By letting $x_1 = x_2$ and $x_3 = x_4$, we get $x_{2n+1} = x_{2n}$ for all n.

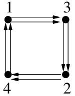
Letting $\{T_n\}$ be the sequence $\{x_{2n}\}_{n \in \mathbb{Z}}$, we obtain a single recurrence.

$$T_n T_{n-2} = 2T_{n-1}^2.$$

If $T_1 = T_2 = 1$, $\{T_n\} = \{1, 1, 2, 8, 64, 1024, 32768, \ldots\} = \left\{2^{\frac{(n-1)(n-2)}{2}}\right\}$.

For $n \geq 3$, $T_n = \#$ (perfect matchings of the $(n-2)$nd Aztec Diamond).

Cluster Algebras and Aztec Diamonds

Let $Q =$ 

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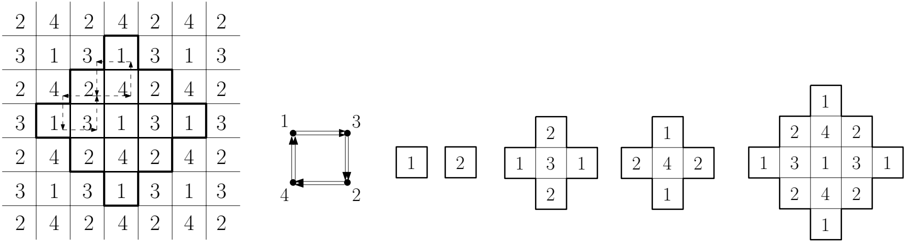

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What is a Brane Tiling (in Physics & Algebraic Geometry)

In physics, **Brane Tilings** are combinatorial models that are used to

Decribe the world volume of both D_3 and M_2 branes, and describe certain $(3 + 1)$-dimensional superconformal field theories arising in string theory (Type II B).

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In Algebraic Geometry, they are used to

Probe certain toric Calabi-Yau singularities, and relate to non-commutative crepant resolutions and the 3-dimensional McKay correspondence.

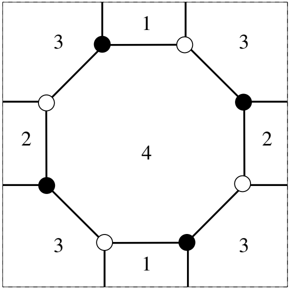
Certain examples of path algebras with relations (Jacobian Algebras) can be constructed by a quiver and potential coming from a brane tiling.

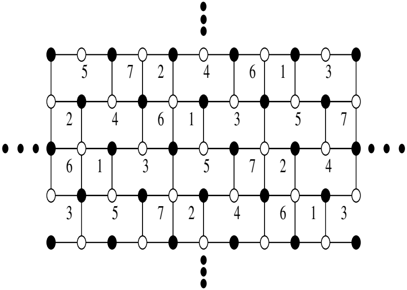
What is a Brane Tiling (Combinatorially)

However, this is a mathematics talk, not a physics talk, so I will henceforth focus on combinatorial motivation instead.

Most simply stated, a Brane Tiling is a **Bipartite graph on a torus**.

We view such a tiling as a doubly-periodic tiling of its universal cover, the Euclidean plane.

Examples: 

, 

Brane Tilings from a Quiver Q with Potential W

A **Brane Tiling** can be associated to a pair (Q, W), where Q is a quiver and W is a potential (called a superpotential in the physics literature).

A quiver Q is a directed graph where each edge is referred to as an arrow, and multiple edges are allowed.

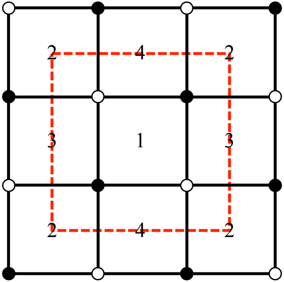
A potential W is a linear combination of cyclic paths in Q (possibly an infinite linear combination).

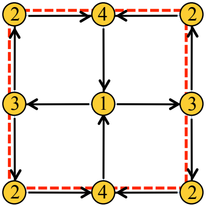
For combinatorial purposes, we assume other conditions on (Q, W), such as

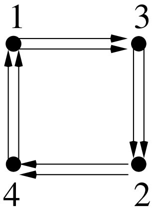
- Each arrow of Q appears in one term of W with a positive sign, and one term with a negative sign.

- The number of terms of W with a positive sign equals the number with a negative sign. All coefficients in W are ± 1.

Example of a Brane Tiling and its Potential







$$W = X_{13}^{(W)}X_{32}^{(S)}X_{24}^{(E)}X_{41}^{(N)} - X_{13}^{(W)}X_{32}^{(N)}X_{24}^{(E)}X_{41}^{(S)}$$
$$+ X_{13}^{(E)}X_{32}^{(N)}X_{24}^{(W)}X_{41}^{(S)} - X_{13}^{(E)}X_{32}^{(S)}X_{24}^{(W)}X_{41}^{(N)}$$

Brane Tilings in Physics

Face $\longleftrightarrow$ Gauge Group $U(N)$

Edge $\longleftrightarrow$ Bifundamental Chiral Fields (Representations)

Vertex $\longleftrightarrow$ Gauge-invariant operator (Term in the Superpotential)

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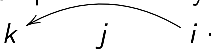
Together, this data yields a **quiver gauge theory**. One can apply Seiberg duality to get a different quiver gauge theory.

Combinatorial connection:
Seiberg duality corresponds to mutation in cluster algebra theory.

Recall: Quiver Mutation (Fomin-Zelevinsky 2001) at vertex j of Q:

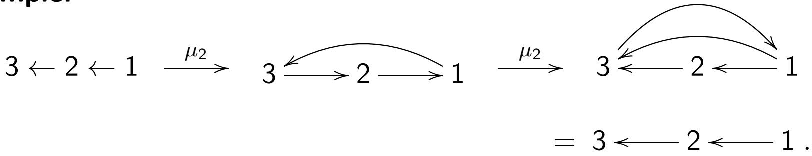
Step 1: Reverse all arrows incident to vertex j.

Step 2: For every 2-path $k \leftarrow j \leftarrow i$ in Q, add a new arrow



Step 3: Delete any 2-cycles created by Steps 1 and 2.

Example:



Mutation of Potentials (Derksen-Weyman-Zelevinsky 2007)

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Given a quiver Q, a potential W is a linear combination of cycles of the quiver Q. With the new data of a potential, we mutate the quiver and potential (Q, W) together (at vertex j):

Step 1: For every arrow $X_{jk} = j \to k$ (resp. $X_{ij} = i \to j$) incident to vertex j, replace it with its dual $X^*_{kj} = k \to j$ (resp. $X^*_{ji} = j \to i$).

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Step 2a: For every 2-path, $i \to j \to k$ in Q, add a new arrow $i \to k$ to Q and a new degree 3 term to W, namely $X_{ik}X^*_{kj}X^*_{ji}$.

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Step 2b: Replace any instances of $X_{ij}X_{jk}$ in W with the new arrow X_{ik}.

Step 3: Letting (Q', W') be the result after Steps 1 and 2, apply a *right-equivalence* to equate

$$(Q', W') \sim (Q'_{red}, W'_{red}) \oplus (Q'_{triv}, W'_{triv})$$

where Q'_{red} has no 2-cycles and W'_{red} has no terms of degree 2.

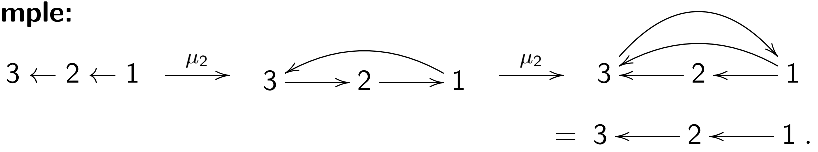
Mutation of Potentials (Derksen-Weyman-Zelevinsky 2007)

Step 1: For every arrow X_{jk} (resp. X_{ij}) incident to vertex j, replace it with its dual X^*_{kj} (resp. X^*_{ji}).

Step 2a, 2b: For every 2-path, $i \to j \to k$ in Q, add $i \to k$ to Q and $X_{ik}X^*_{kj}X^*_{ji}$ to W. Replace instances of $X_{ij}X_{jk}$ in W with the new arrow X_{ik}.

Step 3: Letting (Q', W') be the result after Steps 1 and 2, apply a *right-equivalence* to equate $(Q', W') \sim (Q'_{red}, W'_{red}) \oplus (Q'_{triv}, W'_{triv})$ where Q'_{red} has no 2-cycles and W'_{red} has no terms of degree 2.

Example:



$W = 0$ $\qquad W' = X_{13}X^*_{32}X^*_{21}$ $\qquad W'' = X_{13}(X_{31}) + X_{31}X_{12}X_{23}$

$W''_{red} = 0, \quad W''_{triv} = X_{13}X_{31}.$

Description of Seiberg Duality (from physics)

From **"Brane Dimers and Quiver Gauges Theories (2005)** by Franco, Hanany, Kennaway, Vegh, and Wecht:

After picking a node to dualize at: "Reverse the direction of all arrows entering or exiting the dualized node. This is because Seiberg duality requires that the dual quarks transform in the conjugate flavor representations to the originals. ...

Next, draw in ... bifundamentals which correspond to composite (mesonic) operators. ... the Seiberg mesons are promoted to the fields in the bifundamental representation of the gauge group. ...

It is possible that this will make some fields massive, in which case the appropriate fields should then be integrated out."

Description of Seiberg Duality (rephrased combinatorially)

Pick a vertex j of the quiver Q (equiv. face of the brane tiling $\mathcal{T}_Q$) at which to mutate. Then, reverse the direction of all arrows incident to j, i.e. $A_{ij} \to A^*_{ji}$. Next, for every two-path $i \to j \to k$, *"meson"*, in Q draw in a new arrow $i \to k$, *"the Seiberg mesons are promoted to the fields"*. Let Q' denote this new quiver.

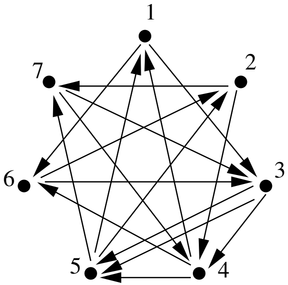
We similarly alter the superpotential W to get W'. For every 2-path $i \to j \to k$ in Q, we replace any appearance of the product $A_{ij}A_{jk}$ in W with the singleton A_{ik} and add or subtract a new degree 3-term $A_{ik}A^*_{kj}A^*_{ji}$.

It is possible, that this will make some of the terms of W' of degree two, *"massive"*, in which case there should be an associated 2-cycle in the mutated quiver Q' that can be deleted, *"the appropriate fields should then be integrated out"*.

This is in fact Mutation of Quivers with potential from cluster algebras (as defined by Derksen-Weyman-Zelevinsky).

In the special case, that we are mutating at a vertex with two arrows in and out, a **toric vertex**, this corresponds to a Urban Renewal of a square face in the brane tiling.

Example ($Q_7^{(2,3)}$):

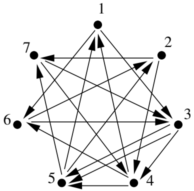


with potential

$$W = A_{13}A_{34}A_{41} + A_{16}A_{63}A_{35}^{(V)}A_{51} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} - A_{16}A_{62}A_{24}A_{41} - A_{34}A_{46}A_{63} - A_{13}A_{35}^{(H)}A_{51} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74}.$$

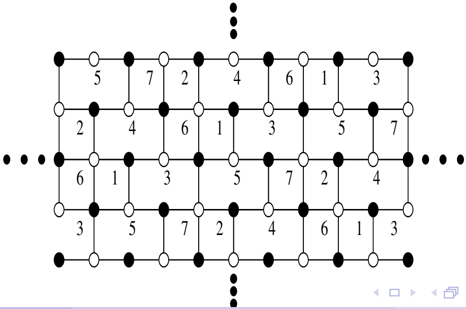
We consider the corresponding Brane Tiling and mutation of (Q, W) at the toric vertex labeled 1.

Description of Seiberg Duality (on the Brane Tiling)

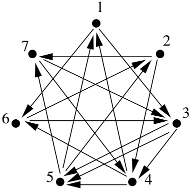
Example ($Q_7^{(2,3)}$): 

with potential

$$
\begin{aligned}
W &= A_{13}A_{34}A_{41} + A_{16}A_{63}A_{35}^{(V)}A_{51} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\
&- A_{16}A_{62}A_{24}A_{41} - A_{34}A_{46}A_{63} - A_{13}A_{35}^{(H)}A_{51} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74}.
\end{aligned}
$$

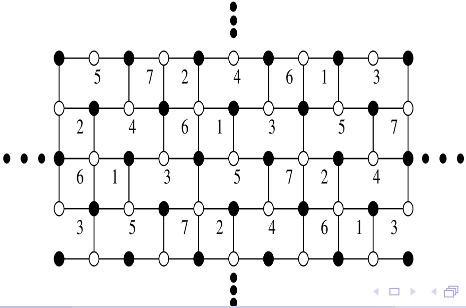


Description of Seiberg Duality (on the Brane Tiling)

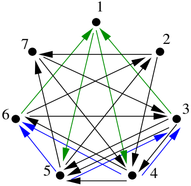


Example ($Q_7^{(2,3)}$): Rotate potential terms containing 1

$$W = A_{41}A_{13}A_{34} + A_{51}A_{16}A_{63}A_{35}^{(V)} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62}$$
$$- A_{41}A_{16}A_{62}A_{24} - A_{34}A_{46}A_{63} - A_{51}A_{13}A_{35}^{(H)} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74}.$$

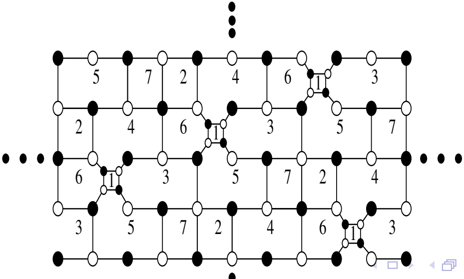


Example ($Q_7^{(2,3)}$):



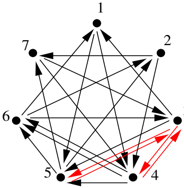
Mutating at 1 yields

$$\begin{aligned} W' &= A_{43}A_{34} + A_{56}A_{63}A_{35}^{(V)} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\ &- A_{46}^{(D)}A_{62}A_{24} - A_{34}A_{46}A_{63} - A_{53}^{(H)}A_{35}^{(H)} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74} \\ &+ A_{14}^{*}A_{46}^{(D)}A_{61}^{*} + A_{15}^{*}A_{53}^{(H)}A_{31}^{*} - A_{14}^{*}A_{43}A_{31}^{*} - A_{15}^{*}A_{56}A_{61}^{*}. \end{aligned}$$



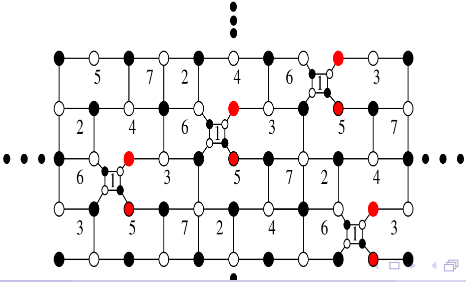
Description of Seiberg Duality (on the Brane Tiling)

Example ($Q_7^{(2,3)}$):



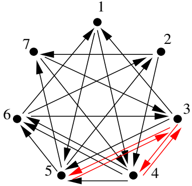
Highlighting Massive terms

$$
\begin{aligned}
W' &= A_{43}A_{34} + A_{56}A_{63}A_{35}^{(V)} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\
&- A_{46}^{(D)}A_{62}A_{24} - A_{34}A_{46}A_{63} - A_{53}^{(H)}A_{35}^{(H)} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74} \\
&+ A_{14}^{*}A_{46}^{(D)}A_{61}^{*} + A_{15}^{*}A_{53}^{(H)}A_{31}^{*} - A_{14}^{*}A_{43}A_{31}^{*} - A_{15}^{*}A_{56}A_{61}^{*}.
\end{aligned}
$$



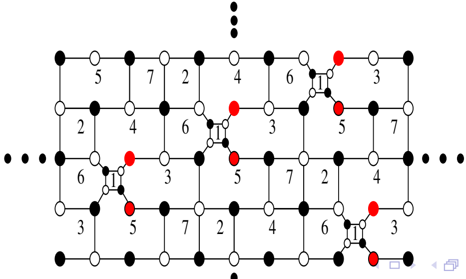
Description of Seiberg Duality (on the Brane Tiling)

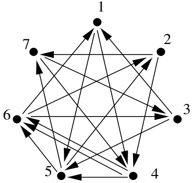
Example ($Q_7^{(2,3)}$):



Highlighting complementary terms

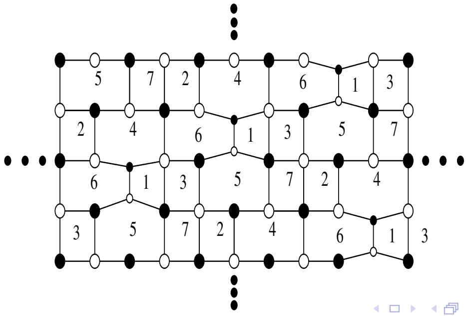
$$\begin{aligned} W' \;=\;& A_{43}A_{34} + A_{56}A_{63}A_{35}^{(V)} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\ -\;& A_{46}^{(D)}A_{62}A_{24} - A_{34}A_{46}A_{63} - A_{53}^{(H)}A_{35}^{(H)} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74} \\ +\;& A_{14}^{*}A_{46}^{(D)}A_{61}^{*} + A_{53}^{(H)}A_{31}^{*}A_{15}^{*} - A_{43}A_{31}^{*}A_{14}^{*} - A_{15}^{*}A_{56}A_{61}^{*}. \end{aligned}$$



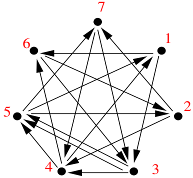


Example ($Q_7^{(2,3)}$): Reduces the potential to

$$\begin{aligned} W'' &= A_{56}A_{63}A_{35}^{(V)} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} - A_{46}^{(D)}A_{62}A_{24} - A_{27}A_{73}A_{35}^{(V)}A_{52} \\ &- A_{45}A_{57}A_{74} + A_{14}^{*}A_{46}^{(D)}A_{61}^{*} - A_{15}^{*}A_{56}A_{61}^{*} - A_{46}A_{63}A_{31}^{*}A_{14}^{*} + A_{31}^{*}A_{15}^{*}A_{57}A_{73}. \end{aligned}$$

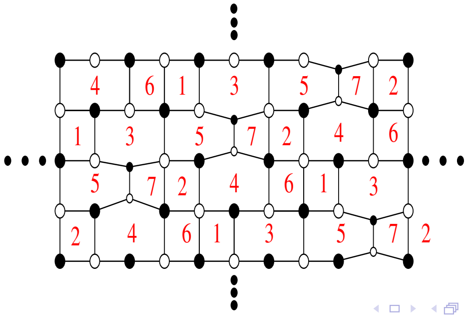


Description of Seiberg Duality (on the Brane Tiling)

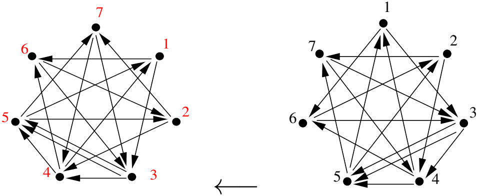


Example ($Q_7^{(2,3)}$): If we cyclically permute vertices

$$W'' = A_{45}A_{52}A_{24}^{(V)} + A_{13}A_{34}A_{41} + A_{16}A_{63}A_{35}A_{51} - A_{35}^{(D)}A_{51}A_{13} - A_{16}A_{62}A_{24}^{(V)}A_{41}$$
$$- A_{34}A_{46}A_{63} + A_{73}^{*}A_{35}^{(D)}A_{57}^{*} - A_{74}^{*}A_{45}A_{57}^{*} - A_{35}A_{52}A_{27}^{*}A_{73}^{*} + A_{27}^{*}A_{74}^{*}A_{46}A_{62}.$$

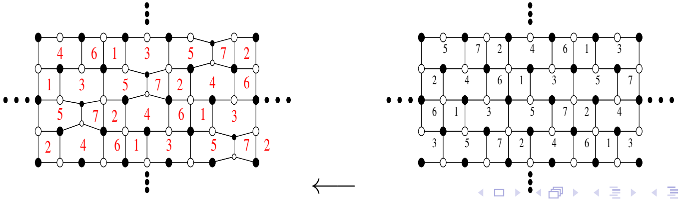


Description of Seiberg Duality (on the Brane Tiling)



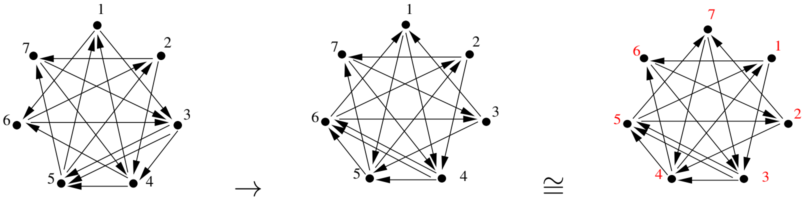
Example ($Q_7^{(2,3)}$):
The cyclic permutation yields the **original** Brane Tiling and (Q, W)!

$$
\begin{aligned}
W'' &= A_{45}A_{52}A_{24}^{(V)} + A_{13}A_{34}A_{41} + A_{16}A_{63}A_{35}A_{51} - A_{35}^{(D)}A_{51}A_{13} - A_{16}A_{62}A_{24}^{(V)}A_{41} \\
&- A_{34}A_{46}A_{63} + A_{73}^{*}A_{35}^{(D)}A_{57}^{*} - A_{74}^{*}A_{45}A_{57}^{*} - A_{35}A_{52}A_{27}^{*}A_{73}^{*} + A_{27}^{*}A_{74}^{*}A_{46}A_{62} \\
W &= A_{13}A_{34}A_{41} + A_{16}A_{63}A_{35}^{(V)}A_{51} + A_{35}^{(H)}A_{57}A_{73} + A_{24}A_{45}A_{52} + A_{27}A_{74}A_{46}A_{62} \\
&- A_{16}A_{62}A_{24}A_{41} - A_{34}A_{46}A_{63} - A_{13}A_{35}^{(H)}A_{51} - A_{27}A_{73}A_{35}^{(V)}A_{52} - A_{45}A_{57}A_{74}.
\end{aligned}
$$



Such Cluster Mutations yield the Gale-Robinson Sequences

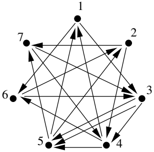
Example ($Q_N^{(r,s)}$): (e.g. $r = 2, \; s = 3, \; N = 7$)

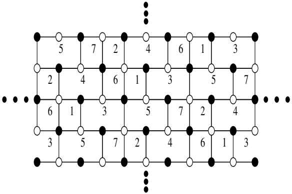


Mutating at $1, 2, 3, \ldots, N, 1, 2, \ldots$ yields the same quiver, up to cyclic permutation, at each step, hence we obtain the infinite sequence of $x_{N+1}, x_{N+2}, \ldots$ satsifying

$$x_n = \left(x_{n-r}x_{n-N+r} + x_{n-s}x_{n-N+s}\right)/x_{n-N} \quad \text{for} \quad n > N.$$

Known as the **Gale-Robinson Sequence** of Laurent polynomials.

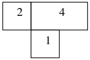
$Q =$ 

$\mathcal{T}_Q =$ 

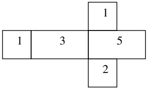
$x_8 \leftrightarrow$ 

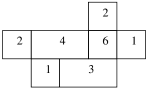
, $x_9 \leftrightarrow$ 

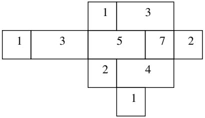
, $x_{10} \leftrightarrow$ 

, $x_{11} \leftrightarrow$ 

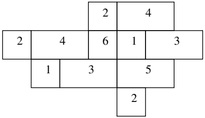
,

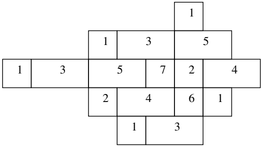
$x_{12} \leftrightarrow$ 

, $x_{13} \leftrightarrow$ 

, $x_{14} \leftrightarrow$ 

,

$x_{15} \leftrightarrow$ 

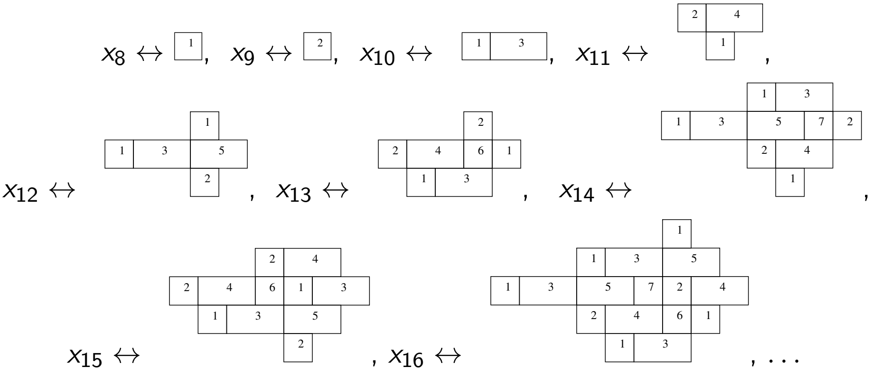
, $x_{16} \leftrightarrow$ 

, ...

FPSAC Proceedings 2013 (Jeong-M-Zhang)

Obtain **pinecone graphs** from Bousquet-Mélou, Propp, and West in terms of Brane Tilings Terminology.

Furthermore, to get cluster variable formulas with coefficients, need only use weights (Goncharov-Kenyon, Speyer) and heights (Kenyon-Propp-...)

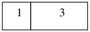


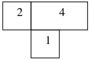
Similar connections (without principal coefficients) also observed in "Brane tilings and non-commutative geometry" by Richard Eager.

Eager uses physics terminology where he looks at $Y^{p,q}$ and $L^{a,b,c}$ quiver gauge theories, and their periodic Seiberg duality (i.e. quiver mutations).

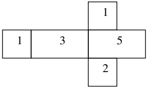
$x_8 \leftrightarrow$ 

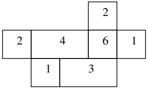
, $x_9 \leftrightarrow$ 

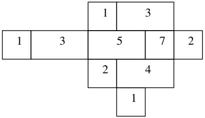
, $x_{10} \leftrightarrow$ 

, $x_{11} \leftrightarrow$ 

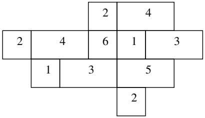
,

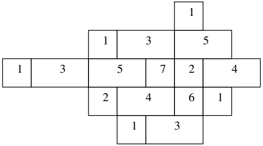
$x_{12} \leftrightarrow$ 

, $x_{13} \leftrightarrow$ 

, $x_{14} \leftrightarrow$ 

,

$x_{15} \leftrightarrow$ 

, $x_{16} \leftrightarrow$ 

, . . .

Recent Extensions of Seiberg Duality by Physicists

Brane Tilings like the above example correspond to a 4-dimensional $N = 1$ super-symmetric quiver gauge theory.

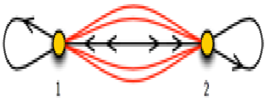
We next consider a 2-dimensional $N = (0, 2)$ SUSY quiver gauge theory.

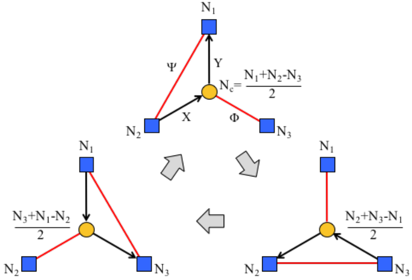
Recent Extensions of Seiberg Duality by Physicists

Brane Tilings like the above example correspond to a 4-dimensional $N = 1$ super-symmetric quiver gauge theory.

We next consider a 2-dimensional $N = (0, 2)$ SUSY quiver gauge theory.

Gadde, Gukov, and Putrov (2013) introduced dynamics which are analogues of Seiberg Duality: GGP $(0, 2)$ Triality.

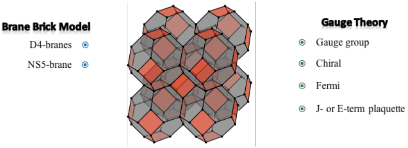


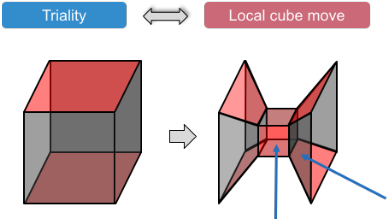


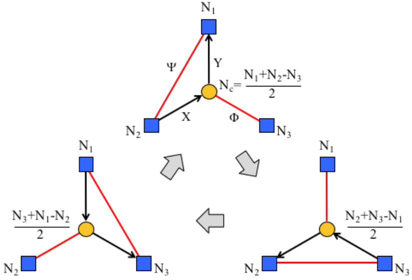
Fermis are undirected arrows. **Chirals** are directed arrows.

Recent Extensions of Seiberg Duality by Physicists

Corresponding geometric and combinatorial model of Brane Bricks developed by Franco-Lee-Seong (2015); an extension of Brane Tilings.

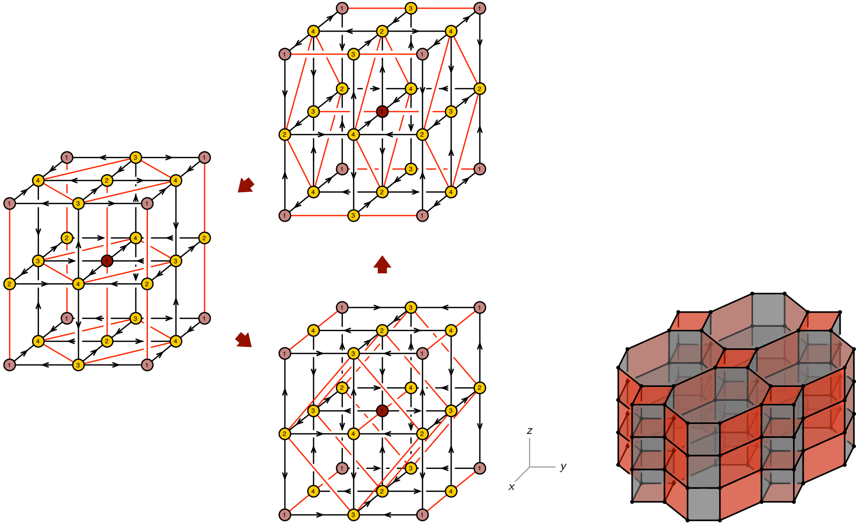


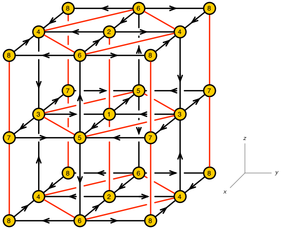




Recent Extensions of Seiberg Duality by Physicists

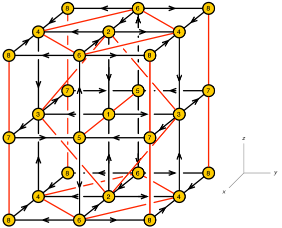
Example $Q^{1,1,1}$**:**





Example $Q^{1,1,1}/\mathbb{Z}_2$ (J-terms and E-terms):

$$
\begin{aligned}
W = {} & \Lambda_{21}^{+}X_{15}^{+}X_{56}^{-}X_{62}^{-} - \Lambda_{21}^{+}X_{15}^{-}X_{56}^{-}X_{62}^{+} + \Lambda_{12}^{-}X_{24}^{+}X_{43}^{+}X_{31}^{-} - \Lambda_{12}^{-}X_{24}^{-}X_{43}^{+}X_{31}^{+} + \Lambda_{21}^{-}X_{15}^{-}X_{56}^{+}X_{62}^{+} - \Lambda_{21}^{-}X_{15}^{+}X_{56}^{+}X_{62}^{-} \\
& + \Lambda_{12}^{+}X_{24}^{+}X_{43}^{-}X_{31}^{-} - \Lambda_{12}^{+}X_{24}^{-}X_{43}^{-}X_{31}^{+} + \Lambda_{78}^{+}X_{84}^{+}X_{43}^{-}X_{37}^{-} - \Lambda_{78}^{+}X_{84}^{-}X_{43}^{-}X_{37}^{+} + \Lambda_{87}^{-}X_{75}^{+}X_{56}^{+}X_{68}^{-} - \Lambda_{87}^{-}X_{75}^{-}X_{56}^{+}X_{68}^{+} \\
& + \Lambda_{78}^{-}X_{84}^{-}X_{43}^{+}X_{37}^{+} - \Lambda_{78}^{-}X_{84}^{+}X_{43}^{+}X_{37}^{-} + \Lambda_{87}^{+}X_{75}^{+}X_{56}^{-}X_{68}^{-} - \Lambda_{87}^{+}X_{75}^{-}X_{56}^{-}X_{68}^{+} + \Lambda_{64}^{++}X_{43}^{+}X_{37}^{-}X_{75}^{-}X_{56}^{-} - \Lambda_{64}^{++}X_{43}^{-}X_{31}^{-}X_{15}^{-}X_{56}^{+} \\
& + \Lambda_{46}^{--}X_{62}^{+}X_{24}^{+} - \Lambda_{46}^{--}X_{68}^{+}X_{84}^{+} + \Lambda_{64}^{--}X_{43}^{+}X_{31}^{+}X_{15}^{+}X_{56}^{-} - \Lambda_{64}^{--}X_{43}^{-}X_{37}^{+}X_{75}^{+}X_{56}^{+} \\
& + \Lambda_{46}^{++}X_{62}^{-}X_{24}^{-} - \Lambda_{46}^{++}X_{68}^{-}X_{84}^{-} + \Lambda_{64}^{+-}X_{43}^{-}X_{31}^{+}X_{15}^{-}X_{56}^{+} - \Lambda_{64}^{+-}X_{43}^{+}X_{37}^{-}X_{75}^{+}X_{56}^{-} \\
& + \Lambda_{46}^{-+}X_{62}^{+}X_{24}^{-} - \Lambda_{46}^{-+}X_{68}^{-}X_{84}^{+} + \Lambda_{64}^{-+}X_{43}^{-}X_{37}^{+}X_{75}^{-}X_{56}^{+} - \Lambda_{64}^{-+}X_{43}^{+}X_{31}^{-}X_{15}^{+}X_{56}^{-} + \Lambda_{46}^{+-}X_{62}^{-}X_{24}^{+} - \Lambda_{46}^{+-}X_{68}^{+}X_{84}^{-} \\
& + \Lambda_{35}^{++}X_{56}^{+}X_{62}^{-}X_{24}^{-}X_{43}^{-} - \Lambda_{35}^{++}X_{56}^{-}X_{68}^{-}X_{84}^{-}X_{43}^{+} + \Lambda_{53}^{--}X_{37}^{+}X_{75}^{+} - \Lambda_{53}^{--}X_{31}^{+}X_{15}^{+} + \Lambda_{35}^{--}X_{56}^{+}X_{68}^{+}X_{84}^{+}X_{43}^{-} - \Lambda_{35}^{--}X_{56}^{-}X_{62}^{+}X_{24}^{+}X_{43}^{+} \\
& + \Lambda_{53}^{++}X_{37}^{-}X_{75}^{-} - \Lambda_{53}^{++}X_{31}^{-}X_{15}^{-} + \Lambda_{35}^{+-}X_{56}^{-}X_{68}^{+}X_{84}^{-}X_{43}^{+} - \Lambda_{35}^{+-}X_{56}^{+}X_{62}^{-}X_{24}^{+}X_{43}^{-} + \Lambda_{53}^{-+}X_{37}^{+}X_{75}^{-} - \Lambda_{53}^{-+}X_{31}^{-}X_{15}^{+} \\
& + \Lambda_{35}^{-+}X_{56}^{-}X_{62}^{+}X_{24}^{-}X_{43}^{+} - \Lambda_{35}^{-+}X_{56}^{+}X_{68}^{-}X_{84}^{+}X_{43}^{-} + \Lambda_{53}^{+-}X_{37}^{-}X_{75}^{+} - \Lambda_{53}^{+-}X_{31}^{+}X_{15}^{-}
\end{aligned}
$$

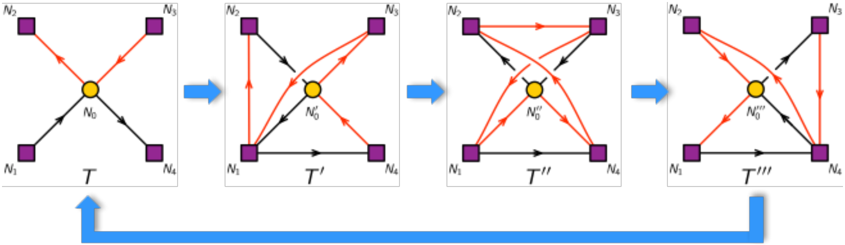


Example $Q^{1,1,1}/\mathbb{Z}_2$ After Mutation at 1:

$$
\begin{aligned}
W' = {} & X^+_{21}\Lambda^+_{15}X^-_{56}X^-_{62} - X^+_{21}\Lambda^-_{15}X^-_{56}X^+_{62} + X^+_{24}X^+_{43}\Lambda^{--}_{32} - X^-_{24}X^+_{43}\Lambda^{+-}_{32} + \Lambda^{-+}_{23}X^+_{37}X^+_{75}X^-_{56}X^-_{62} \\
& + \Lambda^{++}_{23}X^+_{37}X^-_{75}X^-_{56}X^-_{62} - \Lambda^{-+}_{23}X^-_{37}X^+_{75}X^-_{56}X^+_{62} - \Lambda^{++}_{23}X^-_{37}X^-_{75}X^-_{56}X^+_{62} + \Lambda^{--}_{32}X^+_{21}X^+_{13} + \Lambda^{+-}_{32}X^+_{21}X^-_{13} + X^-_{21}\Lambda^-_{15}X^+_{56}X^+_{62} \\
& - X^-_{21}\Lambda^+_{15}X^+_{56}X^-_{62} + X^+_{24}X^-_{43}\Lambda^{-+}_{32} - X^-_{24}X^-_{43}\Lambda^{++}_{32} + \Lambda^{--}_{23}X^-_{37}X^+_{75}X^+_{56}X^+_{62} + \Lambda^{+-}_{23}X^-_{37}X^-_{75}X^+_{56}X^+_{62} - \Lambda^{--}_{23}X^+_{37}X^+_{75}X^+_{56}X^-_{62} \\
& - \Lambda^{+-}_{23}X^+_{37}X^-_{75}X^+_{56}X^-_{62} + \Lambda^{-+}_{32}X^-_{21}X^+_{13} + \Lambda^{++}_{32}X^-_{21}X^-_{13} + \Lambda^+_{78}X^+_{84}X^-_{43}X^-_{37} - \Lambda^+_{78}X^-_{84}X^-_{43}X^+_{37} + \Lambda^-_{87}X^+_{75}X^+_{56}X^-_{68} - \Lambda^-_{87}X^-_{75}X^+_{56}X^+_{68} \\
& + \Lambda^-_{78}X^-_{84}X^+_{43}X^+_{37} - \Lambda^-_{78}X^+_{84}X^+_{43}X^-_{37} + \Lambda^+_{87}X^+_{75}X^-_{56}X^-_{68} - \Lambda^+_{87}X^-_{75}X^-_{56}X^+_{68} + \Lambda^{++}_{64}X^+_{43}X^-_{37}X^-_{75}X^-_{56} - \Lambda^{++}_{64}X^-_{43}X^-_{37}X^+_{75}X^+_{56} \\
& + \Lambda^{--}_{46}X^+_{62}X^+_{24} - \Lambda^{--}_{46}X^+_{68}X^+_{84} + X^-_{37}X^+_{75}\Lambda^+_{51}X^+_{13} + \Lambda^{--}_{64}X^+_{43}X^+_{37}X^-_{75}X^-_{56} - \Lambda^{--}_{64}X^-_{43}X^+_{37}X^+_{75}X^+_{56} + \Lambda^{++}_{46}X^-_{62}X^-_{24} - \Lambda^{++}_{46}X^-_{68}X^-_{84} \\
& - X^+_{37}X^-_{75}\Lambda^-_{51}X^-_{13} + \Lambda^{+-}_{64}X^-_{43}X^-_{37}X^-_{75}X^+_{56} - \Lambda^{+-}_{64}X^+_{43}X^-_{37}X^+_{75}X^-_{56} + \Lambda^{-+}_{46}X^+_{62}X^-_{24} - \Lambda^{-+}_{46}X^-_{68}X^+_{84} \\
& - X^-_{37}X^-_{75}\Lambda^+_{51}X^-_{13} + \Lambda^{-+}_{64}X^-_{43}X^+_{37}X^-_{75}X^+_{56} - \Lambda^{-+}_{64}X^+_{43}X^+_{37}X^+_{75}X^-_{56} + \Lambda^{+-}_{46}X^-_{62}X^+_{24} - \Lambda^{+-}_{46}X^+_{68}X^-_{84} + X^+_{37}X^+_{75}\Lambda^-_{51}X^+_{13}
\end{aligned}
$$

Recent Extensions of Seiberg Duality by Physicists

Franco-Lee-Seong-Vafa (2016) then developed an ($N = 1$) 0-dimensional super-symmetric quiver gauge theory and a mutation known as Quadrality.



Fermis and **Chirals** are both directed arrows in this case.

Notice the new Fermi from $N_3 \to N_1$ after the initial Quadrality.

Question: Mathematical Model for Mutations and associated Relations?

Path Algebra (Example for A_n Quivers)

The A_n quiver Q is $n \longleftarrow n-1 \longleftarrow \ldots \longleftarrow 2 \longleftarrow 1$.

The path algebra kQ has elements given by the paths
p_{ji} : $j \longleftarrow j-1 \longleftarrow \ldots \longleftarrow i+1 \longleftarrow i$ for $1 \leq i < j \leq n$,

and the idempotents e_i. Note $p_{ji} \cdot p_{\ell k} = \begin{cases} p_{jk} \text{ if } i = \ell \\ 0 \text{ otherwise} \end{cases}$.

As an algebra,

$$kQ \cong \{ \text{ lower triangular } n \times n \text{ matrices over } k \}.$$

p_{ji} corresponds to E_{ji} which has a 1 in column i, row j and 0 elsewhere.

e_i corresponds to E_{ii}.

Path Algebra (Example for A_2 Quiver)

The A_2 quiver Q is $2 \leftarrow 1$ with path algebra kQ given by

$$\{e_1, e_2, p_{21} : e_2 \cdot p_{21} = p_{21},\ p_{21} \cdot e_1 = p_{21},\ e_1^2 = e_1,\ e_2^2 = e_2\}$$

with all other products equal to zero.

Under the isomorphism with lower triangular 2×2 matrices,

$$e_1 \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_2 \leftrightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and } p_{21} \leftrightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

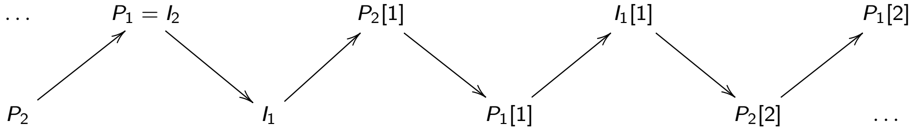
From Path Algebras to Cluster Categories (Acyclic Case)

The **bounded derived category** $\mathcal{D}^b(kQ)$ has indecomposable objects of the form $M[i]$ (M indecomposable of kQ and $i \in \mathbb{Z}$ with shift functor [1]).

Example (A_2 Quiver): $2 \leftarrow 1$ admits three indecomposable modules

$$P_1 = \langle e_1, p_{21} \rangle = I_2, \quad P_2 = \langle e_2 \rangle, \quad I_1 = \langle e_1 \rangle.$$

The indecomposables of $\mathcal{D}^b(k(2 \leftarrow 1))$ can be arranged as



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$\mathcal{D}^b(kQ)$ is also a **triangulated category** meaning there are certain distinguished short exact sequences $0 \to A \to B \to C \to 0$ known as **almost split sequences**. (Correspond to triangles $A \to B \to C \to A[1]$)

An almost split exact sequence is **not split**, i.e. $B \ncong A \oplus C$ be is irreducible (i.e. as close to being split without being split).

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The **Auslander-Reiten translation** τC of indecomposable C is the unique indecomposable such that $0 \to \tau C \to \_ \to C \to 0$ is almost split.

Cluster Categories (Acyclic Case)

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τ has the property that it sends projective indecomposable objects to **zero** and otherwise sends non-projective indecomposables to indecomposables.)

Def. (Buan-Marsh-Reineke-Reiten-Todorov 2004): The **Cluster Category** $\mathcal{C}_1(kQ)$ is defined as $\mathcal{D}^b(kQ)/(\tau^{-1} \circ [1])$ where τ is Auslander-Reiten translation and [1] is the shift functor .

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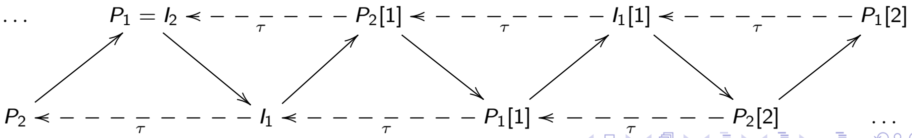

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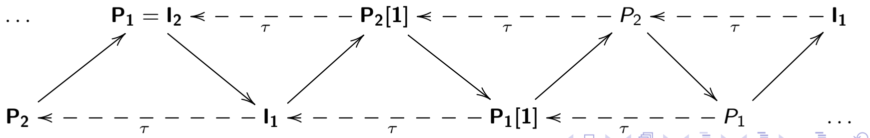

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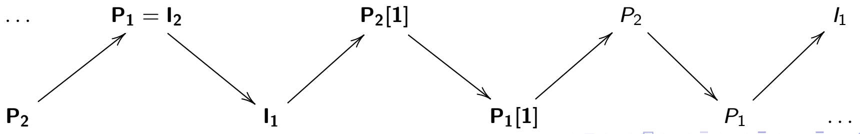

Tilting Objects in a Cluster Category

Given an acyclic quiver Q and the associated cluster algebra $\mathcal{A}(Q)$, then clusters correspond to **Tilting Objects** in the Cluster Category $\mathcal{C}_1(kQ)$.

Tilting Objects $T = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ satisfy

1) $Ext(M_i, M_j) = Ext(M_j, M_j) = 0$ for $i \neq j$ and $Ext(M_i, M_i) = 0$ for all i.

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Letting $\overline{T} = T \setminus M_j$, there is a unique $M_j' \ncong M_j$ such that

$$M_j \to \oplus_i B_i^{(0)} \to M_j' \to M_j[1]$$

$$M_j' \to \oplus_i B_i^{(1)} \to M_j \to M_j'[1]$$

are distinguished triangles (analogues of almost split sequences) in $\mathcal{C}_1(kQ)$.

Corresponds to cluster mutation as $x_j x_j' = \prod_i x_{B_i^{(1)}} + \prod_k x_{B_i^{(0)}}$.

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Tilting Object $M_1 \oplus M_2$ satisfies $Hom(M_i, M_j[1]) = 0$ for $i, j \in \{1, 2\}$.

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$$\begin{array}{ccccc}
\{x_1, x_2\} & \xrightarrow{\mu_1} & \{x_3, x_2\} & \xrightarrow{\mu_2} & \{x_3, x_4\} \\
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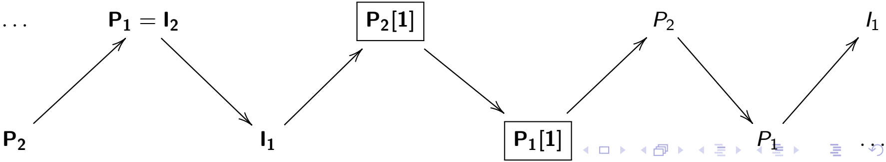
Compared with $\mathcal{C}_1(k(2 \leftarrow 1))$, we have $\{x_1, x_2\} \longleftrightarrow P_1[1] \oplus P_2[1]$.



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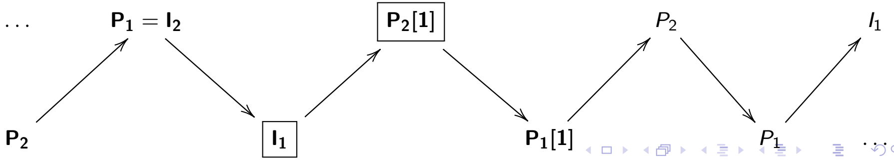

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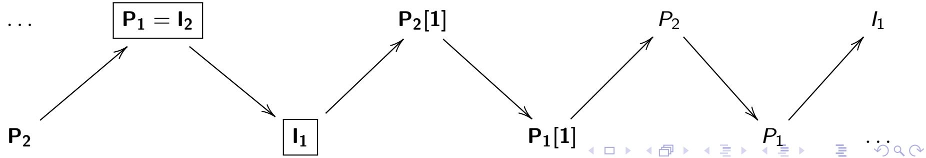

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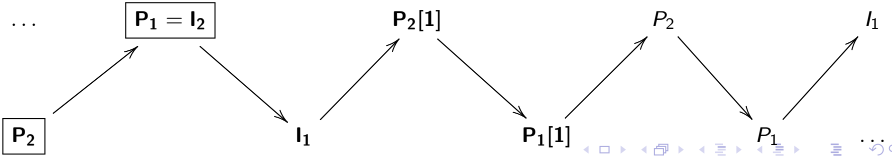

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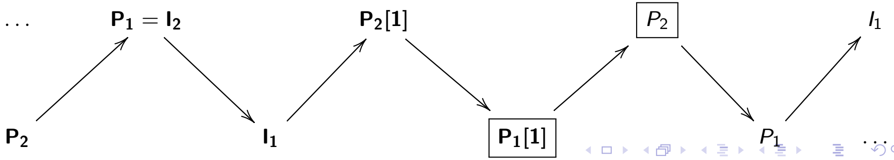

Cluster Algebra and Cluster Category of Type A_2

Observe we have the correpondence

$$x_1 \longleftrightarrow P_1[1]$$

$$x_2 \longleftrightarrow P_2[1]$$

$$x_3 = \frac{x_2 + 1}{x_1} \longleftrightarrow I_1 = \langle e_1 \rangle$$

$$x_4 = \frac{x_1 + x_2 + 1}{x_1 x_2} \longleftrightarrow P_1 = \langle e_1, p_{21} \rangle$$

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There is a general map (Caledro-Chapton's Cluster Character) from rigid indecomposable modules of $\mathcal{C}_1(kQ)$ to cluster variables by $M \to x_M$.

Reading off the Quiver from a Tilting Object

Given a tilting module $T = M_1 \oplus M_2 \oplus \cdots \oplus M_n$, we build the quiver Q_T by starting with n disconnected vertices.

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then we adjoin b_{ij} copies of the arrow $i \to j$ to our quiver Q_T.

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Similarly, if there exists the distinguished triangle

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then we then adjoin b_{jk} copies of the arrow $j \to k$ to Q_T.

Cluster variable mutation of Q_T, i.e. $x_j x_j' = \prod_{i \to j \in Q_T} x_i^{b_{ij}} + \prod_{j \to k \in Q_T} x_k^{b_{kj}}$ agrees with the relation $x_j x_j' = \prod_i x_{B_i^{(1)}} + \prod_i x_{B_i^{(0)}}$ coming from the distinguished triangles

$$M_j' \to \oplus_i B_i^{(1)} \to M_j \to M_j'[1] \quad \text{and} \quad M_j \to \oplus_i B_i^{(0)} \to M_j' \to M_j[1].$$

Higher Cluster Categories/Colored Quivers (Buan-Thomas)

(Thomas 2006) generalizes the cluster category $\mathcal{C}_1(kQ) = \mathcal{D}^b(kQ)/\tau^{-1}[1]$:

Given an acyclic quiver Q, define the m-**Cluster Category** as the quotient category

$$\mathcal{C}_m(kQ) = \mathcal{D}^b(kQ)/\tau^{-1}[m].$$

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Indecomposable Objects of $\mathcal{C}_m(kQ)$ are

$$\left\{M : M \text{ indec.}\right\} \cup \left\{M[1] : M \text{ indec.}\right\} \cup \cdots \cup \left\{M[m-1] : M \text{ indec.}\right\}$$

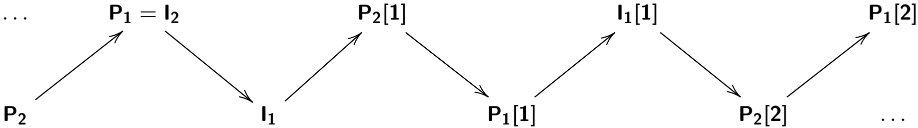
$$\cup \left\{P_1[m], P_2[m], \ldots, P_n[m]\right\}$$

where $P_1, P_2, \ldots, P_n$ are the projective indecomposables of kQ.

(n is the number of vertices of Q)

Higher Cluster Categories/Colored Quivers (Buan-Thomas)

Example (A_2 quiver): The 2-cluster category $\mathcal{C}_2(k(2 \leftarrow 1))$ has indecomposables



where we get periodicity with $P_1[2] \to P_2$.

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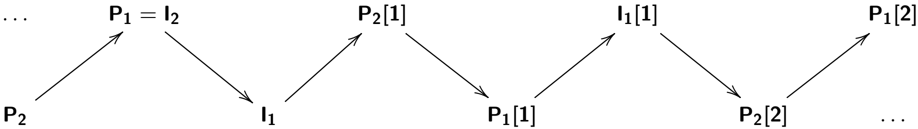

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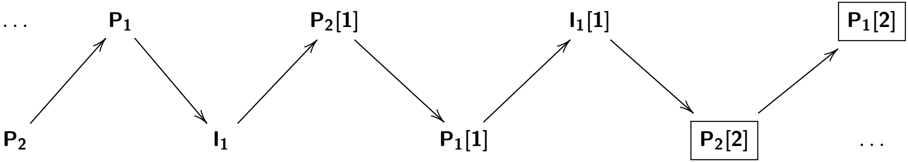
There are 12 higher tilting objects in $\mathcal{C}_2(k(2 \leftarrow 1))$: $P_1[2] \oplus P_2[2]$



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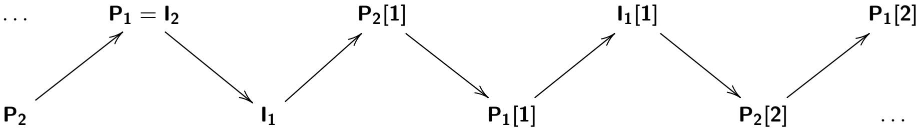

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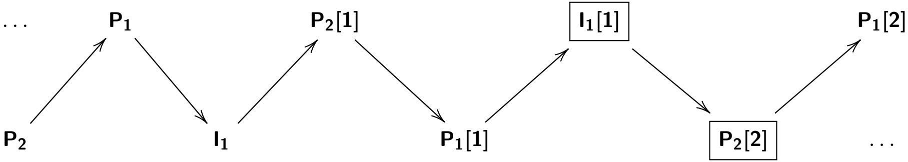

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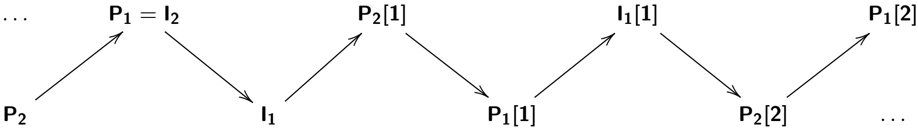

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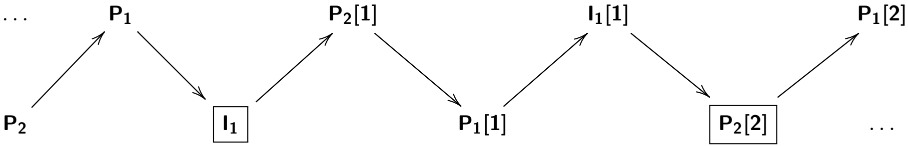

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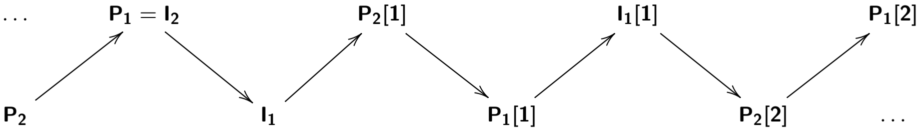

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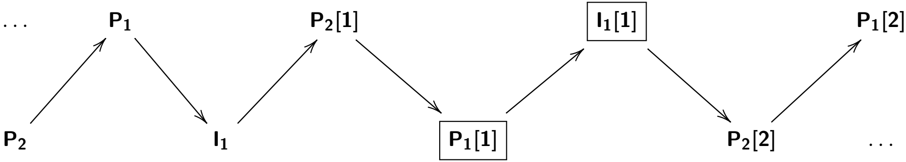

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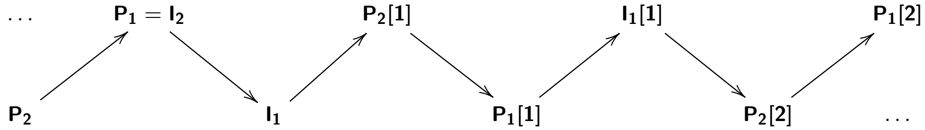

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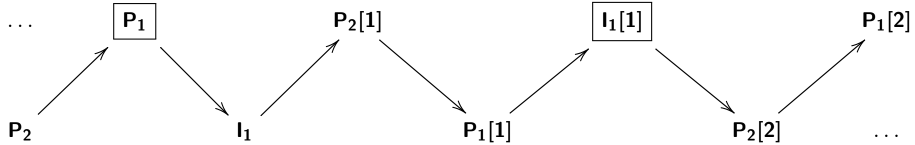

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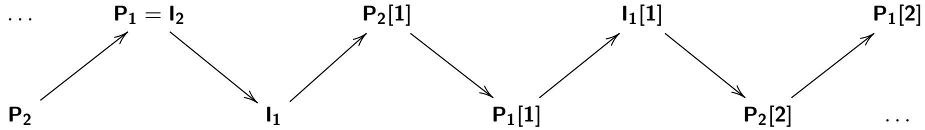

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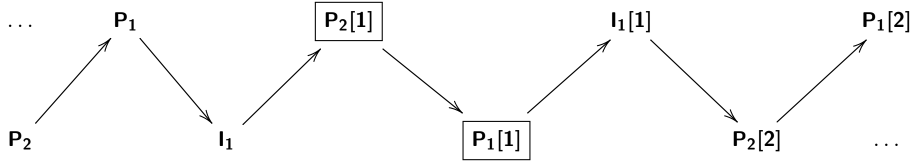

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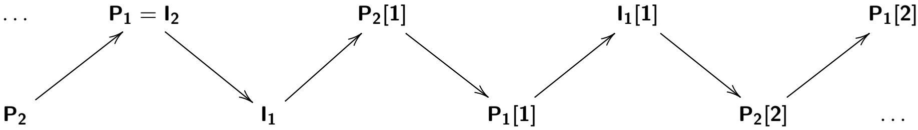

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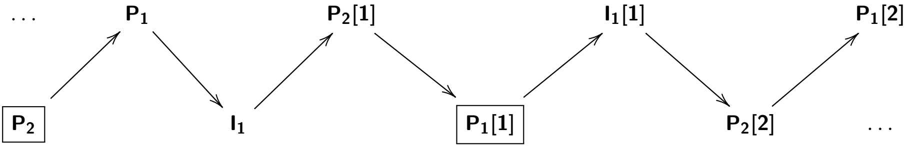

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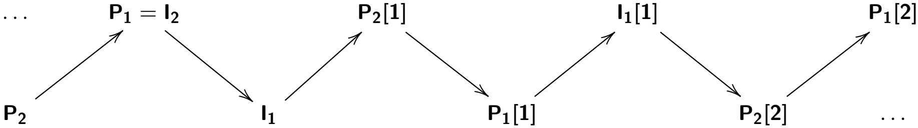

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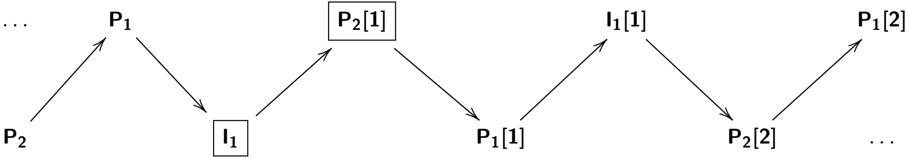

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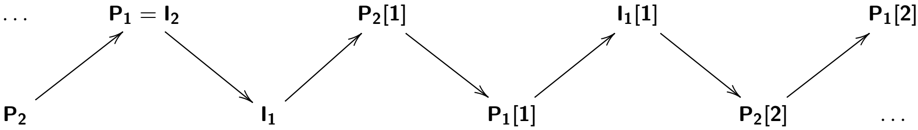

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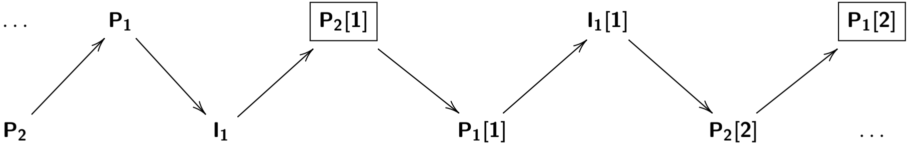

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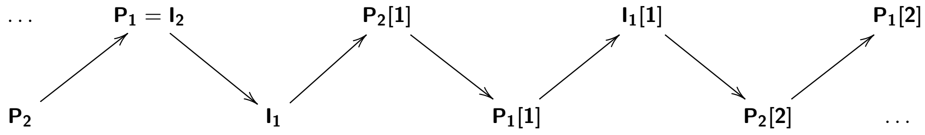

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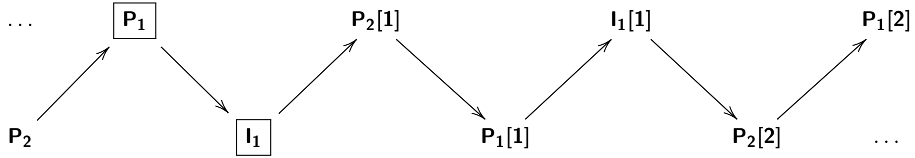

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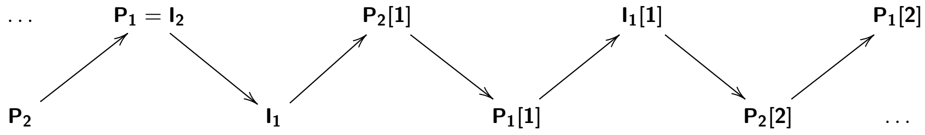

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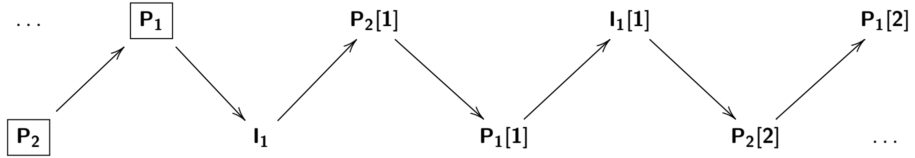

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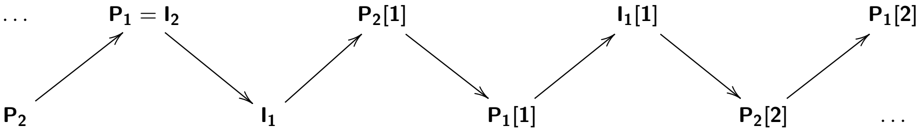

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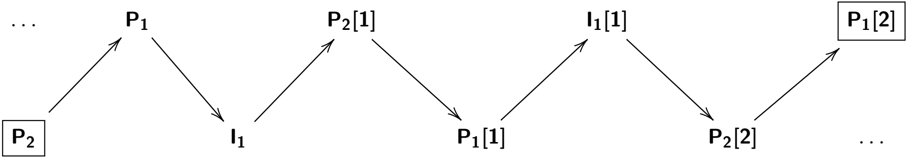

Higher Cluster Categories/Colored Quivers (Buan-Thomas)

We can get a Colored Quiver from a Higher Tilting Object T since there are $(m+1)$ ways to complete $\overline{T} = T \setminus M_j$ in $\mathcal{C}_m(kQ) = \mathcal{D}^b(kQ)/\tau^{-1}[m]$:

$$T = T^{(0)} = \overline{T} \oplus M_j^{(0)}, \quad T^{(1)} = \overline{T} \oplus M_j^{(1)}, \quad T^{(2)} = \overline{T} \oplus M_j^{(2)}, \ \ldots\ , \quad T^{(m)} = \overline{T} \oplus M_j^{(m)}.$$

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These fit together in distinguished triangles (using $M_j^{(m+1)} = M_j^{(0)} = M_j$)

$$\begin{aligned}
M_j &\to \oplus_i B_i^{(0)} \to M_j^{(1)} \to M_j[1] \\
M_j^{(1)} &\to \oplus_i B_i^{(1)} \to M_j^{(2)} \to M_j^{(1)}[1] \\
M_j^{(2)} &\to \oplus_i B_i^{(2)} \to M_j^{(3)} \to M_j^{(2)}[1] \\
&\vdots \\
M_j^{(m-1)} &\to \oplus_i B_i^{(m-1)} \to M_j^{(m)} \to M_j^{(m-1)}[1] \\
M_j^{(m)} &\to \oplus_i B_i^{(m)} \to M_j \to M_j^{(m)}[1]
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Higher Cluster Categories/Colored Quivers (Buan-Thomas)

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M_j^{(m)} &\rightarrow \oplus_i B_i^{(m)} \rightarrow M_j \rightarrow M_j^{(m)}[1]
\end{aligned}$$

Notice that in the special case $m = 1$, we let $M_j^{(1)} = M_j'$ and we get

$$M_j' \rightarrow \oplus_i B_i^{(1)} \rightarrow M_j \rightarrow M_j'[1] \quad \text{and} \quad M_j \rightarrow \oplus_i B_i^{(0)} \rightarrow M_j' \rightarrow M_j[1]$$

as desired.

Higher Cluster Categories/Colored Quivers (Buan-Thomas)

$$M_j \rightarrow \oplus_i B_i^{(0)} \rightarrow M_j^{(1)} \rightarrow M_j[1]$$
$$M_j^{(1)} \rightarrow \oplus_i B_i^{(1)} \rightarrow M_j^{(2)} \rightarrow M_j^{(1)}[1]$$
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$$M_j^{(m)} \rightarrow \oplus_i B_i^{(m)} \rightarrow M_j \rightarrow M_j^{(m)}[1]$$

We build a colored quiver Q_T from T a tilting object of $\mathcal{C}_m(kQ)$ by adjoining $b_{ij}^{(c)}$ **colored arrows** $i \xleftarrow[(c)]{} j$ for every summand M_i in $\oplus_i B_i^{(c)}$.

Higher Cluster Categories/Colored Quivers (Buan-Thomas)

$$\begin{aligned}
M_j &\to \oplus_i B_i^{(0)} \to M_j^{(1)} \to M_j[1] \\
M_j^{(1)} &\to \oplus_i B_i^{(1)} \to M_j^{(2)} \to M_j^{(1)}[1] \\
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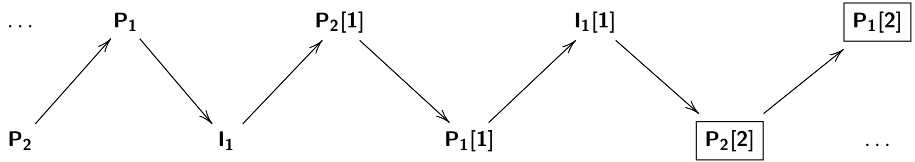
M_j^{(m)} &\to \oplus_i B_i^{(m)} \to M_j \to M_j^{(m)}[1]
\end{aligned}$$

Because we can build the same tower of distinguished triangles using $M_i^{(c)}$'s in place of $M_j^{(c)}$, it follows that

colored arrows come in pairs $i \underset{(m-c)}{\overset{(c)}{\rightleftarrows}} j$. (For $m = 1$, $i \to j = i \underset{(1)}{\overset{(0)}{\rightleftarrows}} j$.)

Example of $\mathcal{C}_2(k(2 \leftarrow 1))$



$$P_1[2] \oplus P_2[2] \quad \longleftrightarrow \quad 2 \underset{(0)}{\overset{(2)}{\rightleftarrows}} 1 \; = \; 2 \longleftarrow 1$$

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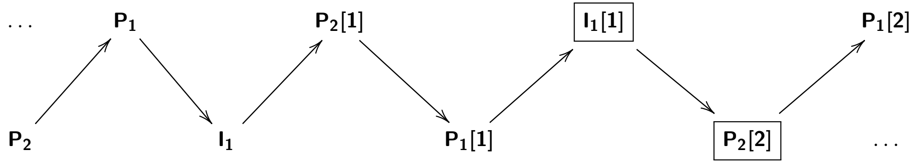

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Mutating by μ_1 yields

$$I_1[1] \oplus P_2[2] \quad \longleftrightarrow \quad 2 \underset{(2)}{\overset{(0)}{\rightleftarrows}} 1 \; = \; 2 \longrightarrow 1\,.$$

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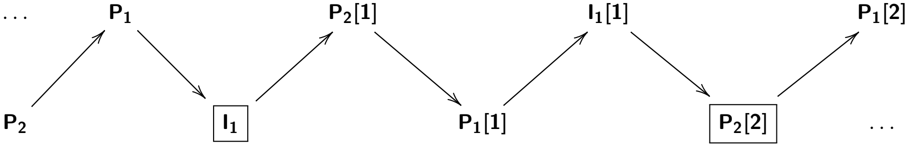

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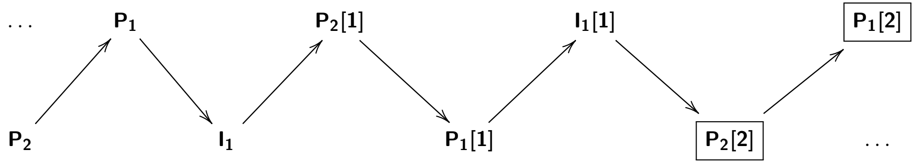
And mutating by μ_1 a second time in a row yields

$$I_1 \oplus P_2[2] \quad \longleftrightarrow \quad 2 \underset{(1)}{\overset{(1)}{\rightleftarrows}} 1 \;=\; 2 - - - 1\,.$$

Example of $\mathcal{C}_2(k(2 \leftarrow 1))$

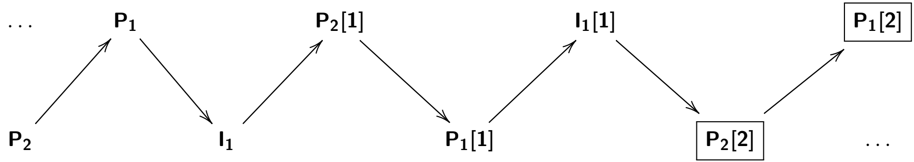


Mutating a third time in a row by μ_1 yields again

$$P_1[2] \oplus P_2[2] \quad \longleftrightarrow \quad 2 \underset{(0)}{\overset{(2)}{\rightleftarrows}} 1 \;=\; 2 \longleftarrow 1\,.$$

i.e. $\mu_i^3 = 1$.

Example of $\mathcal{C}_2(k(2 \leftarrow 1))$



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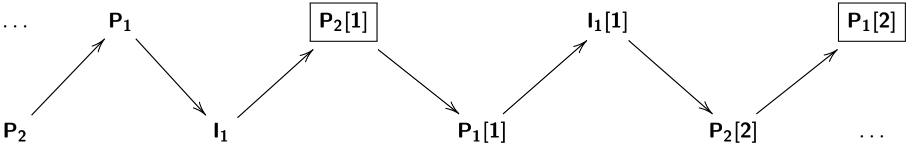
$$P_1[2] \oplus P_2[2] \quad \longleftrightarrow \quad 2 \underset{(0)}{\overset{(2)}{\rightleftarrows}} 1 \;=\; 2 \longleftarrow 1\,.$$

i.e. $\mu_i^3 = 1$. (In general $\mu_i^{m+1} = 1$, which agrees with $\mu^2 = 1$ for the ordinary $m = 1$ case.)

Example of $\mathcal{C}_2(k(2 \leftarrow 1))$



Notice, on the other hand mutating $P_1[2] \oplus P_2[2]$ by μ_2 yields

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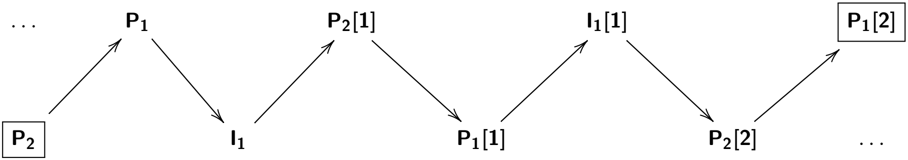

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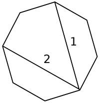
And a second mutation by μ_2 in a row yields

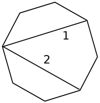
$$P_1[2] \oplus P_2 \quad \longleftrightarrow \quad 2 \underset{(2)}{\overset{(0)}{\rightleftarrows}} 1 \;=\; 2 \longrightarrow 1\,.$$

Example of $\mathcal{C}_2(k(2 \leftarrow 1))$

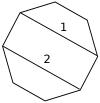
Such higher tilting objects can also be associated to quadrangulations (more generally $(m+2)$-angulations) of a polygon (in the type A_n case).



$$P_1[2] \oplus P_2[2] \quad \longleftrightarrow \quad 2 \underset{(0)}{\overset{(2)}{\rightleftarrows}} 1 \;=\; 2 \longleftarrow 1$$



$$I_1[1] \oplus P_2[2] \quad \longleftrightarrow \quad 2 \underset{(2)}{\overset{(0)}{\rightleftarrows}} 1 \;=\; 2 \longrightarrow 1$$



$$I_1 \oplus P_2[2] \quad \longleftrightarrow \quad 2 \underset{(1)}{\overset{(1)}{\rightleftarrows}} 1 \;=\; 2 - - - 1$$

Colored Quiver Mutation (Buan-Thomas 2008)

A **Colored Quiver** $Q = (Q_0, Q_1) = (Q_0, Q_1^{(0)} \oplus Q_1^{(1)} \oplus Q_1^{(2)} \oplus \cdots \oplus Q_1^{(m)})$ is a collection of vertices and arrows where arrows can have one of $(m+1)$ colors, which we label as $(0), (1), \ldots, (m)$, satisfying three properties:

(i) No loops: There are no arrows which have $i \in Q_0$ as both its starting and ending point.

(ii) Monochromaticity: If there is an arrow $i \xleftarrow{(c)} j$ of color (c) between vertices $i, j \in Q_0$, then there are no arrows $i \xleftarrow{(c')} j$ of any other color (c'), although multiple arrows of the same color are possible.

(iii) Skew-symmetry: If there are $q_{ij}^{(c)}$ arrows $i \xleftarrow{(c)} j$ of color (c), then there are also $q_{ij}^{(c)}$ arrows $i \xrightarrow{(m-c)} j$ of color $(m-c)$.

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Buan-Thomas not only define Colored Quivers, but define a dynamic on them called Colored Quiver Mutation (at vertex j):

Step 1a: Replace every incoming arrow $j \xleftarrow{(c)} i$ with the arrow $j \xleftarrow{(c-1)} i$.

Step 1b: Replace every outgoing arrow $k \xleftarrow{(c)} j$ with an arrow $k \xleftarrow{(c+1)} j$.

Both of these values are taken **modulo** $(m+1)$. As special cases,

$j \underset{(m)}{\overset{(0)}{\leftrightarrows}} i$ mutates to $j \underset{(0)}{\overset{(m)}{\leftrightarrows}} i$ and $k \underset{(1)}{\overset{(m-1)}{\leftrightarrows}} j$ mutates to $k \underset{(0)}{\overset{(m)}{\leftrightarrows}} j$.

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Step 3: Delete two arrows of colors $i \underset{(c)}{\overset{(c-1)}{\leftleftarrows}} k$ as a pair until monochromaticity is achieved again. **(Massive terms)**

Colored Quiver Mutation is **of order** $(m+1)$.

Colored Quiver Mutation Motivation (Buan-Thomas 2008)

Colored Quivers and their mutations motivated by the the Higher Cluster Category, the triangulated $(m+1)$-Calabi-Yau category obtained by the quotient $\mathcal{D}^b(kQ)/(\tau^{-1} \circ [m])$

Recall the Higher Tilting Objects are maximally dimensional direct sums of indecomposables which have no self-extensions.

Example of $\mathcal{C}_2(k(2 \leftarrow 1))$ from above:

$$P_1[2] \oplus P_2[2] \quad \longleftrightarrow \quad 2 \underset{(0)}{\overset{(2)}{\rightleftarrows}} 1 \;=\; 2 \longleftarrow 1$$

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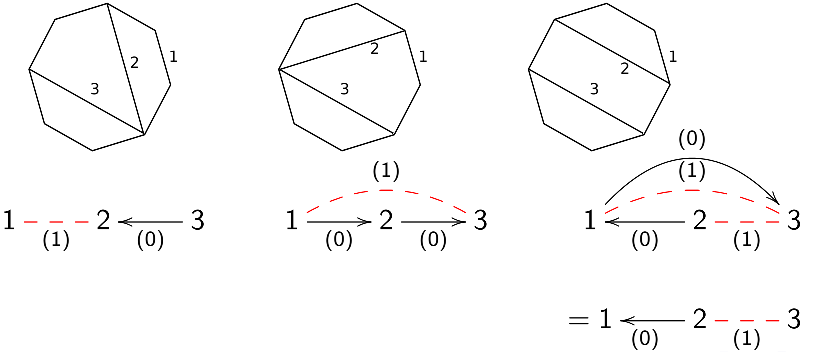
Colored Quiver Mutation Motivation (Buan-Thomas 2008)

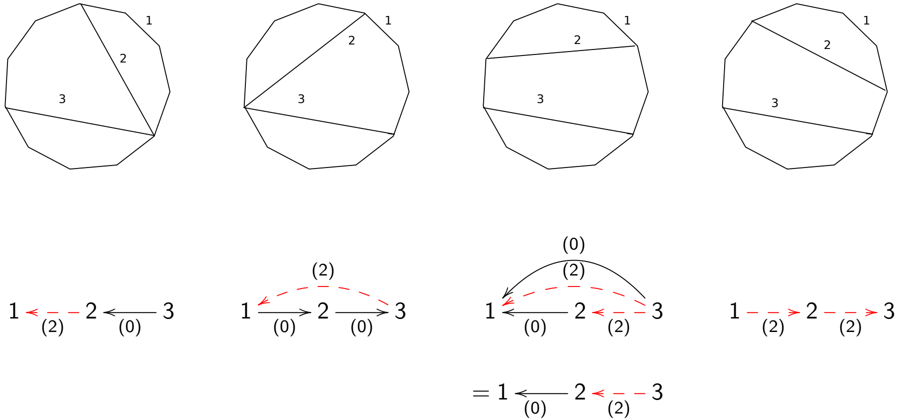
When Q is of type A, the Higher Cluster-Tilting Objects in bijection with $(m+2)$-angulations of polygons. (Draw a colored arrow for number of sides between labeled diagonals counter-clockwise.)

Example (mutating at vertex 2 in $m = 2$ case). We omit arrows of color (2) since $i \xrightarrow{(c)} j = i \xleftarrow{(m-c)} j$.



Colored Quiver Mutation Motivation (Buan-Thomas 2008)

Example (mutating at vertex 2 in $m = 3$ case). We omit arrows of color (1), (3) and set $i \xrightarrow{(c)} j = i \xleftarrow{(m-c)} j$.



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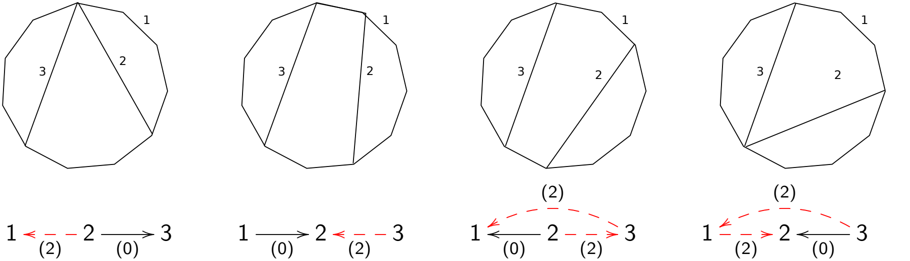
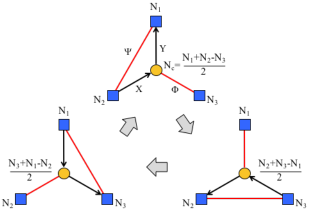


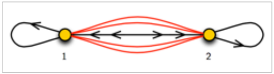
Fourth mutation at vertex 2 yields arrow $1 \overset{(2)}{\dashrightarrow} 3$ that cancels with $1 \overset{(2)}{\dashleftarrow} 3$ and we obtain the first colored quiver again.

Also relates to the Generalized Associahedra of Fomin-Reading (2006).

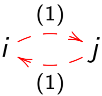
Triality corresponds to $m = 2$ colored quiver mutation.

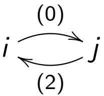


✧ Conifold $\times\mathbb{C}$



	J	E
Λ^1_{12} :	$X_{21}\cdot X_{12}\cdot Y_{21} - Y_{21}\cdot X_{12}\cdot X_{21} = 0$	$\Phi_{11}\cdot Y_{12} - Y_{12}\cdot \Phi_{22} = 0$
Λ^1_{21} :	$X_{12}\cdot Y_{21}\cdot Y_{12} - Y_{12}\cdot Y_{21}\cdot X_{12} = 0$	$\Phi_{22}\cdot X_{21} - X_{21}\cdot \Phi_{11} = 0$
Λ^2_{12} :	$Y_{21}\cdot Y_{12}\cdot X_{21} - X_{21}\cdot Y_{12}\cdot Y_{21} = 0$	$\Phi_{11}\cdot X_{12} - X_{12}\cdot \Phi_{22} = 0$
Λ^2_{21} :	$Y_{12}\cdot X_{21}\cdot X_{12} - X_{12}\cdot X_{21}\cdot Y_{12} = 0$	$\Phi_{22}\cdot Y_{21} - Y_{21}\cdot \Phi_{11} = 0$

Draw Fermis as 

and **Chirals** as 

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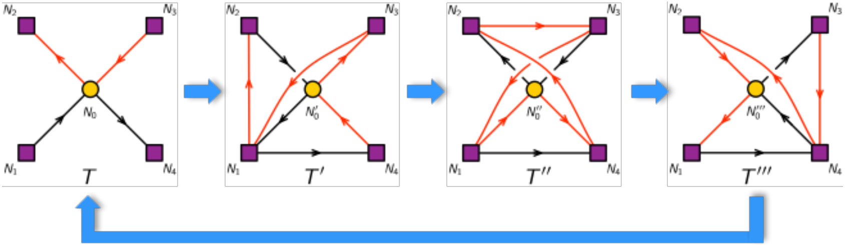
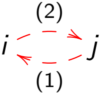
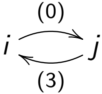
Now Allowed: Loops and Arcs of Different Colors Between Two Vertices.

We wish to deduce J-term and E-term Relations from Potentials for Colored Quivers.

Quadrality corresponds to $m = 3$ colored quiver mutation.



Draw Directed Fermis as 

and **Chirals** as 

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Now Allowed: Loops and Arcs of Different Colors Between Two Vertices.

We wish to deduce J-term Relations from Potentials for Colored Quivers.

Physics also has H-term Relations. What is the Mathematics behind them?

(Franco-M 2017) Potentials for Colored Quivers

Based on the examples of $(m+2)$-angulations and brane bricks, we constructed a combinatorial theory of potentials for colored quivers.

(Franco-M 2017) Potentials for Colored Quivers

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We define a potential W for a colored quiver Q to be a linear combination of terms $\alpha_1\alpha_2\cdots\alpha_k$ of the path algebra, each satisfying

(1) The starting point of α_{i+1} is the ending point of α_i for $1 \leq i \leq k-1$; also the starting point of α_1 is the ending point of α_k.

(2) Letting $c_i \in \{0, 1, 2, \ldots, m\}$ be the color of arrow α_i, we have

$$c_1 + c_2 + \cdots + c_k = m - 1.$$

Theorem: There are simple combinatorial rules so that mutation of potentials is compatible with assignment of potentials to a brane brick model or to an $(m+2)$-angulation of a polygon.

(Franco-M 2017) Potentials for Colored Quivers

We define Mutation of Colored Quivers with Potential (at vertex j).

Step 1: For every incoming arrow $\alpha_{ij}^{(c)} = \ i \xrightarrow{(c)} j$ (resp. outgoing arrow $\alpha_{jk}^{(c)} = \ j \xrightarrow{(c)} k$), replace it with $\alpha_{ij}^{(c-1)} = \ i \xrightarrow{(c-1)} j$ (resp. $\alpha_{jk}^{(c-1)} = \ j \xrightarrow{(c-1)} k$) in Q. Values taken in $\{0, 1, 2, \ldots, m\}$ mod $(m+1)$.

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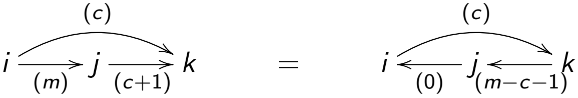
Step 2a: For every 2-path, $i \xrightarrow{(0)} j \xrightarrow{(c)} k$ in Q, where the color of the outgoing arrow is (0), add the new arrow $i \xrightarrow{(c)} k$ in Q and the new degree 3 term $\alpha_{ik}^{(c)} \alpha_{ij}^{(m)} \alpha_{jk}^{(c+1)} = \alpha_{ik}^{(c)} \alpha_{kj}^{(m-c-1)} \alpha_{ji}^{(0)}$ to W.



(Franco-M 2017) Potentials for Colored Quivers

We define Mutation of Colored Quivers with Potential (at vertex j).

Step 2b: Replace instances of $\alpha_{ij}^{(0)}\alpha_{jk}^{(c)}$ in W with $\alpha_{ik}^{(c)}$.

Step 2c: Replace instances of $\alpha_{ij}^{(c)}\alpha_{jk}^{(d)}$ in W with $\alpha_{ij}^{(c-1)}\alpha_{jk}^{(d+1)}$ when $c \neq 0$.

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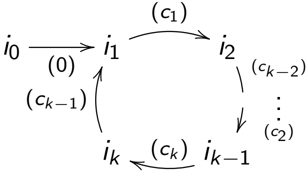
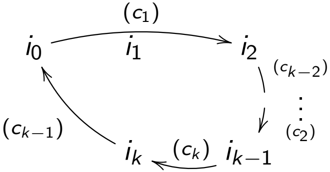
Step 2d: For a local configuration 

where

$\alpha_{i_1,i_2}^{(c_1)} \cdots \alpha_{i_{k-1},i_k}^{(c_{k-1})}\alpha_{i_k,i_1}^{(c_k)}$ is in W, then add a new term to the potential

$\alpha_{i_0,i_2}^{(c_1)} \cdots \alpha_{i_{k-1},i_k}^{(c_{k-1})}\alpha_{i_k,i_0}^{(c_k)}$ 

replacing i_1 with i_0.

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We define Mutation of Colored Quivers with Potential (at vertex j).

Step 3: Apply reductions of massive terms to get an equivalent colored quiver with potential. (Generically, delete massive terms as well as terms sharing an arrow with massive term. In special cases, more complicated.)

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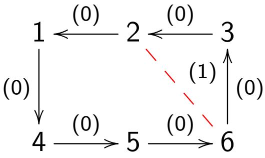
Example (m=2): We omit arrows of color (2) and set $\alpha_{ij}^{(c)} = \alpha_{ji}^{(m-c)}$.



$$W = X_{21}^{(0)} X_{14}^{(0)} X_{45}^{(0)} X_{56}^{(0)} \Lambda_{62}^{(1)} + \Lambda_{26}^{(1)} X_{63}^{(0)} X_{32}^{(0)}.$$

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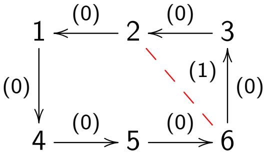

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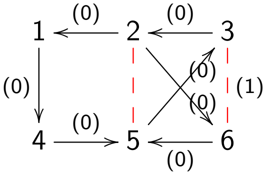
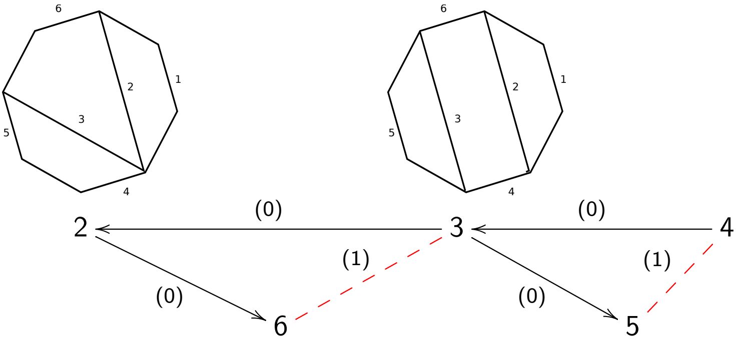
Mutating at vertex 6 via Rules (2a), (2b), (2c), (2d) yields



$$W' = X_{21}^{(0)} X_{14}^{(0)} X_{45}^{(0)} \Lambda_{52}^{(1)} + X_{26}^{(0)} \Lambda_{63}^{(1)} X_{32}^{(0)}$$

$$+ \Lambda_{52}^{(1)} X_{26}^{(0)} X_{65}^{(0)} + X_{53}^{(0)} \Lambda_{36}^{(1)} X_{65}^{(0)} + \Lambda_{25}^{(1)} X_{53}^{(0)} X_{32}^{(0)}$$

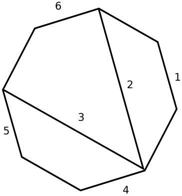
Example (m=2): Mutating at Vertex 3 and want new potential to match new quadrangulation

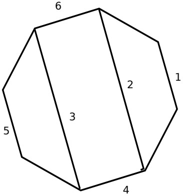


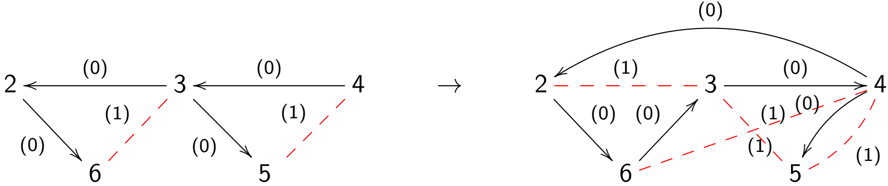
$$W = \Lambda_{63}^{(1)} X_{32}^{(0)} X_{26}^{(0)} \qquad + \Lambda_{54}^{(1)} X_{43}^{(0)} X_{35}^{(0)}$$

(Franco-M 2017) Potentials for Colored Quivers

Example (m=2): Mutating at Vertex 3 via Rules (2a), (2b), (2c), (2d)







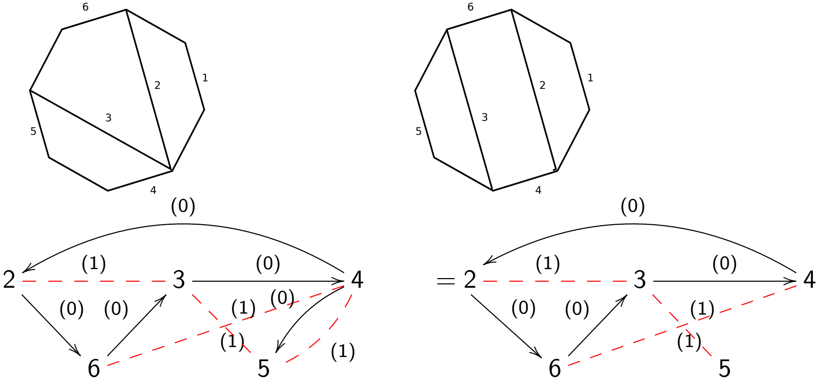
$$W = \Lambda^{(1)}_{63} X^{(0)}_{32} X^{(0)}_{26} + \Lambda^{(1)}_{54} X^{(0)}_{43} X^{(0)}_{35}$$

$$W' = X^{(0)}_{63} \Lambda^{(1)}_{32} X^{(0)}_{26} + \Lambda^{(1)}_{54} X^{(0)}_{45} + \Lambda^{(1)}_{64} X^{(0)}_{42} X^{(0)}_{26}$$

$$+ X^{(0)}_{45} \Lambda^{(1)}_{53} X^{(0)}_{34} + \Lambda^{(1)}_{46} X^{(0)}_{63} X^{(0)}_{34} + X^{(0)}_{42} \Lambda^{(1)}_{23} X^{(0)}_{34}$$

(Franco-M 2017) Potentials for Colored Quivers

Example (m=2): Mutating at Vertex 3 and reducing massive terms



$$W' = X^{(0)}_{63}\Lambda^{(1)}_{32}X^{(0)}_{26} + \Lambda^{(1)}_{54}X^{(0)}_{45} + \Lambda^{(1)}_{64}X^{(0)}_{42}X^{(0)}_{26}$$

$$+ X^{(0)}_{45}\Lambda^{(1)}_{53}X^{(0)}_{34} + \Lambda^{(1)}_{46}X^{(0)}_{63}X^{(0)}_{34} + X^{(0)}_{42}\Lambda^{(1)}_{23}X^{(0)}_{34}$$

$$W'_{red} = X^{(0)}_{63}\Lambda^{(1)}_{32}X^{(0)}_{26} + \Lambda^{(1)}_{64}X^{(0)}_{42}X^{(0)}_{26} + \Lambda^{(1)}_{46}X^{(0)}_{63}X^{(0)}_{34} + X^{(0)}_{42}\Lambda^{(1)}_{23}X^{(0)}_{34}.$$

DG Structures (Ginzburg 2006, Van den Bergh 2015)

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$$d(\alpha) = \begin{cases} 0 \ \text{ if } \alpha \text{ has degree (i.e. color) } (0) \\ \partial_{\alpha^{op}} W \ \text{ if } \alpha \text{ has degree (i.e. color) } \in \{1, 2, 3, \ldots, m\} \\ e_i(\sum_\alpha [\alpha, \alpha_{op}])e_i \ \text{ if } \alpha = \ell_i, \text{ a loop of degree } (m+1) \end{cases} .$$

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For a potential W such that the Kontsevich bracket vanishes, i.e. $\{W, W\} = 0$, the Vacuum Moduli Space agrees with the **Jacobian algebra**, given as

$$k\overline{Q} \Big/ (\{\partial_\alpha W : \alpha \text{ of color } (m-1)\}) = H^0(\widehat{\Gamma}_{m+2}(Q, W)).$$

Mutations of DG Structures (Oppermann 2017)

In an effort to study "Quivers for Silting Mutation", Oppermann gives an algebraic description of how this differential graded structure mutates alongside the colored (a.k.a. graded) quiver mutation.

$$W \to W' = \mathrm{dec}_{\mathrm{cyc}} W + \sum_{\alpha\colon \xrightarrow{(0)} j \,,\ \varphi\colon j \xrightarrow{(c)}} \alpha \; \mathrm{dec}(\varphi\varphi^{op})\alpha^*.$$

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Such a compact expression is possible due to the functions **dec** and $\mathbf{dec_{cyc}}$, which are defined as an action on a cycle $\gamma = \varphi_{i_1,i_2}\varphi_{i_2,i_3}\cdots\varphi_{i_\ell,i_1}$ and then extended linearly to act on a potential.

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Every 2-path $\varphi_{i,j}\varphi_{j,k}$ in γ is replaced with $\left(\varphi_{i,j}\varphi_{j,k} - \sum_{\alpha \colon \xrightarrow{(0)} j} \varphi_{i,j}\alpha^{-1}\alpha\varphi_{j,k} \right)$. The result is $\mathrm{dec}(\gamma)$.

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Such a compact expression is possible due to the functions **dec** and **dec$_{\mathbf{cyc}}$**, which are defined as an action on a cycle $\gamma = \varphi_{i_1,i_2}\varphi_{i_2,i_3}\cdots\varphi_{i_\ell,i_1}$ and then extended linearly to act on a potential.

Every 2-path $\varphi_{i,j}\varphi_{j,k}$ in γ is replaced with

$$\left(\varphi_{i,j}\varphi_{j,k} - \sum_{\alpha:\ \xrightarrow{(0)} j} \varphi_{i,j}\alpha^{-1}\alpha\varphi_{j,k}\right).$$ The result is $\mathrm{dec}(\gamma)$.

For the case of $\mathrm{dec}_{\mathrm{cyc}}(\gamma)$, this operation is taken cyclically, i.e. we also consider 2-paths $\varphi_{i_\ell,i_1}\varphi_{i_1,i_2}$ where $i_1 = j$.

Open Questions and Work in Progress

Work in Progress (with Emily Gunawan and Ana Garcia Elsener): Assigning a potential to general $(m+2)$-angulations of a surface (in the spirt of Daniel Labardini-Fragoso for triangulations) so that this assignment rule is compatible with mutation and diagonal rotation in an $(m+2)$-gon. Can we better analyze certain gentle algebras and/or oriented flip graphs of Al Garver and Thomas McConville using the machinery of colored potentials?

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Big Open Question: Is there an analogue of cluster variables for brane bricks $(m = 2)$ or in the case of higher m. In the case of A_n colored quivers, such variables would correspond to admissible arcs in an $(m+2)$-gon. Or for toric colured quivers, correspond to brick regions of brane brick models (instead of faces of brane tilings). We know the analogue of colored quiver mutation and even colored potential mutation. But what are the analogues of the binomial exchange relations? Currently working with S. Franco, R. Kenyon, D. Speyer, and L. Williams on this and related questions.

Thank You!

http://www.math.umn.edu/~musiker/Higher19.pdf

Seiberg duality (1995) $\longleftrightarrow$ Quiver Mutation (2001)
(Seiberg) (Fomin-Zelevinsky)

Zamolodchikov Periodicity (1991) $\longleftrightarrow$ Y-system Periodicity (2003)
(Zamolodchikov) (Fomin-Zelevinsky)

Superpotentials & Moduli Spaces (2002) $\longleftrightarrow$ Quivers with Potentials (2007)
(Berenstein-Douglas) (Derksen-Weyman-Zelevinsky)

Amplituhedron (2013) $\longleftrightarrow$ Positive Grassmannian (2006)
(Arkani-Hamed-Trnka) (Postnikov)

Brane Tilings & Gauge Theories (2005) $\longleftrightarrow$ Cluster Integrable Systems (2011)
(Franco-Hanany-Kennaway-Vegh-Wecht) (Goncharov-Kenyon)

Brane Bricks & Hyperbricks (2015-2016) $\Longleftrightarrow$ Colored Quiver Mutation (2008)
(Franco-Lee-Seong-Vafa) (Buan-Thomas)

Higher Calabi-Yau Quiver Theories (2017) $\Longleftrightarrow$ Quivers for Silting Mutation (2015)
(Franco-M, arXiv:1711.01270) (Oppermann)