A Graph Theoretic Interpretation for Cluster Algebras of Classical Type

Gregg Musiker

MIT

March 14, 2008

Outline.

Definition [Sergey Fomin and Andrei Zelevinsky 2001] A cluster algebra $\mathcal{A}$ is a certain subalgebra of $k(x_1, \dots, x_m)$, the field of rational functions over $\{x_1, \dots, x_m\}$. Generators constructed by a series of exchange relations, which in turn induce all relations satisfied by the generators.

Definition. A *seed* for $\mathcal{A}$ is an initial *cluster* $\{x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_m\}$ and an m-by-n skew-symmetrizable integral matrix B with $(m \geq n)$. $(d_i b_{ij} = -d_j b_{ji}$ for some positive integers $d_i)$

Columns of B encode the exchanges

$$x_k x_k' = \prod_{b_{ik}>0} x_i^{|b_{ik}|} + \prod_{b_{ik}<0} x_i^{|b_{ik}|}$$

for $k \in \{1, 2, \dots n\}$. Note: If only one sign occurs (e.g. $b_{ik} > 0$), we still get binomial

$$\prod_{b_{ik}>0} x_i^{|b_{ik}|} + 1.$$

Mutation

For all $k \in \{1, 2, \ldots, n\}$, there exists another seed for $\mathcal{A}$ consisting of cluster $\{x_1, \ldots, \widehat{x_k}, \ldots, x_m\} \cup \{x_k'\}$ and matrix $\mu_k(B)$.

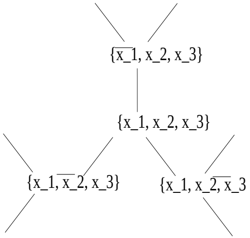
$$\mu_k(B)_{ij} = \begin{cases} -b_{ij} & \text{if} \quad k = i \quad \text{or} \quad k = j \\ b_{ij} & \text{if} \quad b_{ik}b_{kj} \leq 0 \\ b_{ij} + b_{ik}b_{kj} & \text{if} \quad b_{ik}, b_{kj} > 0 \\ b_{ij} - b_{ik}b_{kj} & \text{if} \quad b_{ik}, b_{kj} < 0 \end{cases}$$

Point: Matrix $\mu_k(B)$ is again integral and skew-symmetrizable. Thus $\left(\{x_1, \ldots, \widehat{x_k}, \ldots, x_m\} \cup \{x_k'\}, \mu_k(B)\right)$ is also a cluster algebra seed. Also mutation is an involution, $\mu_k^2(B) = B$.

After all exchanges, the x_k''s obtained this way are the generators of the cluster algebra $\mathcal{A} \subset k(x_1, x_2, \ldots, x_m)$. Relations induced by the exchange relations used to construct the generators.

Exchange Graphs

A priori, get a tree of exchanges:



In practice, often get identifications among clusters.

In extreme cases, get only a finite number of clusters as tree closes up on itself.

Example: B_2

$$\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \qquad \{x_1, x_2\} \;\_\mu_1\_\; \left\{\frac{1+x_2^2}{x_1}, x_2\right\} \;\_\mu_2\_\; \left\{\frac{1+x_2^2}{x_1}, \frac{x_2^2+x_1+1}{x_1x_2}\right\}$$

$$\_\mu_1\_\left\{\frac{x_1^2+2x_1+x_2^2+1}{x_1x_2^2}, \frac{x_2^2+x_1+1}{x_1x_2}\right\}$$

$$\_\mu_2\_\left\{\frac{x_1^2+2x_1+x_2^2+1}{x_1x_2^2}, \frac{x_1+1}{x_2}\right\} \;\_\mu_1\_\left\{x_1, \frac{x_1+1}{x_2}\right\}$$

$\_\mu_2\_\{x_1, x_2\}$. Thus exchange graph is a hexagon.

Cluster Expansion Formulas

Definition. The union of all clusters is the set of *cluster variables*. These are generators of the cluster algebra $\mathcal{A}$ defined by seed $\{x_1, x_2, \ldots, x_n\}, B$.

Example: Cluster algebra with seed $\left(\{x_1, x_2\}, \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}\right)$ has cluster variables

$$\left\{x_1, x_2, \frac{1 + x_2^2}{x_1}, \frac{x_2^2 + x_1 + 1}{x_1 x_2}, \frac{x_1^2 + 2x_1 + x_2^2 + 1}{x_1 x_2^2}, \frac{x_1 + 1}{x_2}\right\}.$$

Theorem. (The Laurent Phenomenon FZ 2001) Given any cluster algebra defined by initial seed $(\{x_1, x_2, \ldots, x_m\}, B)$, all cluster variables of $\mathcal{A}(B)$ are *Laurent polynomials* in $\{x_1, x_2, \ldots, x_m\}$ (with no coefficient $x_{n+1}, \ldots, x_m$ in the denominator).

Thus we can write any cluster variable in the form $x_\alpha = \frac{P_\alpha(x_1, \ldots, x_m)}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}$ where P_α is a polynomial with integer coefficients.

Finite Type

Definition. A cluster algebra $\mathcal{A}(B)$ is of *finite type* if the corresponding set of cluster variables is finite.

Definition. The *bipartite* exchange matrix B_Φ for root system Φ, also called the Cartan counterpart, is constructed as follows:

1) Take the *Cartan Matrix* C_Φ and replace its diagonal of 2's with zeros,

2) Alter the signs of C_Φ so that the resulting matrix is skew-symmetrizable with elements in columns having common signs.

Finite Type (cont.)

Theorem. (FZ 2002) A cluster algebra is of finite type if and only if B is mutation equivalent to B_Φ for a root system Φ.

The non-initial cluster variables of the system are in bijection with the positive roots of Φ. (In particular notation x_α well defined in this case.) When the seed matrix is B_Φ, the denominator vectors are in fact explicitly given by

$$x_\alpha = \frac{P_\alpha(x_1, x_2, \dots, x_m)}{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}$$

where $(\alpha_1, \dots, \alpha_n)$ is a positive root, i.e. if $s_1, \dots, s_n$ are the simple roots of Φ, then $\alpha_1 \cdot s_1 + \cdots + \alpha_n \cdot s_n$ is a positive root of Φ.

For cluster algebras of finite type, the coefficients of P_α are *nonnegative* integers.

Examples of B_Φ's

$$B_{A_5} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad B_{B_5} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B_{C_5} = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad B_{D_5} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

For D_n, I use the indexing $1, \overline{1}, 2, 3, 4, \ldots, (n-1)$.

Positivity Conjecture

Conjecture. (FZ 2001) Given any cluster variable

$$x_\alpha = \frac{P_\alpha(x_1, \ldots, x_m)}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}},$$

the polynomial $P_\alpha(x_1, \ldots, x_n)$ has *nonnegative* integer coefficients.

This conjecture is still wide open for general cluster algebras. Work of [Carroll-Price 2002] gave expansion formulas for case of Ptolemy algeras, examples of cluster algebras of type A_n with coefficients. [FZ 2002] proved positivity for finite type with bipartite seed. Positivity also proven for those cluster variables in cluster algebra with acyclic seed [Caldero-Reineke 2006] and cluster algebras arising from unpunctured surfaces [Schiffler-Thomas 2007]. Work of Schiffler-Thomas also includes expansion formulas.

A different approach to proving positivity for $\mathcal{A}(B_\Phi)$ (Φ classical) follows, yielding explicit combinatorial interpretations for expansions.

Perfect Matchings and their weightings

Given a simple undirected graph $G = (V, E)$, a *perfect matching* $M \subseteq E$ is a set of distinguished edges so that every vertex of V is covered exactly once.

We let the edges of our graph have weights $w(e)$ which are each either 1 (unweighted) or some variable x_i.

The weight of a matching M is the product of the weights of the constituent edges, i.e. $w(M) = \prod_{e \in M} w(e)$.

Definition. The *perfect matching enumerator* of a weighted graph G is given by the polynomial

$$P(G) = \sum_{M \text{ is a matching of } G} w(M).$$

A Framework for Graph Theoretic Interpretations of Cluster Expansions

Notice that starting with a seed $\{x_1, \ldots, x_n\}, B$ that there are n elementary exchanges, which lead to cluster variables of the form

$$\frac{Binomial(x_1, \ldots, \widehat{x_k}, \ldots, x_n)}{x_k}.$$

If the corresponding Binomial has degree d, then the cluster variable $x_{s_k} = \frac{P_{s_k}(x_1,\ldots,x_n)}{x_k}$ can be expressed as $\frac{P(T_k)}{x_k}$ where graph T_k is a weighted cycle graph of even length, which is greater than or equal to $2d$.

We wish to generalize this interpretation to other positive roots α.

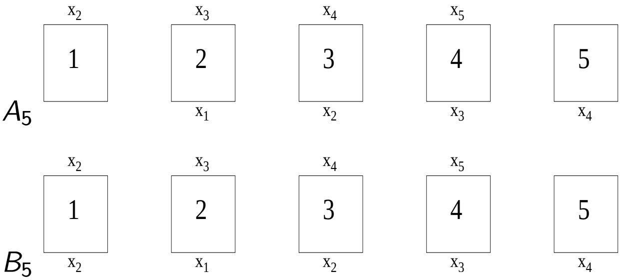
Main Theorem

Theorem. (M 2007) For every classical root system there exists a family of graphs $\mathcal{G}_\Phi = \{G_\alpha\}_{\alpha \in \Phi_+}$ such that x_α, the cluster variable of $\mathcal{A}(B_\Phi)$ corresponding to $\alpha \in \Phi_+$, can be expressed as

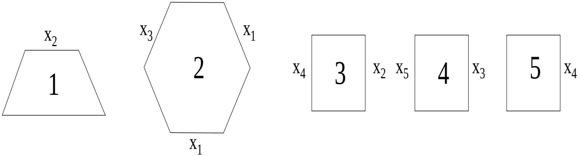
$$x_\alpha = \frac{P_{G_\alpha}(x_1, \ldots, x_n)}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}.$$

Further, we will construct the graphs in a very simple manner using the tiles T_k.

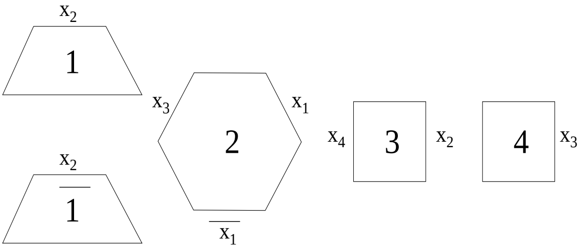
Tiles for the four classical types



Tiles for the four classical types (cont.)



C_5



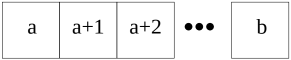
D_5

Graphs for A_n and B_n

We will construct the graphs G_α for other cluster variables x_α (for α a positive root of Φ) by gluing together these tiles.

Example: A_n, positive roots look like $(0, \dots, 0, 1, 1, \dots, 1, 0, \dots, 0) = s_a + \cdots + s_b$.

We glue tiles T_a through T_b together horizontally.

$G_\alpha =$ 

Cluster variable $x_\alpha = \frac{P(G_\alpha)}{x_a x_{a+1} \cdots x_b}$ where $P(G_\alpha)$ is the perfect matching enumerator.

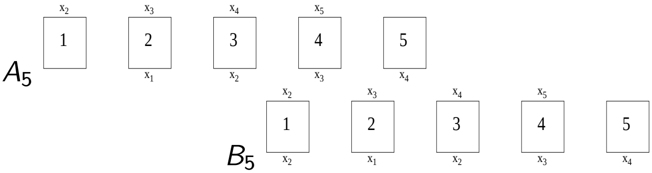
Graphs for A_n and B_n (cont.)

Example: B_n, positive roots are of the form $s_a + \cdots + s_b$ as in the A_n case, or

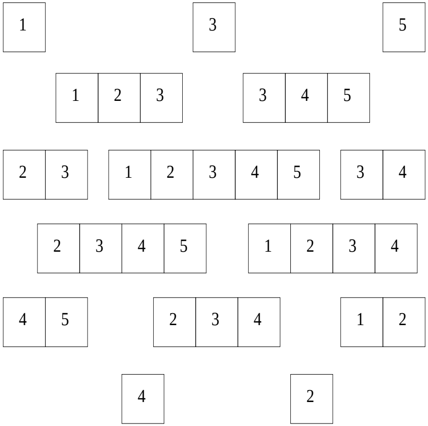
$$s_a + s_{a-1} + \cdots + s_2 + s_1 + s_2 + \cdots + s_b$$

with $a \leq b$.

We again glue tiles together horizontally in this order.

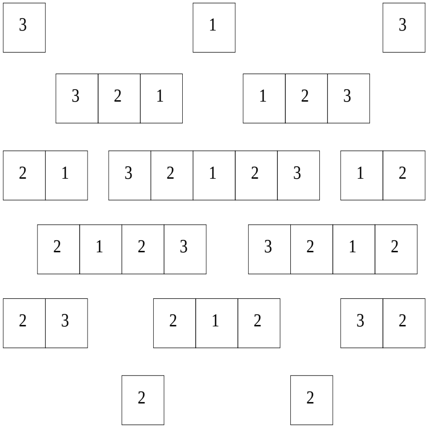


A_5



Graphs for A_n and B_n (cont.)

B_3 folds onto A_5 (Take right-half including middle)



Sketch of proof for A_n and B_n

Not only is this lattice a useful visualization of the set of cluster variables, it also provides a helpful graphical description of the proof.

In particular, what I have drawn are also known as the layers of the bipartite belt.

Since matrices B_Φ are bipartite (in fact stratified by odds versus evens), we can mutate all odd indicies independently, followed by a mutation of all even indicies.

The mutated matrix will always be $\pm B_\Phi$ at the end of a row.

Thus, these lattices are frieze patterns defined completely by the diamond condition.

$$ad = bc + 1 \qquad \begin{array}{ccc} & a & \\ b & & c \\ & d & \end{array}$$

3	4	5

Example.

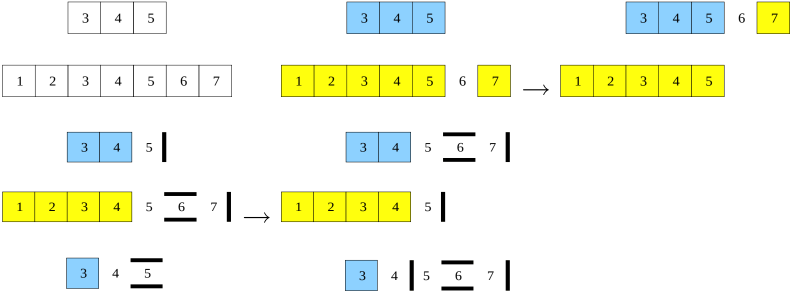
1	2	3	4	5

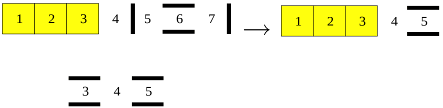
3	4	5	6	7

1	2	3	4	5	6	7

$$P(\boxed{3\,|\,4\,|\,5})P(\boxed{1\,|\,2\,|\,3\,|\,4\,|\,5\,|\,6\,|\,7}) = P(\boxed{1\,|\,2\,|\,3\,|\,4\,|\,5})P(\boxed{3\,|\,4\,|\,5\,|\,6\,|\,7}) + x_1x_2x_3^2x_4^2x_5^2x_6x_7$$

Sketch of proof for A_n and B_n (cont.)





Only one matching left:

does not decompose, contributes

$$(x_1 x_3)(x_2 x_4)(x_3 x_5)(x_4 x_6)(x_5 x_7) = x_1 x_2 x_3^2 x_4^2 x_5^2 x_6 x_7.$$

Sketch of proof for A_n and B_n (cont.)

Thus, these lattices are frieze patterns defined completely by the diamond condition.

$$ad = bc + 1 \qquad \begin{array}{ccc} & a & \\ b & & c \\ & d & \end{array}$$

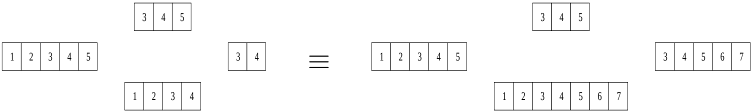
Example.

$$\begin{array}{ccc} & \begin{array}{|c|c|c|}\hline 3&4&5\\\hline\end{array} & \\ \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\\hline\end{array} & & \begin{array}{|c|c|c|c|c|}\hline 3&4&5&6&7\\\hline\end{array} \\ & \begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&3&4&5&6&7\\\hline\end{array} & \end{array}$$

$$P(\begin{array}{|c|c|c|}\hline 3&4&5\\\hline\end{array})P(\begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&3&4&5&6&7\\\hline\end{array}) = P(\begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\\hline\end{array})P(\begin{array}{|c|c|c|c|c|}\hline 3&4&5&6&7\\\hline\end{array}) + x_1 x_2 x_3^2 x_4^2 x_5^2 x_6 x_7$$

Secondly, to avoid boundary behavior, we use excision.

Example.

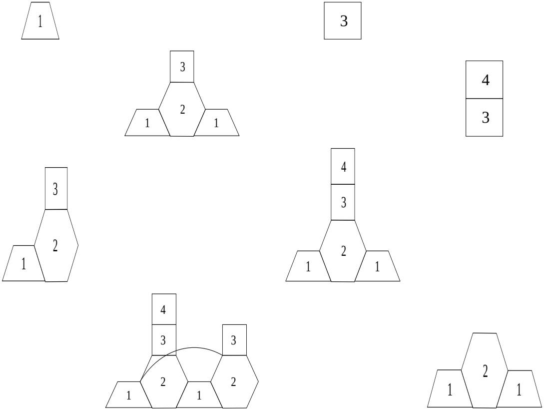


In A_5, we let $x_6 = 1$, $x_7 = 0$, $x_8 = -1$, and pattern continues with $x_9 = -x_5$, $x_{10} = -x_4$, $\ldots$, thereby obtaining

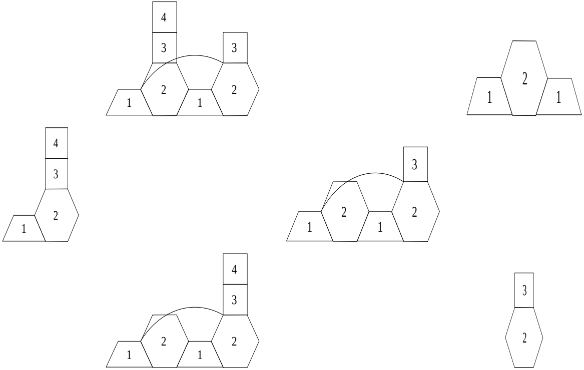
$$\frac{P(\boxed{1|2|3|4})}{x_1 x_2 x_3 x_4} = \lim_{y_0 \to 0} \frac{P(\boxed{1|2|3|4|5|6|7})}{x_1 x_2 x_3 x_4 (1)(y_0)(-1)}$$

$$\frac{P(\boxed{3|4})}{x_3 x_4} = \lim_{y_0 \to 0} \frac{P(\boxed{3|4|5|6|7})}{x_3 x_4 (1)(y_0)(-1)}$$

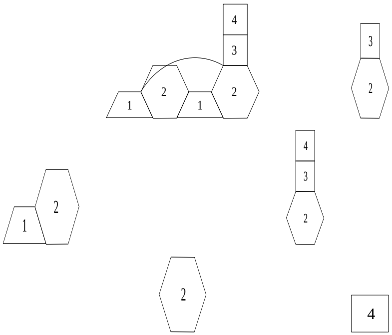
C_4 After mutating with respect to x_1 and x_3 (x_2 and x_4), we obtain



The C_n and D_n cases (cont.)

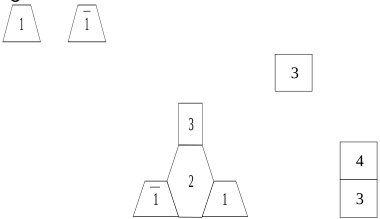


The C_n and D_n cases (cont.)

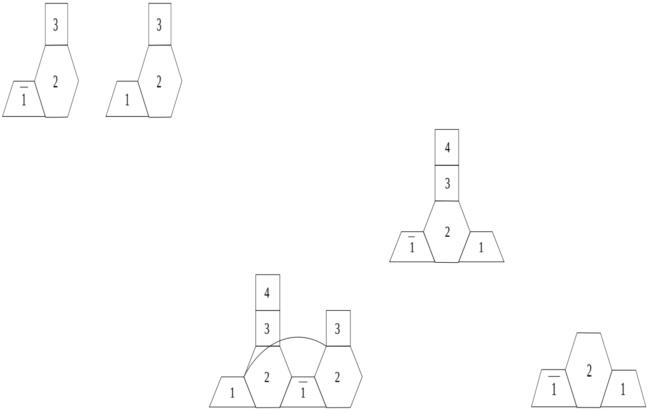


The C_n and D_n cases (cont.)

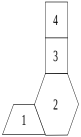
D_5

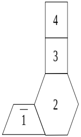


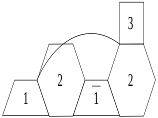
D_5 (cont.)

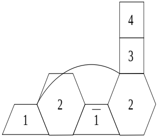


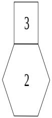
D_5 (cont.)





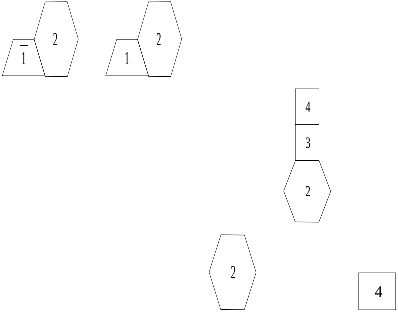






The C_n and D_n cases (cont.)

D_5 (cont.)



The C_n and D_n cases (cont.)

The proof comes down to superpositions similar to the A_n and B_n cases. We deal with boundary behavior by excision.

We use a different frieze pattern, which is identical except for the first two columns.

For C_n and b in the first column

$$ad = b^2c + 1 \qquad \begin{array}{ccc} & a & \\ b & & c \\ & d & \end{array}$$

The C_n and D_n cases (cont.)

The proof comes down to superpositions similar to the A_n and B_n cases. We deal with boundary behavior by excision.

We use a different frieze pattern, which is identical except for the first two columns.

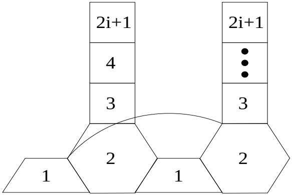
For D_n and $b, \overline{b}$ in the first column

$$ad = b\overline{b}c + 1 \qquad \begin{array}{cccc} & & a & \\ b & \overline{b} & & c \\ & & d & \end{array}$$

We let $\tilde{b} = b^2$ or $b\overline{b}$, respectively.

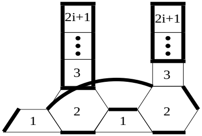
The C_n and D_n cases (cont.)

Suffices to show that numerator of $\tilde{b} = b^2$ corresponds to perfect matching enumerator of

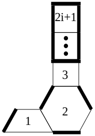


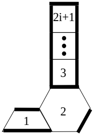
.

There is a weight-preserving bijection between matchings of



and matchings of





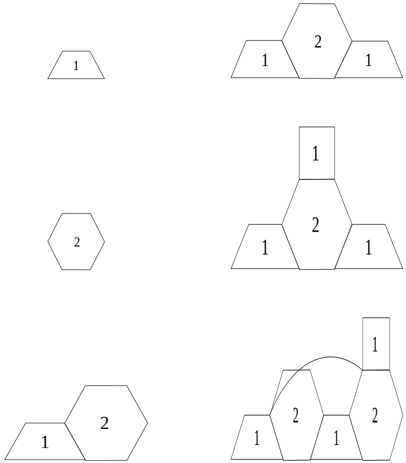
.

The right hexagon is rotated clockwise 120°, and so we in fact obtain a weight of $x_1^2 x_2 x_3$ from the forced arcs in both graphs.

Seed matrix is $B = \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}$

Hexagon has x_1 on NW, NE, and S sides,

Trapezoid has x_2 on N side.



Joint work with Jim Propp.

Let $B = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$.

Here we also exploit invariance of matrices B under mutation.

So we are considering (b, c)-sequence

$$x_n x_{n-2} = \begin{cases} x_{n-1}^b + 1 & \text{if } n \text{ odd} \\ x_{n-1}^c + 1 & \text{if } n \text{ even} \end{cases}$$

for $(b, c) = (2, 2)$ or $(1, 4)$.

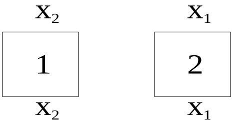
Affine Rank 2 (cont.)

Since cluster algebra structure, (b, c) sequence consists of Laurent polynomials.

Work of Sherman and Zelevinsky verifies positive coefficients for $(1, 4)$ and $(2, 2)$ using Newton polytope, and Caldero-Zelevinsky give another proof of positivity for $(2, 2)$ case via Quiver Grassmannians.

We give proof of positivity via graph theoretical interpretation similar to above.

$(2,2)$: all cluster variables have denominators $x_1^d x_2^{d+1}$ (resp. $x_1^{d+1} x_2^d$)
We string together corresponding number of sqares



in an intertwining fashion.

Examples:

$$\frac{{x_2}^4+2\,{x_2}^2+1+{x_1}^2}{{x_1}^2 x_2} \leftrightarrow$$

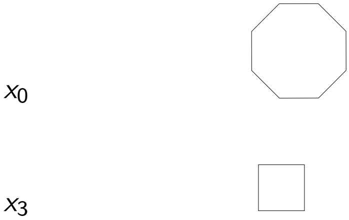
1	2	1

$$\frac{{x_1}^6+3\,{x_1}^4+3\,{x_1}^2+2\,{x_2}^2{x_1}^2+{x_2}^4+1+2\,{x_2}^2}{{x_2}^3{x_1}^2} \leftrightarrow$$

2	1	2	1	2

Affine Rank 2 (cont.)

$(1, 4)$: Tiles are a square and an octagon:



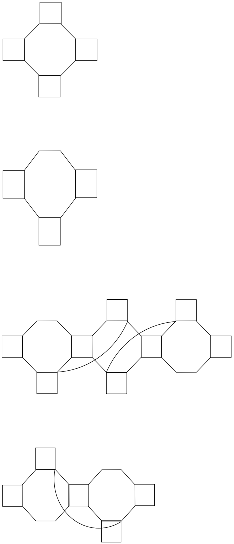
Sequnce Continues

x_4 17 terms

x_5 9 terms

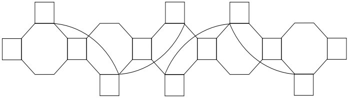
x_6 386 terms

x_7 43 terms

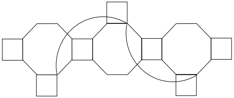


Sequnce Continues (cont.)

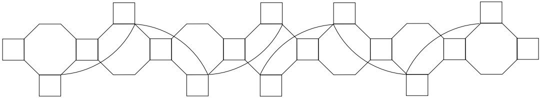
x_8 8857 terms



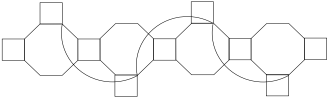
x_9 206 terms



x_{10} 203321 terms

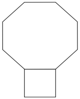


x_{11} 987 terms

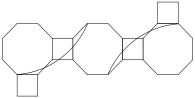


Running the $(1,4)$ sequence backwards

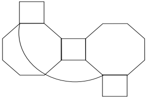
x_{-1} 3 terms



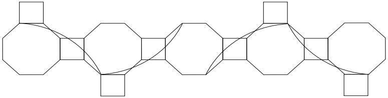
x_{-2} 41 terms



x_{-3} 14 terms

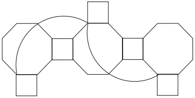


x_{-4} 937 terms

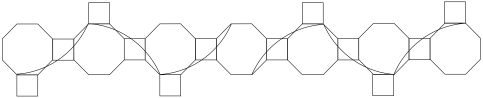


Running the $(1, 4)$ sequence backwards (cont.)

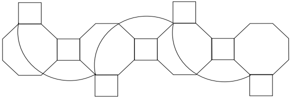
x_{-5} 67 terms



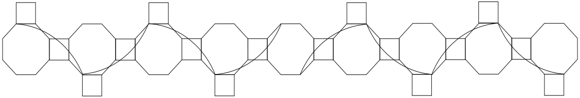
x_{-6} 21506 terms



x_{-7} 321 terms



x_{-8} 493697 terms



Markoff polynomials

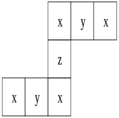
Joint work by Carroll, Itsara, Le, M, Price, Thurston, and Viana under Propp in REACH program.

$$B = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix},$$ Exchange graph is free ternary tree.

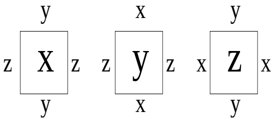
B invariant under mutation. All exchanges have form $(x, y, z) \mapsto (x', y, z)$ where $xx' = y^2 + z^2$.

These also have graph theoretic interpretation: Snake Graphs, .e.g

$$\frac{Polynomial(x,y,z)}{x^4y^2z^1} \longleftrightarrow$$



with tiles



Further directions

I am investigating how to push these interpretations further: i.e. different seeds, with coefficients, other cases of infinite type.

Recent work with Ralf Schiffler seems to indicate similar interpetations for cluster algebras from unpunctured triangulated surfaces, which includes more cases of affine cluster algebras.

References

A Graph Theoretic Expansion Formula for Cluster Algebras of Classical Type, http://www-math.mit.edu/∼ musiker/Finite.pdf

Combinatorial Interpretations for Rank-Two Cluster Algebras of Affine Type (with Jim Propp), Electronic Journal of Combinatorics. Vol. 14 (R15), 2007.

The Combinatorics of Frieze Patterns and Markoff Numbers (by Jim Propp), arXiv:math.CO/0511633

Thanks for Listing

Happy π Day.