Super Fibonacci and Super Markov Numbers

Gregg Musiker (University of Minnesota)

Paris Algebra Seminar

February 10, 2025

Some of this joint work with Nicholas Ovenhouse and Sylvester Zhang.

`http://www-users.math.umn.edu/~musiker/Paris25.pdf`

Paper in preparation ... coming soon

Motivation

April 2021 Talk: Combinatorial formulas for λ-lengths and μ-invariants in decorated super-Teichmüller spaces associated to polygons, and their relationship to superfriezes, joint work with N. Ovenhouse and S. Zhang.

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February 2025 Talk: Super Markov Numbers and **signed** enumeration formulas associated to the once-punctured torus. Leads to new avenues in algebra, combinatorics, geometry, and number theory, and the like.

What is a Cluster Algebra?

Definition (Sergey Fomin and Andrei Zelevinsky 2001)

A cluster algebra $\mathcal{A}$ (of geometric type) is a subalgebra of $k(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ constructed cluster by cluster by certain exchange relations.

Generators:

Specify an initial finite set of them, a Cluster, $\{x_1, x_2, \ldots, x_{n+m}\}$.

Construct the rest via Binomial Exchange Relations:

$$x_\alpha x'_\alpha = \prod x_{\gamma_i}^{d_i^+} + \prod x_{\gamma_i}^{d_i^-}.$$

The set of all such generators are known as Cluster Variables, and the initial pattern B of exchange relations determines the Seed.

Relations:

Induced by the Binomial Exchange Relations.

Example: Coordinate Ring of Grassmannian$(2, n+3)$

Let $Gr_{2,n+3} = \{V | V \subset \mathbb{C}^{n+3},\ \dim V = 2\}$ planes in $(n+3)$-space

Elements of $Gr_{2,n+3}$ represented by 2-by-$(n+3)$ matrices of full rank.

Plücker coordinates $p_{ij}(M) =$ det of 2-by-2 submatrices in columns i and j.

The coordinate ring $\mathbb{C}[Gr_{2,n+3}]$ is generated by all the p_{ij}'s for $1 \leq i < j \leq n+3$ subject to the Plücker relations given by the 4-tuples

$$p_{ik}p_{j\ell} = p_{ij}p_{k\ell} + p_{i\ell}p_{jk} \text{ for } i < j < k < \ell.$$

Claim. $\mathbb{C}[Gr_{2,n+3}]$ has the structure of a cluster algebra. Clusters are each maximal algebraically independent sets of p_{ij}'s.

Each have size $(2n+3)$ where $(n+3)$ of the variables are frozen and n of them are exchangeable.

Example: Coordinate Ring of Grassmannian$(2, n+3)$

Cluster algebra structure of $Gr_{2,n+3}$ as a triangulated $(n+3)$-gon.

Frozen Variables / Coefficients $\longleftrightarrow$ sides of the $(n+3)$-gon

Cluster Variables $\longleftrightarrow \{p_{ij} : |i-j| \neq 1 \mod (n+3)\} \longleftrightarrow$ diagonals

Seeds $\longleftrightarrow$ triangulations of the $(n+3)$-gon

Clusters $\longleftrightarrow$ Set of p_{ij}'s corresponding to a triangulation

Can exchange between various clusters by flipping between triangulations.

This is called mutation, and we will present a detailed example later.

Another Example: Rank 2 Cluster Algebras

Let $B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$, $b, c \in \mathbb{Z}_{>0}$. $(\{x_1, x_2\}, B)$ is a seed for a cluster algebra $\mathcal{A}(b, c)$ of rank 2.

$$\mu_1(B) = \mu_2(B) = -B \text{ and } x_1 x_1' = x_2^c + 1, \quad x_2 x_2' = 1 + x_1^b.$$

Thus the cluster variables in this case are

$$\{x_n : n \in \mathbb{Z}\} \text{ satisfying } x_n x_{n-2} = \begin{cases} x_{n-1}^b + 1 \text{ if } n \text{ is odd} \\ x_{n-1}^c + 1 \text{ if } n \text{ is even} \end{cases}.$$

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Example $(b = c = 1)$:

$$x_3 = \frac{x_2 + 1}{x_1}. \quad x_4 = \frac{x_3 + 1}{x_2} = \frac{\frac{x_2+1}{x_1} + 1}{x_2} = \frac{x_1 + x_2 + 1}{x_1 x_2}.$$

$$x_5 = \frac{x_4 + 1}{x_3} = \frac{\frac{x_1+x_2+1}{x_1 x_2} + 1}{(x_2 + 1)/x_1} = \frac{x_1(x_1 + x_2 + 1 + x_1 x_2)}{x_1 x_2 (x_2 + 1)} = \frac{x_1 + 1}{x_2}. \quad x_6 = x_1.$$

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Example $(b = c = 2)$:

$$x_3 = \frac{x_2^2 + 1}{x_1}. \quad x_4 = \frac{x_3^2 + 1}{x_2} = \frac{x_2^4 + 2x_2^2 + 1 + x_1^2}{x_1^2 x_2}.$$

$$x_5 = \frac{x_4^2 + 1}{x_3} = \frac{x_2^6 + 3x_2^4 + 3x_2^2 + 1 + x_1^4 + 2x_1^2 + 2x_1^2 x_2^2}{x_1^3 x_2^2}, \ \ldots$$

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If we let $x_1 = x_2 = 1$, we obtain $\{x_3, x_4, x_5, x_6\} = \{2, 5, 13, 34\}$.

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The next number in the sequence is $x_7 = \frac{34^2+1}{13} = \frac{1157}{13} = 89$, an integer!

In fact, the sequence $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, \ldots\}$ ends up being

Every-other Fibonacci Number by Cassini's identity.

Teichmüller and Decorated Teichmüller Spaces

Let $S = S_g^n$ be a smooth oriented surface (possibly with boundary) of genus g equipped with a collection of marked points $p_1, p_2, \ldots, p_n$.

Here $n \geq 0$. The marked points either lie on boundary components, or in the interior of S, in which case they are called punctures.

Roughly speaking, the *Teichmüller space* of such a surface is

$$T(S) = \text{the set of hyperbolic structures on S/isotopy.}$$

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Definition

Define the *Teichmüller space* of S to be the quotient space

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Definition (Penner)

When $n > 0$, any such surface $S = S_g^n$ also admits a *decorated Teichmüller space*, which is a trivial $\mathbb{R}^n_{>0}$-bundle over $T(S)$, denoted $\tilde{T}(S)$.

Decorated Teichmüller Theory

We will usually let $S = S_0^n$ be a disk with n marked points on its unique boundary (i.e. a polygon). Such surfaces admit the *Poincaré disk* $\mathbb{D}$ model as a hyperbolic structure.

$\mathbb{D} := \{z = x + yi \in \mathbb{C} : |z| < 1\}$, with metric $ds = 2\frac{\sqrt{dx^2+dy^2}}{1-|z|^2}$.

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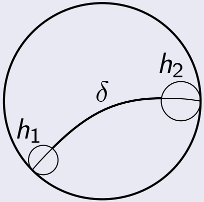
Definition (λ-length via horocycles)



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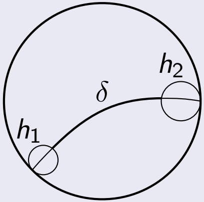

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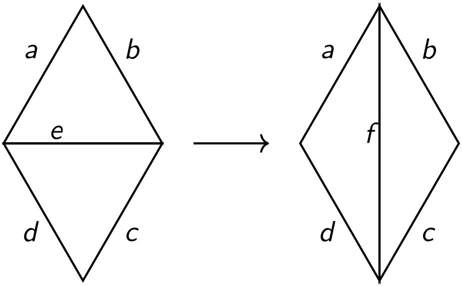
For a pair of horocycles h_1, h_2, the λ-*length* between them is

$$\lambda(h_1, h_2) = e^{\delta/2}$$

where δ is the hyperbolic distance between the two intersections.

Ptolemy Relations

Given a quadruple of horocycles with distinct centers (a decorated ideal quadrilateral), one has the **Ptolemy transformation** induced by flipping the diagonal of the quadrilateral.



At the level of λ-lengths, this induces the identity

$$\lambda(e)\lambda(f) = \lambda(a)\lambda(c) + \lambda(b)\lambda(d).$$

Note that we will often abbreviate this as $ef = ac + bd$.

Structural Theorems for Cluster Algebras

Theorem (Fomin-Zelevinsky 2001, The Laurent Phenomenon)

For any cluster algebra defined by initial seed $(\{x_1, x_2, \ldots, x_{n+m}\}, B)$*, all cluster variables of* $\mathcal{A}(B)$ *are* *Laurent polynomials* *in* $\{x_1, x_2, \ldots, x_{n+m}\}$ *(with no coefficient* $x_{n+1}, \ldots, x_{n+m}$ *in the denominator).*

Because of the Laurent Phenomenon, any cluster variable x_α can be expressed as $\frac{P_\alpha(x_1,\ldots,x_{n+m})}{x_1^{\alpha_1}\cdots x_n^{\alpha_n}}$ where $P_\alpha \in \mathbb{Z}[x_1, \ldots, x_{n+m}]$ and the α_i's $\in \mathbb{Z}$.

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Theorem (Lee-Schiffler 2014, Gross-Hacking-Keel-Kontsevich 2015, Prooof of the Positivity Conjecture)

For any cluster variable x_α and any initial seed (i.e. initial cluster $\{x_1, \ldots, x_{n+m}\}$ and initial exchange pattern B), the polynomial $P_\alpha(x_1, \ldots, x_n)$ has nonnegative integer coefficients.

Cluster Algebras from Surfaces

Theorem (Fomin-Shapiro-Thurston 2006)

Given a Riemann surface with marked points (S, M), one can define a corresponding cluster algebra $\mathcal{A}(S, M)$.

$$\textit{Seed} \leftrightarrow \textit{Triangulation } T = \{\tau_1, \tau_2, \ldots, \tau_n\}$$

$$\textit{Cluster Variable} \leftrightarrow \textit{Arc } \gamma \; (x_i \leftrightarrow \tau_i \in T)$$

$$\textit{Cluster Mutation (Binomial Exchange Relations)} \leftrightarrow \textit{Flipping Diagonals.}$$

(Based on earlier work of Gekhtman-Shapiro-Vainshtein and Fock-Goncharov.)

From the perspective of hyperbolic geometry, Laurent expansions of cluster variables may be expressed as λ-lengths of arcs, which can be measured by choosing a point in Penner's decorated Teichmüller space.

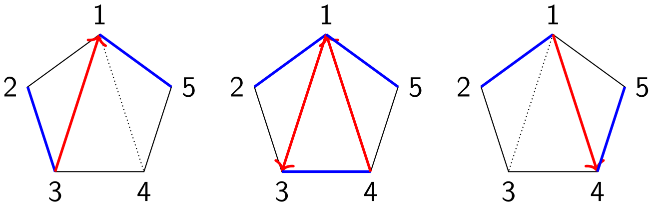
Positivity of Cluster Algebras from Surfaces

Theorem (Schiffler 2006)

Let $\mathcal{A}$ be any cluster algebra of type A_n, i.e. with a seed Σ defined by a triangulation T of an $(n+3)$-gon.

Then the Laurent expansion of every cluster variable with respect to the seed Σ has non-negative coefficients.

Proof via explicit combinatorial formulas in terms of **T-paths**.



$$\lambda_{25} = \frac{x_{23}x_{15}}{x_{13}} + \frac{x_{12}x_{34}x_{15}}{x_{13}x_{14}} + \frac{x_{12}x_{45}}{x_{14}} = \frac{x_{23}x_{14}x_{15} + x_{12}x_{34}x_{15} + x_{12}x_{13}x_{45}}{x_{13}x_{14}}.$$

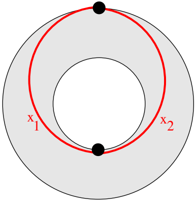
Positivity of Cluster Algebras from Surfaces

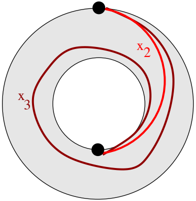
Theorem (Schiffler-Thomas 2007, Schiffler 2008)

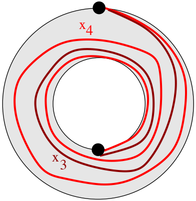
Let $\mathcal{A}(S, M)$ be any cluster algebra arising from an unpunctured surface S with marked points M, with principal coefficients, and let Σ be any initial seed. Here Σ correponds to a triangulation of S with respect to the marked points M.

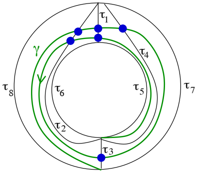
Then the Laurent expansion of every cluster variable with respect to the seed Σ has non-negative coefficients.

Proof via explicit combinatorial formulas in terms of **T-paths**.









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Theorem (M-Schiffler 2008)

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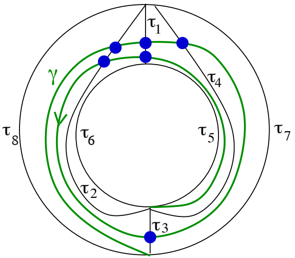
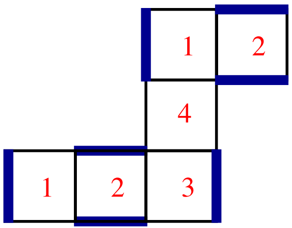
Proof via explicit combinatorial formulas in terms of **snake graphs**.





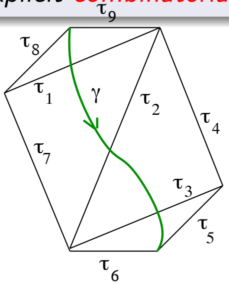
Positivity of Cluster Algebras from Surfaces

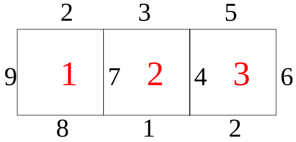
Theorem (M-Schiffler-Williams 2009)

Let $\mathcal{A}(S, M)$ be any cluster algebra arising from a surface (with or without punctures), where the coefficient system is of geometric type, and let Σ be any initial seed.

Then the Laurent expansion of every cluster variable with respect to the seed Σ has non-negative coefficients.

Proof via explicit combinatorial formulas in terms of **snake graphs**.





Superalgebras (and towards Superspace)

A super algebra is a $\mathbb{Z}_2$-graded algebra.

i.e. $A = A_0 \oplus A_1$, (the "*even*" and "*odd*" parts) and

$$A_i A_j \subseteq A_{i+j} \text{ for } i, j \in \{0, 1\} \bmod 2$$

The algebra A generated by $x_1, \cdots, x_n, \theta_1, \cdots, \theta_m$, subject to the following relations

$$x_i x_j = x_j x_i \quad x_i \theta_j = \theta_j x_i \quad \theta_i \theta_j = -\theta_j \theta_i$$

is a superalgebra. In particular $\theta_i^2 = 0$.

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Here A_0 is spanned by monomials with an even number of θ's and A_1 is spanned by monomials with an odd number of θ's.

E.g. $x_1 x_2 + x_1 \theta_1 \theta_3 + x_2 \theta_1 \theta_2 \theta_3 \theta_4 \in A_0$, $x_1 \theta_1 \theta_2 \theta_3 + x_1 x_4 \theta_2 + \theta_4 \in A_1$

Decorated Super-Teichmüller Spaces [PZ19]

- By replacing $\mathrm{PSL}(2,\mathbb{R})$ with $\mathrm{OSp}(1|2)$, Penner and Zeitlin define the **super-Teichmüller space** of a surface S to be
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- But unlike the bosonic case, we need additional invariants to accommodate for the extra degree of freedom coming from the odd dimension.
- They associate an odd variable to each triangle (triple of ideal points), and call them the μ-invariants.

Spin Structures

Components of $ST(S)$ are indexed by the set of **spin structures** on S.

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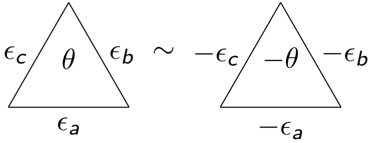
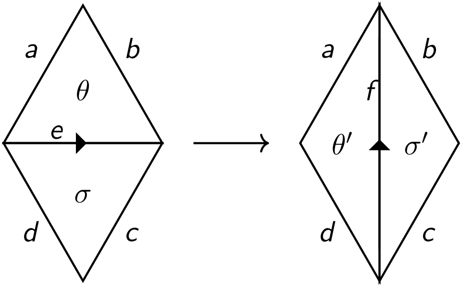
Dual to this formulation, we consider the set of spin structures on S to be the set of equivalence classes of orientations on triangulations of S of the following equivalence relation.



where $\epsilon_a, \epsilon_b, \epsilon_c$ are orientations on the edges, and θ is the μ-invariant associated to the triangle.

Super Ptolemy Relation

The Ptolemy transformation on super λ-length coordinates is given as follows.



$$ef = (ac + bd)\left(1 + \frac{\sigma\theta\sqrt{\chi}}{1+\chi}\right), \quad \chi = \frac{ac}{bd}$$

$$\sigma' = \frac{\sigma - \sqrt{\chi}\theta}{\sqrt{1+\chi}} \quad \text{and} \quad \theta' = \frac{\theta + \sqrt{\chi}\sigma}{\sqrt{1+\chi}}$$

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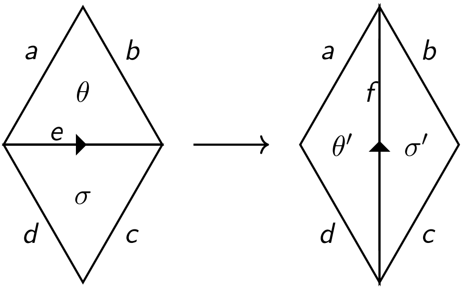


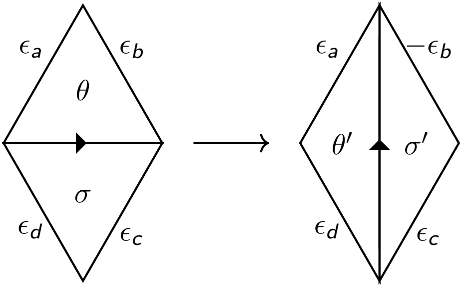
$$ef = ac + bd + \sqrt{abcd}\,\sigma\theta$$

$$\sigma' = \frac{\sigma\sqrt{bd} - \theta\sqrt{ac}}{\sqrt{ac+bd}} \quad \text{and} \quad \theta' = \frac{\theta\sqrt{bd} + \sigma\sqrt{ac}}{\sqrt{ac+bd}}$$

$$\sigma\theta = \sigma'\theta'$$

Super Ptolemy Relation

Super-flip also reverses the orientation of the edge b.

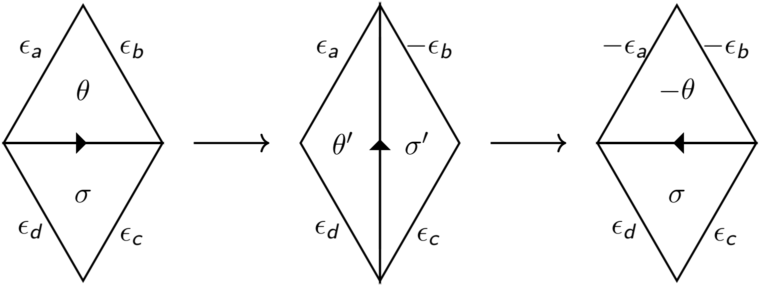


Remark

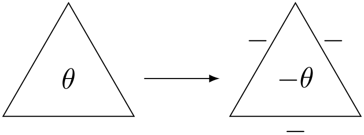
- Super Ptolemy moves are not involutions: $\mu_i^8 = I$.
- The even-degree-0 terms of a super λ-length are exactly the (ordinary) λ-length in the bosonic decorated space.

Super Ptolemy Relation

If we flip a diagonal twice

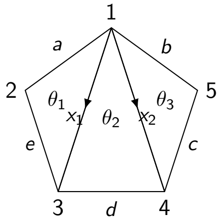


the orientations of the triangle θ are reversed and θ is changed to $-\theta$.



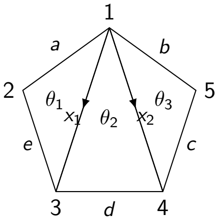
This orientation is equivalent to the original one, i.e. both the first and third pictures represent the same spin structure.

Super Ptolemy Relation - Example



Start with a Pentagon with given orientation.

Super Ptolemy Relation - Example



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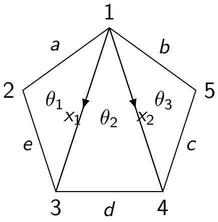

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What are λ_{24}, λ_{25}, and λ_{35}?

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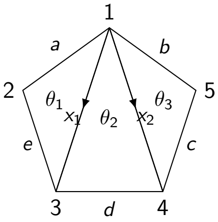

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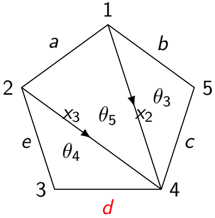
What are λ_{24}, λ_{25}, and λ_{35}?

We first flip the edge x_1.

Super Ptolemy Relation - Example

After flipping x_1 to x_3, we get:



$$x_3 = \frac{ad + ex_2}{x_1} + \frac{\sqrt{adex_2}}{x_1}\theta_1\theta_2$$

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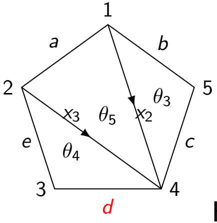

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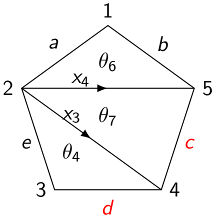
Here the red color indicates that the orientation on the boundary edge has been reversed.

Next we flip x_2.

Super Ptolemy Relation - Example

After flipping x_2 to x_4, we have:

$$x_4 = \frac{ac + bx_3}{x_2} + \frac{\sqrt{acbx_3}}{x_2}\,\theta_5\theta_3$$



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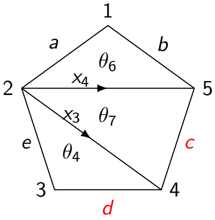

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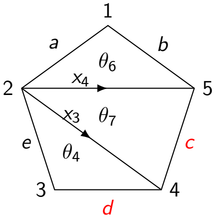

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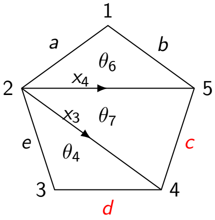

Question: If we now flip x_3 to x_5, what do we expect x_5 to look like?

Main Question

In a cluster algebra A, any cluster variable can be expressed as a positive Laurent polynomial in the initial cluster, i.e.

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Questions

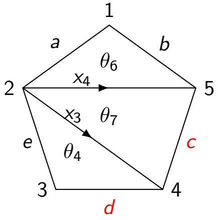
- Does the super λ-length satisfy some Laurent phenomenon?
- Is there a "positivity" for terms with anti-commuting variables?

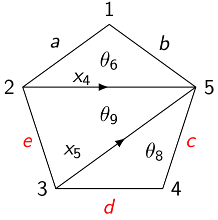
Answers (Spoiler Alert)

- Super λ-lengths live in $\mathbb{R}[x_1^{\pm\frac{1}{2}}, \cdots, x_1^{\pm\frac{1}{2}} | \theta_1, \cdots, \theta_{n+1}]$.
- **For the case of polygons,** there exists an ordering on the odd variables, called *positive ordering*, such that if we multiply θ's in the positive ordering then the coefficients are positive.

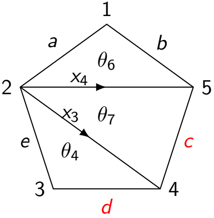
Super Ptolemy Relation - Example Continued

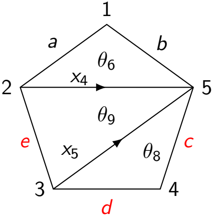
Before giving the general answer, we illustrate the result of flipping x_3 to x_5:





Super Ptolemy Relation - Example Continued





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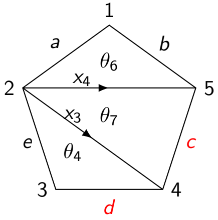
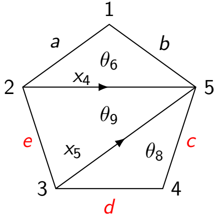
$$\theta_7 = \frac{\sqrt{ac}\,\theta_5 - \sqrt{bx_3}\,\theta_3}{\sqrt{x_2x_4}} =$$

$$\frac{1}{\sqrt{cx_3x_4}}\left(c\sqrt{\frac{ae}{x_1}}\theta_1 + ac\sqrt{\frac{d}{x_1x_2}}\theta_2 - x_3\sqrt{\frac{bc}{x_2}}\theta_3\right).$$

Super Ptolemy Relation - Example Continued





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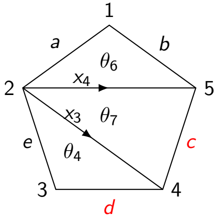

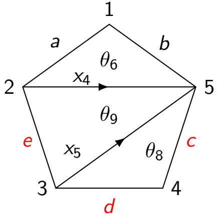

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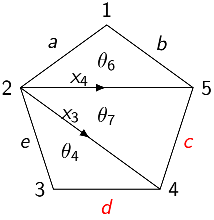
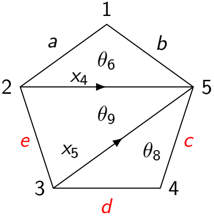
Super Ptolemy Relation - Example Continued





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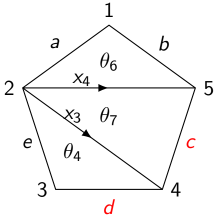

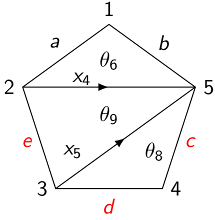

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Continuing with super-flips of x_4 and x_5, in order, yields x_1 and x_2, respectively.

A Combinatorial Interpretation using Snake Graphs

A **snake graph** is a planar graph consisting of a sequence of square tiles, each connected to either the top or right side of the previous tile.

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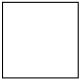
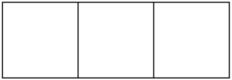
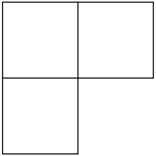
Given a snake graph G, the *word* of G, denoted $W(G)$, is a string in the alphabet $\{$R,U$\}$ (standing for "*right*" and "*up*") indicating how each tile is connected to the previous.



$W(G) = \varnothing$



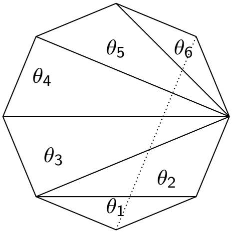
$W(G) = \mathrm{RR}$

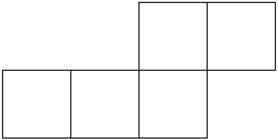


$W(G) = \mathrm{UR}$

A Combinatorial Interpretation using Snake Graphs

Every square tile in a snake graph represents two triangles in the triangulation. We label tiles with the odd variables of those triangles.





A Combinatorial Interpretation using Snake Graphs

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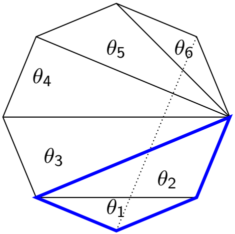

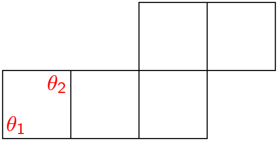

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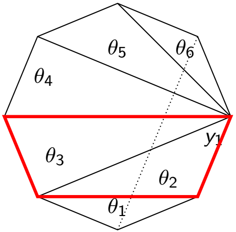

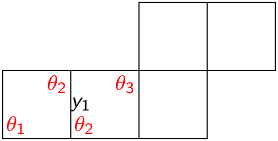

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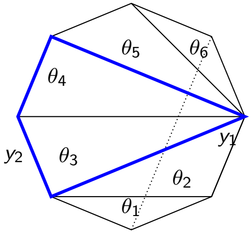

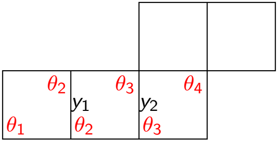

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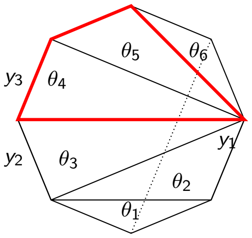

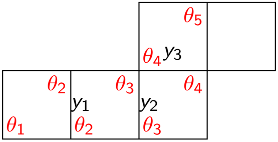

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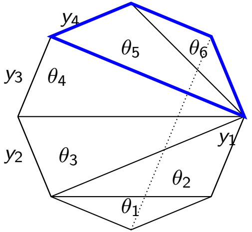

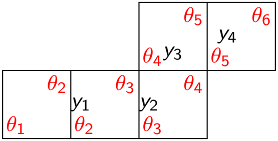

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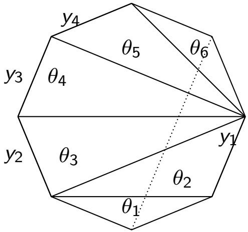
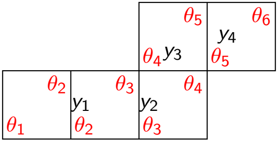


$$W(G) = RRUR$$



We built the snake graph from this triangulation traversing the dashed line from bottom-to-top, gluing tiles together based on boundary edges shared by adjacent quadrilaterals.

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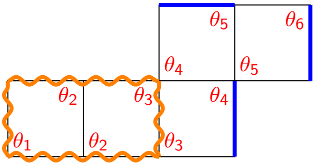

A **double dimer cover** of a graph is the union of two dimer covers. It is composed of cycles and doubled edges.

Dimer covers will be drawn as wavy orange lines, and double dimer covers will be drawn as straight blue lines.

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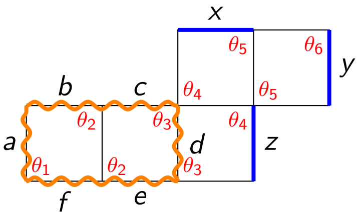

$$\text{weight} = xyz\sqrt{abcdef}\,\theta_1\theta_3$$

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The **weight** of a double dimer cover is the product of the square roots of the edge weights, times the odd variables at the beginning and end of cycles.

A Combinatorial Interpretation using Snake Graphs

Theorem (M-Ovenhouse-Zhang [MOZ22])

Consider a triangulation where f is the longest edge, we follow the construction of [MSW11] to build the snake graph G corresponding to the arc f.

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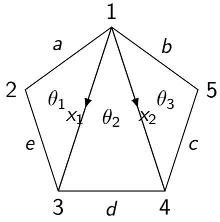
Additionally each cycle around tiles appearing in M contributes a weight of $\theta_i\theta_j$ to wt_θ, where θ_i and θ_j label the first and last triangles of that cycle

A Combinatorial Interpretation using Snake Graphs



$$\textit{Recall } \lambda_{2,5} = \frac{acx_1}{x_1x_2} + \frac{abd}{x_1x_2} + \frac{bex_2}{x_1x_2} + \frac{b\sqrt{ade}}{x_1\sqrt{x_2}}\theta_1\theta_2 + \frac{a\sqrt{bcd}}{\sqrt{x_1}x_2}\theta_2\theta_3 + \frac{\sqrt{abce}}{\sqrt{x_1x_2}}\theta_1\theta_3$$

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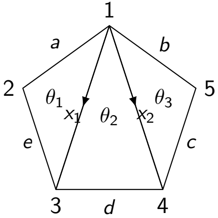

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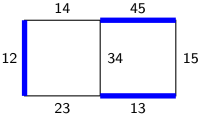
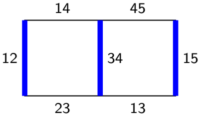


$\frac{acx_1}{x_1x_2}$



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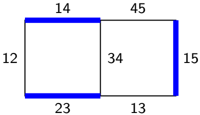

$\frac{bex_2}{x_1x_2}$

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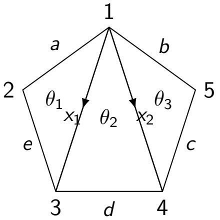

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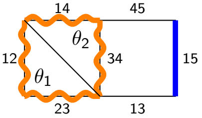

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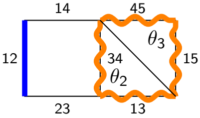

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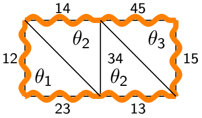

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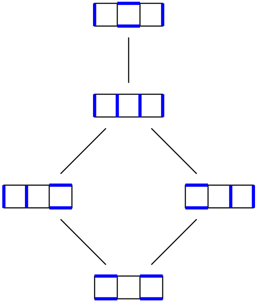
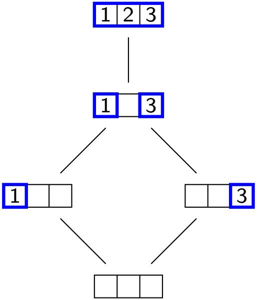
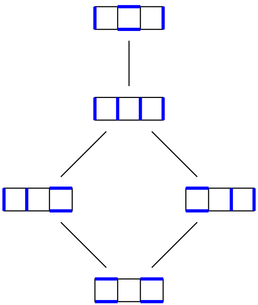
Lattice Structure in Dimer Case

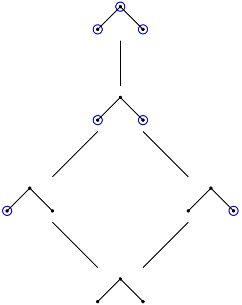




Superimpose the minimal dimer cover (but do not draw doubled edges) to see this is isomorphic to a lattice of subsets ordered under inclusion.

Lattice Structure in Dimer Case

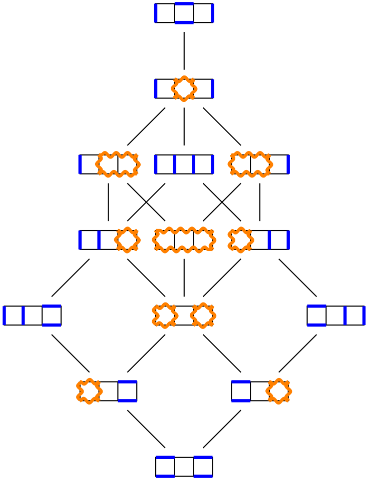




Lattice isomorphism

There is a poset isomorphism $L(G) \cong J(P(G))$, between the set of dimer covers on G and the lattice of lower order ideals in $P(G)$, the fence poset corresponding to the snake graph G.

Application: Lattice Structure in Double Dimer Case



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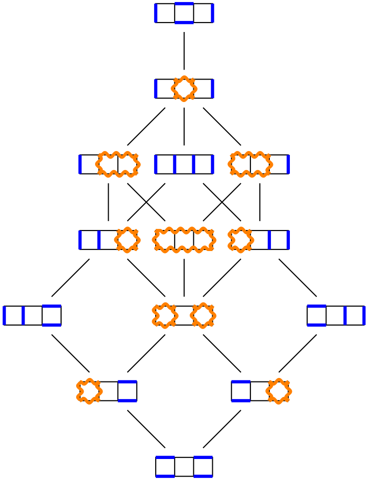
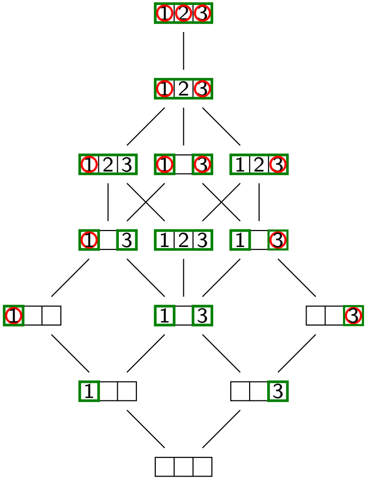
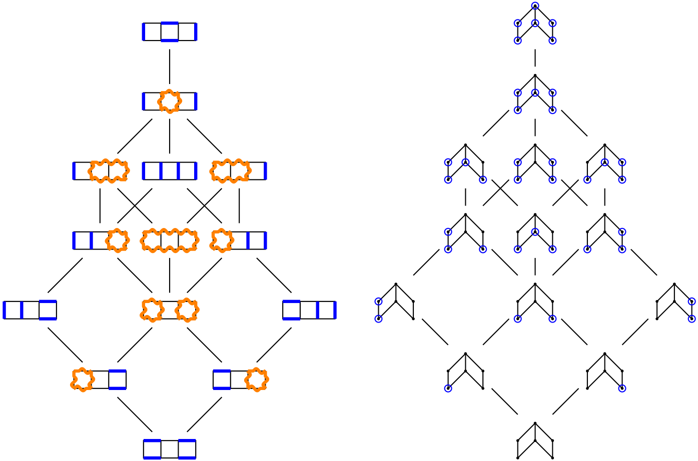




Application: Lattice Structure in Double Dimer Case

Theorem

There is a poset isomorphism $L(G) \cong J(\mathbb{P}(G))$, between double dimer covers on G and lower order ideals in $\mathbb{P}(G) := P(G) \times \{0, 1\}$.



Related Work: Super Caldero-Chapoton Map for Type A

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In the special case that Λ is the path algebra KQ of an acyclic type A_n quiver, then $\widetilde{\Lambda}$ is that path algebra (with relations) of the quiver with a loop ϵ_i at every vertex and relations $\epsilon_i^2 = 0$ and $\epsilon_i\alpha = \alpha\epsilon_j$ for arrows $i \to j$.

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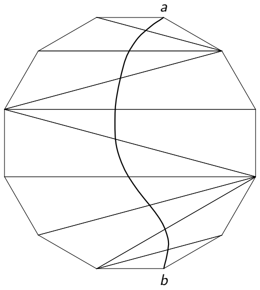
Motivated by how indecomposable rigid modules M_G of Λ in the ordinary type A case correspond to cluster variables and hence snake graphs G:

Theorem 5.18 of [ÇFES24]: The lattice of double dimer covers of snake graph G is in bijection with the submodule lattice of module $\widetilde{M_G}$ of $\widetilde{\Lambda}$.

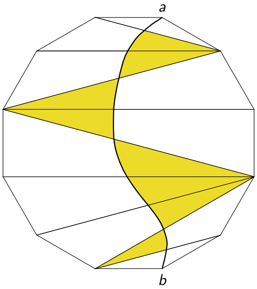
Theorem 6.6 of [ÇFES24]: The Caldero-Chapoton Formula $CC(\widetilde{(}M_G)$ adapted to involve square-root weights and odd elements associated to sectional paths (AR-triangles) matches the super lambda length formulas.

Features of Polygons: Fan Decomposition



For a triangulation T, we will define a canonical fan decomposition.

Features of Polygons: Fan Decomposition



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The arc (a, b) intersect with internal diagonals, and create smaller triangles (colored yellow).

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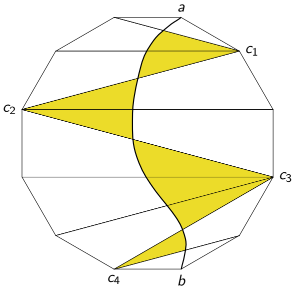

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Vertices of these yellow triangles are called fan centers, denoted $c_1, \cdots, c_n$, ordered by their distance from a. And we further denote $a = c_0$ and $b = c_{n+1}$.

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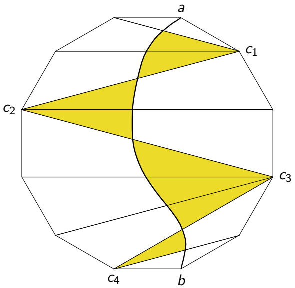

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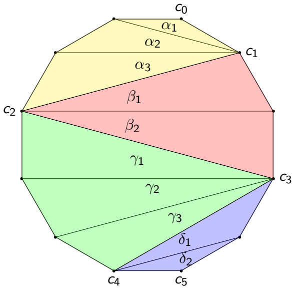
The sub-triangulation bounded by c_{i-1}, c_i, c_{i+1} is called the i-th fan segment of T.

Features: Default Orientation and Positive Ordering



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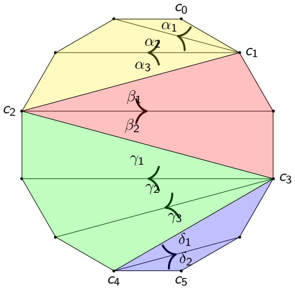

We define a default orientation on the interior diagonals.

- Edges inside each fan segment are directed away from the center.

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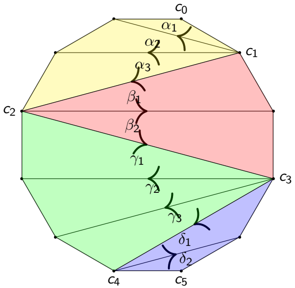

We define a default orientation on the interior diagonals.

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- Others are oriented as $c_1 \to c_2 \to \cdots \to c_n$.

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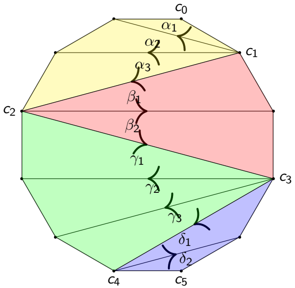

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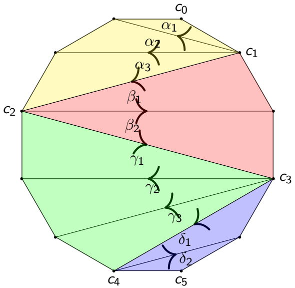

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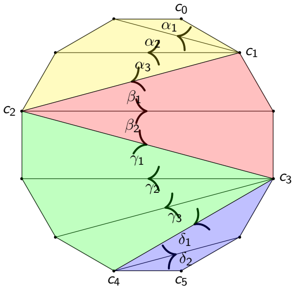

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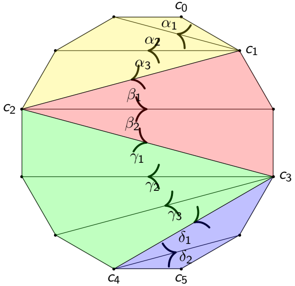
- "Alternate" across the fans.

Features: Default Orientation and Positive Ordering



We define a default orientation on the interior diagonals.

- Edges inside each fan segment are directed away from the center.
- Others are oriented as $c_1 \to c_2 \to \cdots \to c_n$.

We define a positive ordering on μ-invariants.

- μ-invariants in a fan are ordered counterclockwise around the center.
- "Alternate" across the fans.

$\delta_2 > \delta_1$

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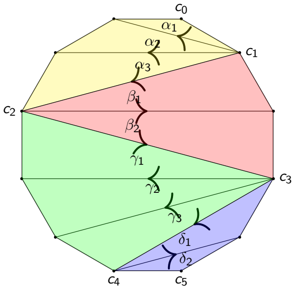

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$$\gamma_1 > \gamma_2 > \gamma_3 > \delta_2 > \delta_1$$

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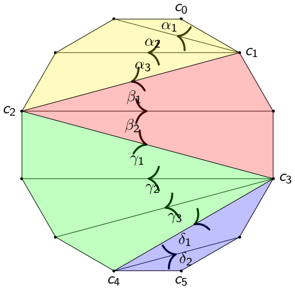

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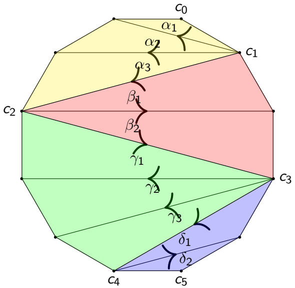

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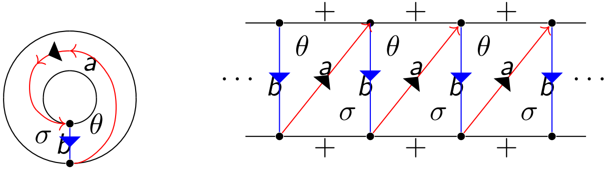
$$\alpha_1 > \alpha_2 > \alpha_3 > \gamma_1 > \gamma_2 > \gamma_3 > \delta_2 > \delta_1 > \beta_2 > \beta_1$$

Application: Super Fibonacci Numbers

Given a triangulation of an annulus, we consider the periodic mutation sequence $a, b, a, b, a, b, \ldots$ in the universal cover. (Arrows indicate default orientation and $\cdots > \sigma > \sigma > \theta > \theta > \cdots$ in positive ordering)



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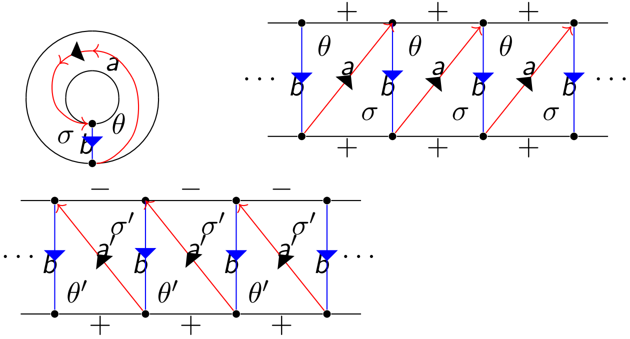

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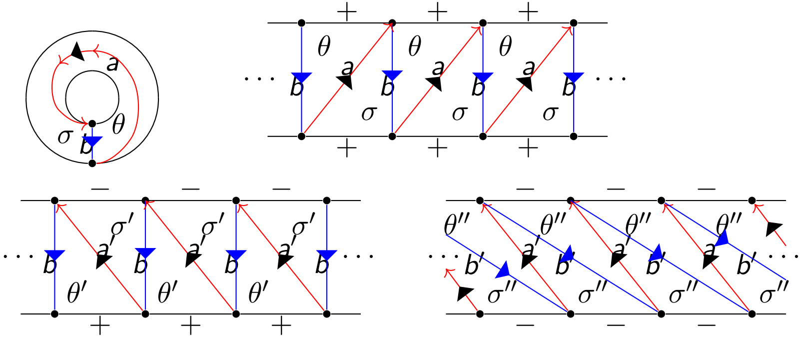

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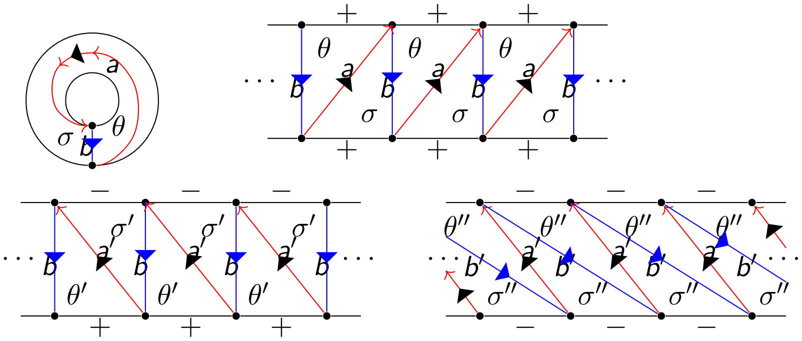

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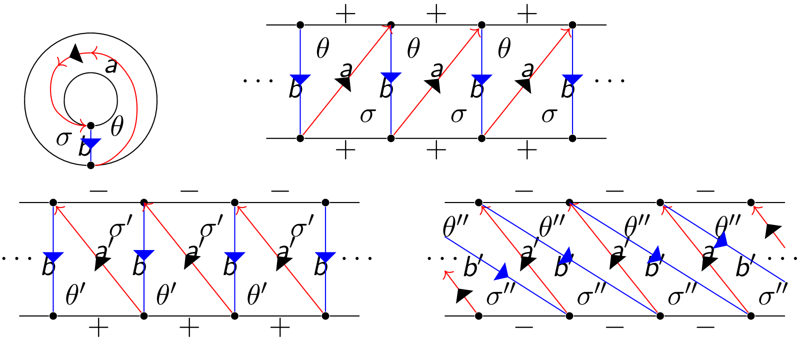
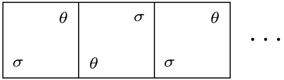


Since $\sigma\theta = \sigma'\theta' = \sigma''\theta'' = \dots$, if we let $\epsilon = \sigma\theta$, the Super Ptolemy Relation will always have the form $ef = a^2 + b^2 + ab\epsilon$. Thus letting $Z_1 = a$, $Z_2 = b$, we get the recurrence $Z_m Z_{m-2} = Z_{m-1}^2 + Z_{m-1}\epsilon + 1$ for the resulting infinite sequence of super λ-lengths.

Application: Super Fibonacci Numbers

Letting G_m denote the snake graph for the word $W(G) = RR \ldots R$, i.e. with m tiles in a horizontal row, where all edges have weight 1, and all tiles alternate between the same two μ-invariants σ and θ



we obtain that the Z_m's are the double dimer partition functions for the snake graphs G_{2m-5}.

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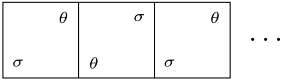

we obtain that the Z_m's are the double dimer partition functions for the snake graphs G_{2m-5}.

Further, when we initialize $Z_1 = a = 1$ and $Z_2 = b = 1$, we get for $m \geq 3$

$$Z_m = F_{2m-3} + \left(\sum_{k=0}^{m-3} (2k+1) \binom{m+k-1}{2k+2} \right) \epsilon,$$

where F_k is the kth Fibonacci number such that $F_1 = F_2 = 1$.

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Examples:

$$Z_3 = 2 + \epsilon$$

$$Z_4 = 5 + 6\epsilon$$

$$Z_5 = 13 + 26\epsilon$$

$$Z_6 = 34 + 97\epsilon$$

Application: Super Fibonacci Numbers

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We also can let $W_m = F_{2m-2} + \left(\sum_{k=0}^{m-3} (2k) \binom{m+k-2}{2k+1} \right) \epsilon$, which is the double dimer partition function for G_{2m-4}.

Examples: `https://oeis.org/A054454`

$$Z_3 = 2 + \epsilon$$
$$W_3 = 3 + 2\epsilon$$
$$Z_4 = 5 + 6\epsilon$$
$$W_4 = 8 + 12\epsilon$$
$$Z_5 = 13 + 26\epsilon$$
$$W_5 = 21 + 50\epsilon$$
$$Z_6 = 34 + 97\epsilon$$

Application: Super Fibonacci Numbers

Question

If we let $W_1 = W_2 = 1$ (or if we let $W_1 = a$ and $W_2 = b$), and set W_m to be the double dimer partition function of G_{2m-4}, then what does W_m correspond to in the context of the decorated super-Teichmüller space?

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We will demonstrate that the W_m's are the super λ-lengths of a peripheral arc in an annulus that wind around $(m-2)$ times, see [MOZ23, Remark 6.3], but this will utilize a description using super matrices.

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Related Work: V. Ovsienko [Ovs23] defines **shadow sequences** including Shadow Fibonacci Numbers of the form $F_m + G_m\epsilon$ where

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We note that if we write our Super Fibonacci Numbers as $X_m + Y_m\epsilon$, e.g.

$$2+\epsilon, 3+2\epsilon, 5+6\epsilon, 8+12\epsilon, 13+26\epsilon, 21+50\epsilon, 34+97\epsilon,$$

then his G_m's equal our $(X_m + Y_m - 1)$ or $(X_m + Y_m)$, up to parity.

Super Markov Numbers

Given an oriented triangulation of the once-punctured torus, and allowing flips in all three directions, the resulting λ-lengths of such arcs correspond to the Markov numbers, which are the integer solutions to the Diophantine equation $x^2 + y^2 + z^2 = 3xyz$.

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Such numbers, and their triples (x, y, z) are of interest in number theory including in the theory of quadratic forms. Markov numbers also count dimer covers in snake graphs as described in [Pro05, Sec. 7] or [RS20].

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In [HPZ19], Huang, Penner, and Zeitlin studied **Super** Decorated Teichmüller space associated to the once-punctured torus and showed that **super** lambda lengths on this surface satisfy

$$x^2 + y^2 + z^2 + (xy + yz + xz)\sigma\theta = 3(1 + \sigma\theta)xyz.$$

Question: Do they have combinatorial interpretations using double dimer covers of the snake graphs appearing in Section 7 of [Propp 2005] in the presence of appropriately specialized μ-invariants?

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Note also that if we fix $z = 1$, the triples $(x, y, 1)$ involve Super-Fibonacci numbers as previously studied, but we also get other triples such as:

$$(2 + \sigma\theta,\ 5 + 6\sigma\theta,\ 29 + 74\sigma\theta),\quad (2 + \sigma\theta,\ 29 + 74\sigma\theta,\ 169 + 668\sigma\theta),$$

The Supergroup OSp(1|2)

Consider the set of $2|1 \times 2|1$ super matrix over $\mathbb{R}$:

$$M = \left(\begin{array}{cc|c} a & b & \gamma \\ c & d & \delta \\ \hline \alpha & \beta & e \end{array}\right)$$

Let J denote the following matrix

$$J = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right),$$

then elements in the group OSp(1|2) can be realized as $2|1 \times 2|1$ supermatrices g with $\mathrm{Ber}(g) = 1$ satisfying the relation

$$g^{\mathrm{st}} J g = J$$

where g^{st} denotes the super-transpose of g and $Ber(g)$ is the Berezinian, a super-analogue of the determinant.

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These constraints $Ber(g) = 1$ and $g^{\mathrm{st}} J g = J$ can also be written down explicitly as the following system of equations.

$$e = 1 + \alpha\beta \tag{1}$$

$$e^{-1} = ad - bc \tag{2}$$

$$\alpha = c\gamma - a\delta \tag{3}$$

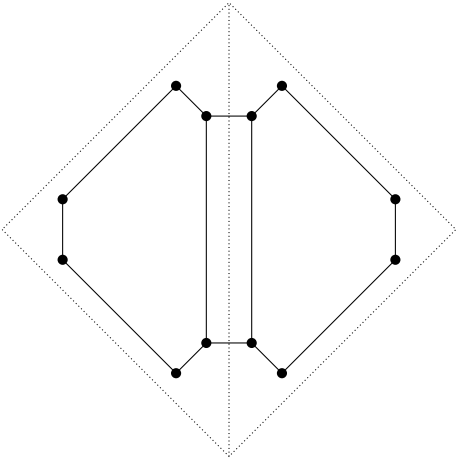
$$\beta = d\gamma - b\delta \tag{4}$$

$$\gamma = a\beta - b\alpha \tag{5}$$

$$\delta = c\beta - d\alpha \tag{6}$$

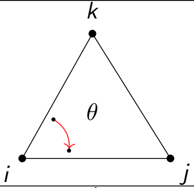
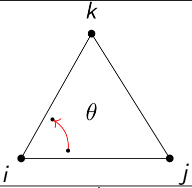
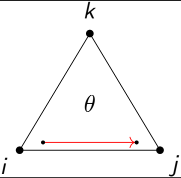
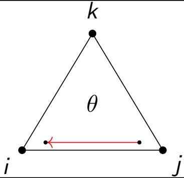
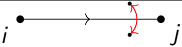
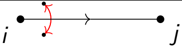
Flat OSp(1|2) Connection

Given a triangulation T, we build a planar graph Γ_T on top of it and then define a flat connection on Γ_T using these super matrices.



1. $A(h^i_{jk}|\theta) =$
$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ -h & 1 & \sqrt{h}\theta \\ \hline -\sqrt{h}\theta & 0 & 1 \end{array}\right]$$ where
$$h = h^i_{jk} = \frac{\lambda_{jk}}{\lambda_{ij}\lambda_{ik}}.$$
2. $$E(\lambda_{ij}) = \left[\begin{array}{cc|c} 0 & -\lambda_{ij} & 0 \\ \lambda_{ij}^{-1} & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right].$$
3. $$\rho = \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right].$$

Flat OSp(1|2) Connection

Type (i)		$A\left(h^{i}_{jk}\|\theta\right)^{-1}$		$A\left(h^{i}_{jk}\|\theta\right)$
Type (ii)		$E(\lambda_{ij})^{-1}$		$E(\lambda_{ij})$
Type (iii)		ρ		id

Matrix Interpretations of super λ-lengths and μ-invariants

Theorem 4.3 of [MOZ23]: Let T be a generic triangulation endowed with an orientation (spin structure) and fan centers labeled as $c_0, c_1, \ldots, c_{N+1}$, with $a = c_0$ and $b = c_{N+1}$. Then the holonomy matrix for the connection from a to b is $H_{a,b} =$

$$\left(\begin{array}{cc|c} -\frac{\lambda_{c_1,c_{N+1}}}{\lambda_{c_0,c_1}} & (-1)^{\epsilon_a}\lambda_{c_0,c_{N+1}} & \nabla^{c_{N+1}}_{c_0,c_1} \\ (-1)^{\epsilon_b}\frac{\lambda_{c_1c_N}}{\lambda_{c_0,c_1}\lambda_{c_Nc_{N+1}}} & (-1)^{\epsilon_a+\epsilon_b-1}\frac{\lambda_{c_0c_N}}{\lambda_{c_Nc_{N+1}}} & (-1)^{\epsilon_b-1}\frac{1}{\lambda_{c_Nc_{N+1}}}\nabla^{c_N}_{c_0,c_1} \\ \hline \frac{1}{\lambda_{c_0,c_1}}\nabla^{c_1}_{c_N,c_{N+1}} & (-1)^{\epsilon_a-1}\nabla^{c_0}_{c_N,c_{N+1}} & 1+\star \end{array}\right)$$

Here the formula for the $(3,3)$-entry (i.e. $1+\star$) can be given as

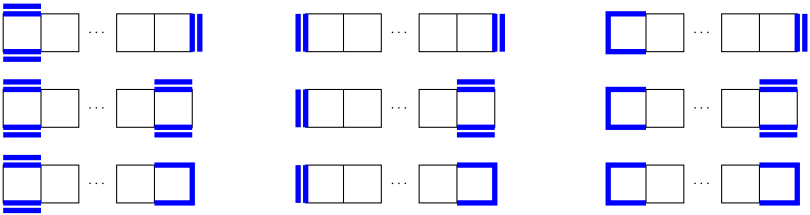
$$1+\star = 1+(-1)^{\epsilon_a-1}\frac{1}{\lambda_{c_0,c_1}}\nabla^{c_1}_{c_N,c_{N+1}}\nabla^{c_0}_{c_N,c_{N+1}} = 1+(-1)^{\epsilon_b-1}\frac{1}{\lambda_{c_Nc_{N+1}}}\nabla^{c_{N+1}}_{c_0,c_1}\nabla^{c_N}_{c_0,c_1}$$

and the signs $(-1)^{\epsilon_a}$ and $(-1)^{\epsilon_b}$ depend on the orientations of initial and final triangles (c_0, c_1, c_2) and (c_{N-1}, c_N, c_{N+1}).

Applications to Super Fibonacci Numbers

The Holonomy Matrix $H_n = \left(E^{-1}A_\theta^{-1}\rho E A_\sigma \rho\right)^{n-2} E^{-1}$ for the bridging arc winding around the annulus equals $H_n =$

$$\left(\begin{array}{cc|c} -W_{n-1} & Z_n & (Z_n - 1)\sigma + W_{n-1}\theta \\ -Z_{n-1} & W_{n-1} & (Z_{n-1} - 1)\theta + W_{n-1}\sigma \\ \hline (Z_{n-1} - 1)\sigma - W_{n-1}\theta & (Z_n - 1)\theta - W_{n-1}\sigma & 1 - (\ell_{2n-4} - 2)\sigma\theta \end{array}\right)$$



where ℓ_k denotes the kth Lucas number defined by $\ell_1 = 1$, $\ell_2 = 3$, and $\ell_k = \ell_{k-1} + \ell_{k-2}$ for $k \geq 3$.

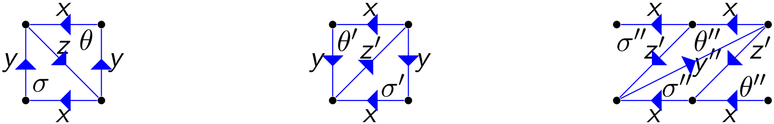
This quantity $\ell_{2n-4} - 2$ also equals $\kappa(W_{n-2})$, the number of spanning trees on the wheel graph with $(n-1)$ vertices.

Applications to Super Markov Numbers

We can also define holonomy matrices for arcs on the once-punctured torus to directly compute Super Markov Numbers.

Lifting the torus to its universal cover (the Euclidean plane), arcs on the torus correspond to lines of rational slope, plus the vertical line of slope $\frac{1}{0}$.

We start with the standard initial triangulation with lines of slope $\frac{0}{1}, \frac{1}{0}, \frac{-1}{1}$ and then apply a sequence of diagonal flips (w.l.o.g. we assume our first two flips are at $\frac{-1}{1} \to \frac{1}{1}$, and then $\frac{1}{0} \to \frac{1}{2}$).



From the triple of arcs of slopes $\{\frac{0}{1}, \frac{1}{1}, \frac{1}{2}\}$, assuming we never backtrack, we use the Farey fraction rule

$$\left\{\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right\} \to \left\{\frac{a}{b}, \frac{a+e}{b+f}, \frac{e}{f}\right\} \text{ or } \to \left\{\frac{c+e}{d+f}, \frac{c}{d}, \frac{e}{f}\right\}.$$

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Given $0 < \frac{p}{q} \leq 1$ with $\gcd(p, q) = 1$, we draw γ of slope p/q on the universal cover of the torus.

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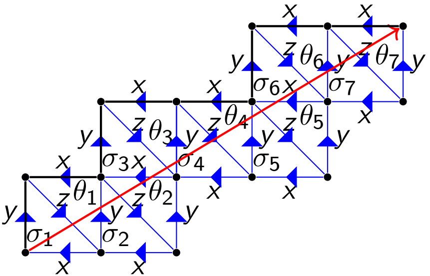
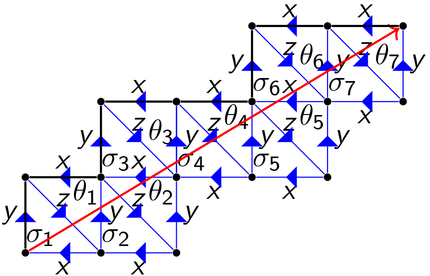
Example: If $p/q = 3/5$, we get $(a_0, a_1, \ldots, a_8) = (0, 5, 2, 7, 4, 1, 6, 3, 0)$ and hence the upper Christoffel word is 10100100, corresponding to the lattice path given by $NENEENEE$, lying just above the arc γ of slope $3/5$.

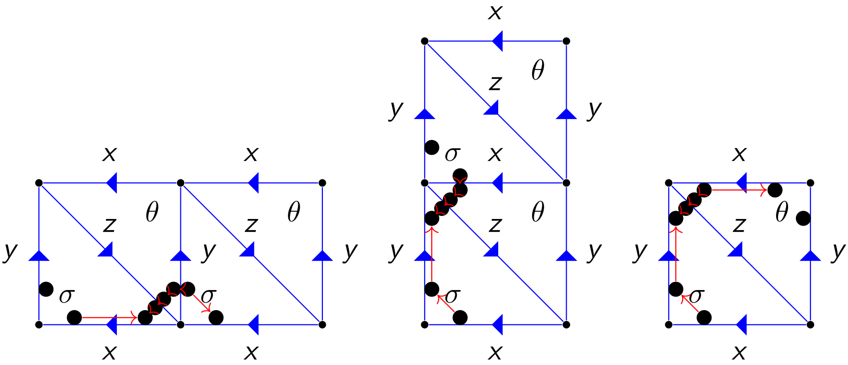


The upper Christoffel word of length $(p+q)$ is $w_1 w_2 \cdots w_{p+q}$ via the characteristic function $w_n = \chi(a_{n-1} < a_n)$.

This determines a lattice path from $(0,0)$ to (p,q) so that the nth step is a north step N (resp. east step E) whenever $a_{n-1} < a_n$ (resp. $a_{n-1} > a_n$), using the residues $a_n = nq \mod (p+q)$ for $0 \leq n \leq p+q$.

Example: If $p/q = 3/5$, we get $(a_0, a_1, \ldots, a_8) = (0,5,2,7,4,1,6,3,0)$ and hence the upper Christoffel word is 10100100, corresponding to the lattice path given by $NENEENEE$, lying just above the arc γ of slope $3/5$.





Following a canonical path based on the upper Christoffel word yields holonomy matrix $H_{\frac{p}{q}}$ so that the Super Markov Number is the $(1, 2)$-entry.

The first (resp. second) path corresponds to $w_i = 0$ (resp. $w_i = 1$), while the third is the canonical path in the final quadrilateral.

Application to Super Markov Numbers

If $p/q = 1/2$, the Upper Christoffel word is 100 and the corresponding Holonomy matrix is

$$\begin{bmatrix} 1 & 2+\sigma\theta & -\sigma+\theta \\ -1 & -1 & \sigma \\ -\sigma & -\sigma-\theta & 1-\sigma\theta \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 3+2\sigma\theta & \theta \\ 0 & \theta & 1 \end{bmatrix} = \begin{bmatrix} 2+\sigma\theta & 5+6\sigma\theta & -\sigma+3\theta \\ -1 & -2-\sigma\theta & \sigma-\theta \\ -\sigma-\theta & -3\sigma-\theta & 1-2\sigma\theta \end{bmatrix}$$

which has $(1,2)$-entry $5+6\sigma\theta = SM_{1/2}$.

If $p/q = 2/3$, the Upper Christoffel word is 10100 and the corresponding Holonomy matrix is

$$\begin{bmatrix} 1 & 2+\sigma\theta & -\sigma+\theta \\ -1 & -1 & \sigma \\ -\sigma & -\sigma-\theta & 1-\sigma\theta \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 3+2\sigma\theta & \theta \\ 0 & \theta & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -\sigma \\ 1 & 2+\sigma\theta & -\sigma+\theta \\ -\theta & -\sigma-\theta & 1-\sigma\theta \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 3+2\sigma\theta & \theta \\ 0 & \theta & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 12+22\sigma\theta & 29+74\sigma\theta & -8\sigma+20\theta \\ -5-6\sigma\theta & -12-22\sigma\theta & 4\sigma-8\theta \\ -8\sigma-4\theta & -20\sigma-8\theta & 1-16\sigma\theta \end{bmatrix}$$

with $(1,2)$-entry $29+74\sigma\theta = SM_{2/3}$.

Application to Super Markov Numbers

For our running example of $p/q = 3/5$, we get truncated Christoffel word 010010 and holonomy matrix

$$\begin{bmatrix} 179 + 706\sigma\theta & 433 + 2032\sigma\theta & -112\sigma + 303\theta \\ -74 - 237\sigma\theta & -179 - 706\sigma\theta & 47\sigma - 125\theta \\ -125\sigma - 47\theta & -303\sigma - 112\theta & 1 - 241\sigma\theta \end{bmatrix}.$$

The $(1,2)$-entry is $SM_{3/5} = 433 + 2032\sigma\theta$.

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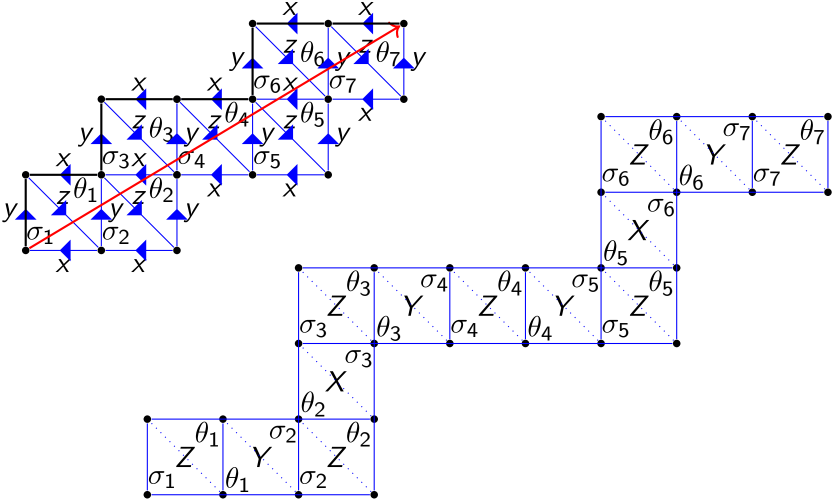
$SM_{p/q} = M_{p/q} + \hat{M}_{p/q}\sigma\theta$, where $M_{p/q}$ is the number of dimer covers on snake graph $G_{p/q}$ (equiv. double dimer covers with only doubled edges), and $\hat{M}_{p/q}$ is the signed enumeration of double dimer covers on $G_{p/q}$ with exactly one cycle, necessarily of odd length.

Combinatorics of Super Markov Numbers

Use the upper Christoffel word to build G_γ corresponding to the arc γ cutting through the universal cover of the once-punctured torus.



Combinatorics of Super Markov Numbers

Theorem in Progress (Musiker 2025): $SM_{p/q} = M_{p/q} + \hat{M}_{p/q}\sigma\theta$, where $\hat{M}_{p/q}$ is the signed enumeration of double dimer covers on $G_{p/q}$ containing a single cycle of odd length as follows:

($\#$ containing a single cycle whose leftmost tile is X and whose rightmost tile is also X)

$+$ ($\#$ containing a single cycle whose leftmost tile is Y and whose rightmost tile is also Y)

$+$ ($\#$ containing a single cycle whose leftmost tile is Z and whose rightmost tile is also Z)

- ($\#$ containing a single cycle whose leftmost tile is X but whose rightmost tile is Y)

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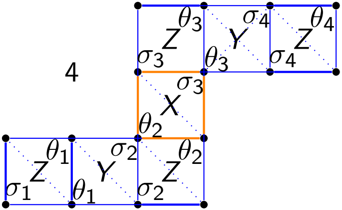
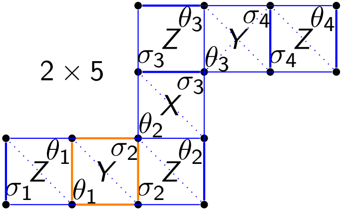
Combinatorics of Super Markov Numbers

Example of double dimer covers contributing positively to $SM_{2/3} = M_{2/3} + \hat{M}_{2/3}\sigma\theta = 29 + 74\sigma\theta = 29 + (78-4)\sigma\theta$, which up to flips of doubled edges, break into classes as $78\sigma\theta = (4+5+5+12+12+10+10+1+5+4+5+2+2+1)\sigma\theta$.

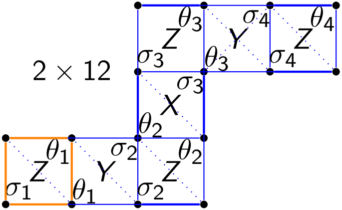
4



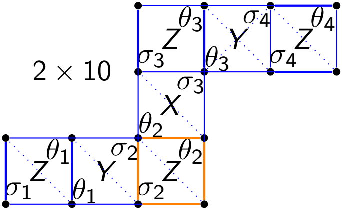
2×5



2×12



2×10



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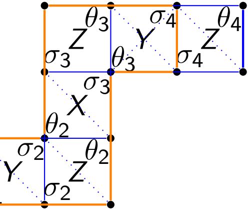
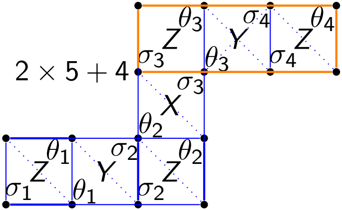
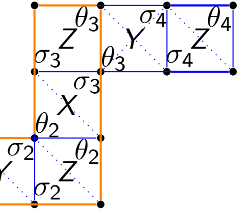
1



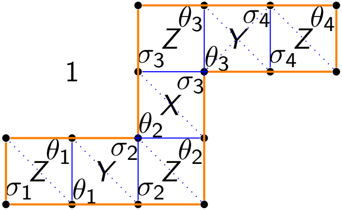
$2 \times 5 + 4$



2×2



1



The four double dimer covers contributing negatively to
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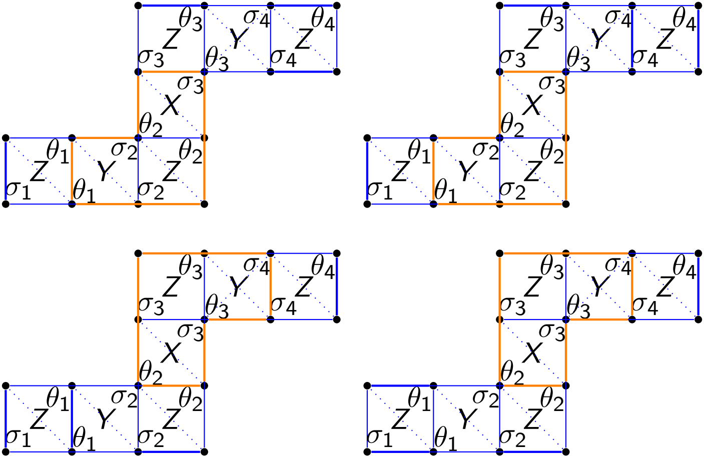
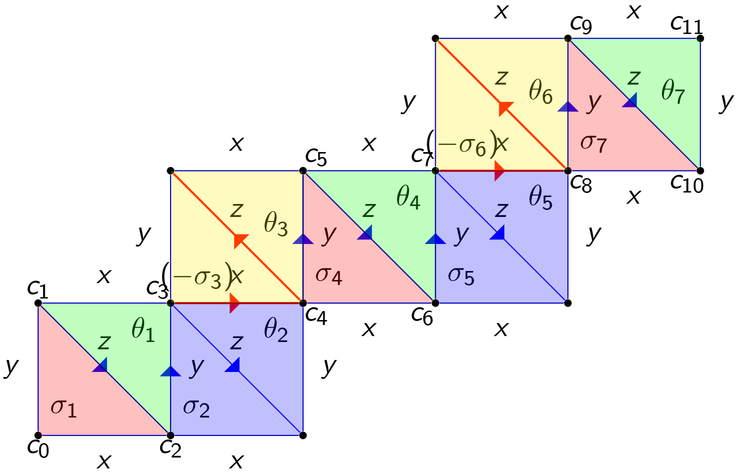


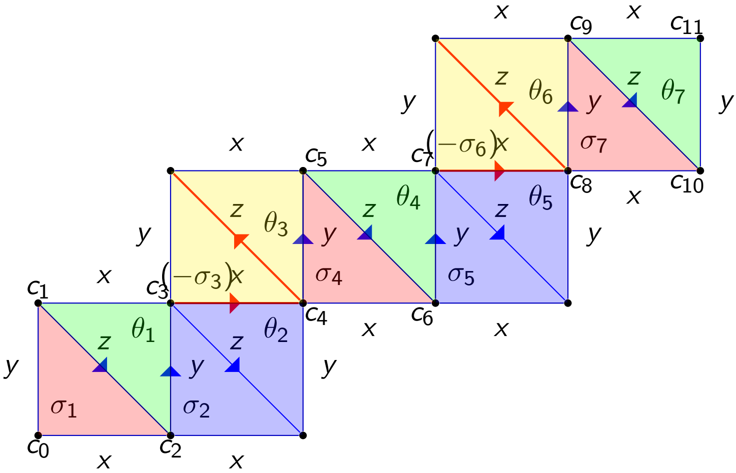
Nuances in the Super Markov Proof

Unlike the disk or annulus (Super Fibonacci) examples, we cannot use a default orientation on the universal cover because it would not be consistent when we project down to the torus.



Nuances in the Super Markov Proof

Instead we use cyclic orientation of the two triangles on the torus and lift that up to the universal cover. Then for a given arc γ, we must apply equivalences triangle-by-triangle thereby negating some μ-invariants.



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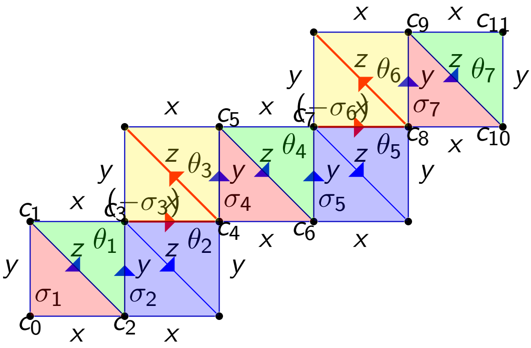

This also affects the positive ordering. For arc $\gamma = \gamma_{3/5}$, the upper Christoffel word is 10100100, which has non-initial 1's in positions 3 and 6. The positive ordering associated to this default orientation is

$$\sigma_1 > \sigma_2 > \theta_2 > \sigma_4 > \sigma_5 > \theta_5 > \sigma_7 > \theta_7 > \theta_6 > (-\sigma_6) > \theta_4 > \theta_3 > (-\sigma_3) > \theta_1.$$

Open Questions

Observation: In all examples done thus far, even when plugging in x, y, and z, the Super Markov Numbers only seem to have positive coefficients despite the above signed formulas.

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Observation: In all examples done thus far, even when plugging in x, y, and z, the Super Markov Numbers only seem to have positive coefficients despite the above signed formulas.

Question: Does this positivity always continue for the super λ-lengths on the once-punctured torus?

Question: If so, is there an adaptation of the above combinatorial interpretation via signed enumeration that would require only a positive expansion formula? In particular, can one further restrict which double dimer covers contribute to the partition function (similar to how Lindstrom-Gessel-Viennot often allows one to interpret the signed expansion formula of a matrix determinant as a positive generating function for non-intersecting lattice paths) therefore manifesting positivity with no cancellation of signed terms needed?

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Question: What about other surfaces like annuli with more than one marked point on each of boundary or once-punctured disks?

Open Questions

Question: What is the relationship between our family of Super Markov numbers and Ovsienko's shadow Markov sequence [Ovs23]?

Example:

$$2 + 4\epsilon \text{ vs } SM_{1/1} = 2 + \sigma\theta$$

$$5 + 13\epsilon \text{ vs } SM_{1/2} = 5 + 6\sigma\theta$$

$$13 + 40\epsilon \text{ vs } SM_{1/3} = 13 + 26\sigma\theta$$

$$34 + 120\epsilon \text{ vs } SM_{1/4} = 34 + 97\sigma\theta$$

$$89 + 354\epsilon \text{ vs } SM_{1/5} = 89 + 332\sigma\theta$$

$$29 + 117\epsilon \text{ vs } SM_{2/3} = 29 + 74\sigma\theta$$

$$169 + 921\epsilon \text{ vs } SM_{3/4} = 169 + 688\sigma\theta$$

$$194 + 976\epsilon \text{ vs } SM_{2/5} = 194 + 801\sigma\theta$$

$$433 + 2592\epsilon \text{ vs } SM_{3/5} = 433 + 2032\sigma\theta$$

Thanks for Listening! References I

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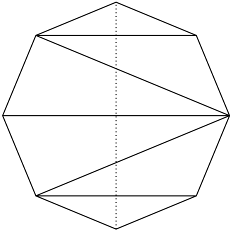
Thanks to the support of the NSF grants DMS-1745638 and DMS-1854162, as well as the University of Minnesota UROP program.

http://www-users.math.umn.edu/∼musiker/Paris25.pdf

What about odd variables?

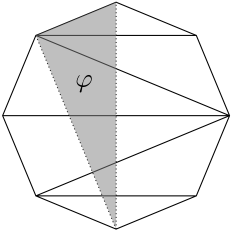
Consider an arc γ as before.



What about odd variables?

Consider an arc γ as before.

Let φ be a triangle with γ as a side, and also a boundary side.

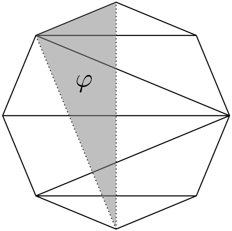


What about odd variables?

Consider an arc γ as before.

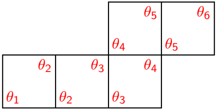
Let φ be a triangle with γ as a side, and also a boundary side.

Can we express the μ-invariant θ_φ in terms of the initial triangulation?



The Toggle Involution

Recall that snake graphs are labelled with odd variables.



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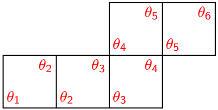

If θ_n is the label on the upper-right of the last tile, define an involution $x \mapsto x^*$ on monomials which adds/removes θ_n.

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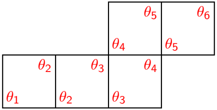
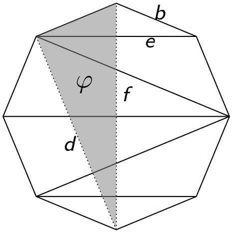


If θ_n is the label on the upper-right of the last tile, define an involution $x \mapsto x^*$ on monomials which adds/removes θ_n.

Examples:

$$(\theta_1\theta_2)^* = \theta_1\theta_2\theta_6, \qquad (\theta_4\theta_6)^* = \theta_4$$

Formula for Odd Variables



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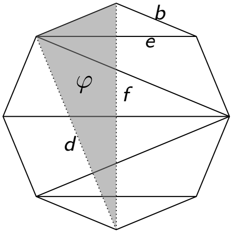
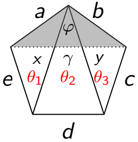


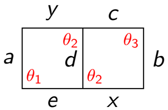
Theorem [M-Ovenhouse-Zhang 2021]

$$\sqrt{df}\ \theta_{\varphi} = \frac{1}{\mathrm{cross}(f)} \frac{\sqrt{e}}{\sqrt{b}} \sum_{M \in D_t(G_f)} \mathrm{wt}(M)^*$$

where D_t is the set of double dimer covers using the **top** edge of the last tile (as long as the polygon has an odd number of triangles; otherwise use the **right** edge on the last tile instead).

Example of μ-invariant Formula





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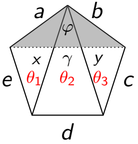
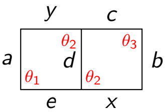
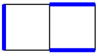
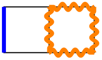
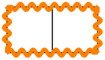




$D_t(G)$:







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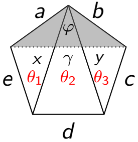

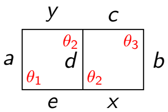

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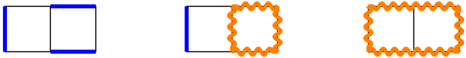

$$\sum_{M \in D_t(G)} \mathrm{wt}(M) = \quad acx \quad + a\sqrt{bcdx}\,\theta_2\theta_3 + \sqrt{abcexy}\,\theta_1\theta_3$$

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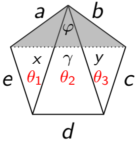

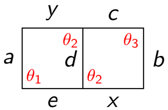

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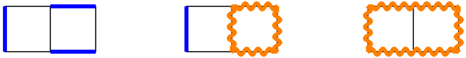
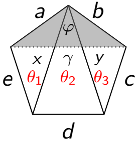
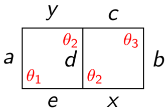


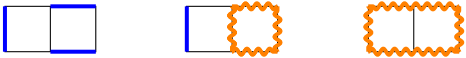
$$\sum_{M \in D_t(G)} \mathrm{wt}(M) = \quad acx \quad + a\sqrt{bcdx}\,\theta_2\theta_3 + \sqrt{abcexy}\,\theta_1\theta_3$$

$$\sum_{M \in D_t(G)} \mathrm{wt}(M)^* = \quad acx\,\theta_3 \quad + \; a\sqrt{bcdx}\,\theta_2 \; + \; \sqrt{abcexy}\,\theta_1$$





$D_t(G)$:

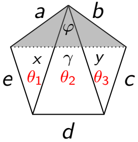


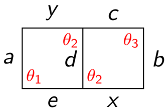
$$\sum_{M\in D_t(G)} \mathrm{wt}(M) = acx + a\sqrt{bcdx}\,\theta_2\theta_3 + \sqrt{abcexy}\,\theta_1\theta_3$$

$$\sum_{M\in D_t(G)} \mathrm{wt}(M)^* = acx\,\theta_3 + a\sqrt{bcdx}\,\theta_2 + \sqrt{abcexy}\,\theta_1$$

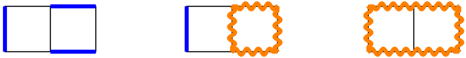
$$\frac{\sqrt{y}}{\sqrt{c}} \sum_{M\in D_t(G)} \mathrm{wt}(M)^* = ax\sqrt{cy}\,\theta_3 + a\sqrt{bdxy}\,\theta_2 + y\sqrt{abex}\,\theta_1$$

Example of μ-invariant Formula





$D_t(G)$:



$$\sum_{M\in D_t(G)} \mathrm{wt}(M) = acx + a\sqrt{bcdx}\,\theta_2\theta_3 + \sqrt{abcexy}\,\theta_1\theta_3$$

$$\sum_{M\in D_t(G)} \mathrm{wt}(M)^* = acx\,\theta_3 + a\sqrt{bcdx}\,\theta_2 + \sqrt{abcexy}\,\theta_1$$

$$\frac{\sqrt{y}}{\sqrt{c}} \sum_{M\in D_t(G)} \mathrm{wt}(M)^* = ax\sqrt{cy}\,\theta_3 + a\sqrt{bdxy}\,\theta_2 + y\sqrt{abex}\,\theta_1$$

$$\sqrt{a\gamma}\,\theta_\varphi = \frac{1}{xy}\Big(ax\sqrt{cy}\,\theta_3 + a\sqrt{bdxy}\,\theta_2 + y\sqrt{abex}\,\theta_1\Big)$$