A new characterization for the m-quasiinvariants of S_n and explicit basis for two row hook shapes

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January 30, 2008

Abstract

In 2002, Feigin and Veselov [4] defined the space of m-quasiinvariants for any Coxeter group, building on earlier work of [2]. While many properties of those spaces were proven in [3, 4, 5, 7] from this definition, an explicit computation of a basis was only done in certain cases. In particular, in [4], bases for m-quasiinvariants were computed for dihedral groups, including S_3, and Felder and Veselov [5] also computed the non-symmetric m-quasiinvariants of lowest degree for general S_n. In this paper, we provide a new characterization of the m-quasiinvariants of S_n, and use this to provide a basis for the isotypic component indexed by the partition $[n-1,1]$. This builds on a previous paper, [1], in which we computed a basis for S_3 via combinatorial methods.

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1 Introduction

A permutation $\sigma \in S_n$ acts on a polynomial in $\mathbf{R} = \mathbb{Q}[x_1, \ldots, x_n]$ by permutation of indices:

$$\sigma P(x_1, \ldots, x_n) = P(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

The S_n-invariant polynomials are known as symmetric functions, and denoted by Λ_n. It is well known that Λ_n is generated by the elementary symmetric functions $\{e_1, \ldots, e_n\}$ where

$$e_j = \sum_{i_1 < i_2 < \cdots < i_j} x_{i_1} \ldots x_{i_j}.$$

The ring of coinvariants of S_n is the quotient

$$\mathbf{R}/\langle e_1, \ldots, e_n \rangle.$$

As an S_n-module, the ring of coinvariants is known to be isomorphic to the left regular representation. It is also known that $\mathbf{R}$ is free over Λ_n which implies that if we choose a basis $\mathcal{B} = \{b_1, \ldots, b_{n!}\}$ for the ring of coinvariants, any element of $P \in \mathbf{R}$ has a unique expansion

$$P = \sum_{i=1}^{n!} b_i f_i$$

where the f_i are symmetric functions. More information is given by the Hilbert series for the isotypic component of $\mathbf{R}$ corresponding to λ, namely

$$\frac{\sum_{T \in ST(\lambda)} f_\lambda \; q^{cocharge(T)}}{(1-q)(1-q^2)\ldots(1-q^n)}.$$

Known bases for the ring of coinvariants with very combinatorial descriptions include the Artin monomials and the descent monomials.

In [2, 4], Chalykh, Feigin and Veselov introduced a generalization of invariance known as "m-quasiinvariance". For the symmetric group the m-quasiinvariants are the polynomials $P \in \mathbb{Q}[x_1, \ldots, x_n]$ which have the divisibility property

$$(x_i - x_j)^{2m+1} \;\Big|\; \Big(1 - (i,j)\Big) P$$

for every transposition (i, j). We set

$$\mathbf{QI}_m = \{m\text{-quasiinvariants of } S_n\}.$$

The m-quasiinvariants of S_n form a ring and an S_n module, and we have the following containments:

$$\mathbf{R} = \mathbf{QI}_0 \supset \mathbf{QI}_1 \supset \cdots \supset \mathbf{QI}_m \supset \cdots \supset \Lambda_n.$$

For all m, the ring of coinvariants $\mathbf{QI}_m/\langle e_1, \ldots, e_n \rangle$ was conjectured in [4], and proved in [3], to be isomorphic as an S_n-module to the left regular representation. In fact, Etingof and Ginzburg further proved that $\mathbf{QI}_m$ is free over the symmetric functions. The Hilbert series of the isotypic component indexed by λ is given by [5] to be

$$\frac{\sum_{T \in ST(\lambda)} f_\lambda \, q^{m\left(\binom{n}{2} - content(\lambda(T))\right) + cocharge(T)}}{(1-q)(1-q^2)\ldots(1-q^n)}. \tag{1.1}$$

Here *content* and *cocharge* are two statistics on tableaux–we will not need the precise definitions. In fact *content* only depends on the shape of T hence it is actually a function on partitions.

In light of the simple combinatorial descriptions of a basis for the coinvariants in the classical (or $m = 0$) case, the authors have looked for a basis for larger m. In [1] and [4] a basis was given for the case $n = 3$. (The work [4] specifically described the quasiinvariants for dihedral groups, so in particular for $D_3 \cong S_3$.) Further, in [5], Felder and Veselov provide integral expressions, $\phi^{(j)}(x)$ for $2 \leq j \leq n$, for the lowest degree (non-symmetric) m-quasiinvariants, i.e. those of degree $mn + 1$. In the present work, we give a complete basis of the isotypic component given by the partition $[n-1, 1]$ for any n. This is accomplished by means of a new characterization of $\mathbf{QI}_m$:

Theorem 1. *The vector space of quasiinvariants has the following direct sum decomposition:*

$$\mathbf{QI}_m = \bigoplus_{T \in ST(n)} \left(\gamma_T \mathbf{R} \cap V_T^{2m+1} \mathbf{R}\right)$$

where $ST(n)$ is the set of standard tableaux of size n, γ_T is a projection opertor due to Young (defined in full detail in the next section) and V_T is the polynomial given by the product over the columns of T of the associated "Vandermonde determinants" (this is also defined in detail below). This characterization is proved using completely elementary methods (namely, computations in the group algebra of the symmetric group) in section 4. In section 5 we use this characterization to construct the basis for the $[n-1, 1]$ isotypic component. Precisely, for T a standard Young tableau of shape $[n-1, 1]$ with j the entry in the second row, we set

$$Q_T^{k,m} = \int_{x_1}^{x_j} t^k \prod_{i=1}^{n} (t - x_i)^m \, dt.$$

With this definition, we have

Theorem 2. *The set*

$$\{Q_T^{0,m}, Q_T^{1,m}, Q_T^{2,m}, \ldots, Q_T^{n-2,m}\}$$

is a basis for $\gamma_T \left(\mathbf{QI}_m/\langle e_1, \ldots, e_n \rangle\right)$.

In section 6 we evaluate the integrals that represent these polynomials in a more explicit form.

Along the journey to these results, we have discovered other interesting facts about the ring $\mathbf{QI}_m$. In section 7, we show that the Calogero-Moser operator

$$L_m = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - 2m \sum_{1 \le i < j \le n} \frac{1}{x_i - x_j} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right)$$

acts on our basis by the simple formula

$$L_m Q_T^{k,m} = k(k-1) Q_T^{k-2,m}.$$

Finally, in section 8 we show that if we think of $\mathbf{QI}_{m+1}$ and $\mathbf{QI}_m$ as modules over the ring Λ_n, the determinant of the respective change of basis matrix is the Vandermonde determinant to the power $n!$, regardless of the value of m. We hope that these results prove as suggestive to others as to ourselves, and spur further investigations into this newly discovered territory.

2 Definitions and Notation

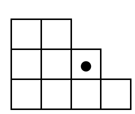
Throughout this paper, we will write elements of the symmetric group S_n using cycle notation. We will perform many calculations in the group algebra of S_n, and as such it will be useful to have shorthand notation for many commonly occurring elements. For a given subgroup A of S_n, we set

$$[A] = \sum_{\sigma \in A} \sigma \qquad \text{and}$$
$$[A]' = \sum_{\sigma \in A} \operatorname{sgn}(\sigma)\sigma.$$

We will extend this notation, abusing it slightly, and also define, for any set U whatsoever,

$$[U] = \sum_{\sigma \in S_U} \sigma \qquad \text{and}$$
$$[U]' = \sum_{\sigma \in S_U} \operatorname{sgn}(\sigma)\sigma.$$

The Young diagram of a partition λ is a subset of the boxes in the positive integer lattice, indexed by ordered pairs (i, j), where i is the row index and j is the column index. For example, in the following Young diagram of $[4, 3, 2]$, the cell $(2, 3)$ is marked:



A *tableau* of shape $\lambda \vdash n$ is a function from the cells of the Young diagram of λ to the set $\{1, \ldots, n\}$. We write the $T(i, j)$ for the value of T at the cell (i, j). For example, if T is the following tableau, $T(2, 3) = 8$:

6	7		
4	5	8	
1	2	3	9

We call a tableau *standard* if it is injective and the entries increase across the rows and up the columns. For example, the tableau above is standard. We denote the set of standard tableaux of shape λ by $ST(\lambda)$ and the set of all standard tableaux with n boxes by $ST(n)$.

Given a tableau T we let C_i be the set of elements in the i^{th} column and we define R_i similarly for the rows. We also set

$$\begin{aligned} C(T) &= \{(i, j) \in S_n \mid i, j \text{ are in the same column of } T\} \\ R(T) &= \{(i, j) \in S_n \mid i, j \text{ are in the same row of } T\} \\ N(T) &= \prod_i [C_i]' \\ P(T) &= \prod_i [R_i] \\ f_\lambda &= \text{ the number of standard tableaux of shape } \lambda \\ \gamma_T &= \frac{f_\lambda \; N(T)P(T)}{n!} \\ \lambda(T) &= \text{ the shape of tableau } T. \end{aligned}$$

Finally, we define the following useful polynomial associated with a tableau T:

$$V_T = \prod_{(i,j) \in C(T)} (x_i - x_j).$$

3 Useful Facts About $\mathbb{Q}S_n$ modules

The fundamental theorem of representation theory states

Proposition 1. *For W a finite dimensional S_n-module,*

$$W \cong \bigoplus_{\lambda \vdash n} V_\lambda^{\oplus m_\lambda}$$

where the V_λ are the irreducible representations of S_n and the m_λ are non-negative integers.

The vector space and S_n-module $V_\lambda^{\oplus m_\lambda}$ is known as the *isotypic component* of V indexed by λ. Now, $\mathbf{QI}_m$is infinite dimensional, but it is the direct sum of homogeneous components, each of which are finite dimensional. So we have that each homogeneous component of $\mathbf{QI}_m$decomposes into a direct sum of irreducibles. The direct sum of all copies of V_λ occuring in this decomposition is still itself an S_n-module, and is still referred to as the isotypic component indexed by λ. However, we will find the following decomposition of V more useful.

Proposition 2. *On any S_n module W, the group algebra elements $\{\gamma_T\}_{T\in ST(n)}$ act as projection operators. In symbols, we have the conditions*

1. $\gamma_T^2 = \gamma_T$
2. $W = \bigoplus_{T\in ST(n)} \gamma_T W$.

Note that in this decomposition, unlike the previous one, the direct summands are not themselves S_n-modules. We do have the following proposition, however, nicely relating the previous two.

Proposition 3. *For any S_n module W,*

$$\bigoplus_{T\in ST(\lambda)} \gamma_T W$$

is the isotypic component of W indexed by λ.

In the case of the quasiinvariants, we have the following

Proposition 4. *The $\mathbb{Q}$-vector space of m-quasiinvariants has the following direct sum decomposition:*

$$\mathbf{QI}_m = \bigoplus_{T\in ST(n)} \gamma_T \mathbf{QI}_m.$$

Our goal will be to use the decomposition $\mathbf{QI}_m/\langle e_1, \dots e_n\rangle = \bigoplus_T \gamma_T \left(\mathbf{QI}_m/\langle e_1, \dots e_n\rangle\right)$ to find a basis for this quotient module.

4 A New Characterization of S_n-Quasiinvariants

In this section we prove the following theorem:

Theorem 1. *The vector space of quasiinvariants has the following direct sum decomposition:*

$$\mathbf{QI}_m = \bigoplus_{T\in ST(n)} \left(\gamma_T \mathbf{R} \cap V_T^{2m+1}\mathbf{R}\right).$$

We will prove this by showing

$$\gamma_T \mathbf{QI}_m = \gamma_T \mathbf{R} \cap V_T^{2m+1}\mathbf{R}. \tag{4.1}$$

Combining (4.1) with Proposition 4 will prove the theorem. Equation (4.1) is proved by considering some relations in the group algebra of S_n. We begin with the following simple proposition:

Proposition 5. *Let $f = \sum_{\sigma \in S_n} f_\sigma \sigma \in \mathbb{Q}S_n$, and $P, Q \in \mathbb{Q}[x_1, \dots x_n]$ with P a symmetric function. Then we have $f(PQ) = Pf(Q)$.*

Proof. We have the following calculation:

$$\begin{aligned} f(PQ) &= (\sum_{\sigma \in S_n} f_\sigma \sigma)(PQ) \\ &= \sum_{\sigma \in S_n} f_\sigma (\sigma P)(\sigma Q) \\ &= P \sum_{\sigma \in S_n} f_\sigma (\sigma Q) \\ &= Pf(Q). \end{aligned}$$

□

Lemma 1. *The group algebra element $[S_n]$ can be written as*

$$\Big(1 + (i_1, i_2)\Big)\Big(1 + (i_1, i_3) + (i_2, i_3)\Big) \cdots \Big(1 + (i_1, i_n) + (i_2, i_n) + \cdots + (i_{n-1}, i_n)\Big)$$

where $\{i_1, \dots, i_n\}$ is any permutation of $\{1, \dots, n\}$. Similarly, $[S_n]'$ can be written as

$$\Big(1 - (i_1, i_2)\Big)\Big(1 - (i_1, i_3) - (i_2, i_3)\Big) \cdots \Big(1 - (i_1, i_n) - (i_2, i_n) - \cdots - (i_{n-1}, i_n)\Big).$$

Proof. The statement is trivial for $n = 1$. Now assume the statement is true for S_{n-1}. Let H be the subgroup of S_n consisting of all permutations which leave i_n fixed. Right coset decomposition gives

$$S_n = H + H(i_1, i_n) + H(i_2, i_n) + \cdots + H(i_{n-1}, i_n).$$

Thus

$$\begin{aligned} [S_n] &= [H]\Big(1 + (i_1, i_n) + (i_2, i_n) + \cdots + (i_{n-1}, i_n)\Big) \text{ and} \\ [S_n]' &= [H]'\Big(1 - (i_1, i_n) - (i_2, i_n) - \cdots - (i_{n-1}, i_n)\Big). \end{aligned}$$

As H is isomorphic to S_{n-1} the statement is proved. □

Remark 1. *Note that left coset decomposition could just as easily have been used in this proof, which would give the factors in the opposite order.*

For the following, we fix the following:

- T a tableau of shape $\lambda \vdash n$,
- i, j with $1 \leq i < j \leq \lambda_1$.

With T fixed, we use the boldface notation $\mathbf{a_b}$ as shorthand for $T(a,b)$, the element in the a^{th} row and b^{th} column of T. In the following, we will make much use of elements of $\mathbb{Q}[S_n]$ of the form $[C_i \cup \{\mathbf{k_j}\}]'$; the signed sum of all permutations of the elements of column i, and a single element $\mathbf{k_j}$ in column j to the right of i. We first note that elements of this form kill $P(T)$:

Lemma 2. *For any $k \in \{1, \dots, |C_j|\}$ we have*

$$[C_i \cup \{\mathbf{k_j}\}]' P(T) = 0.$$

Proof. Since the rows consist of disjoint elements, all factors of the form $[R_k]$ in $P(T)$ commute, and we have

$$\begin{aligned}
[C_i \cup \{\mathbf{k_j}\}]' P(T) &= [C_i \cup \{\mathbf{k_j}\}]' [R_k] \prod_{l \neq k} [R_l] \\
\text{(by Lemma 1)} &= [C_i \cup \{\mathbf{k_j}\}]' \Big(1 + \big(\mathbf{k_i}, \mathbf{k_j}\big)\Big) \text{(other factors)} \\
&= \Big([C_i \cup \{\mathbf{k_j}\}]' - [C_i \cup \{\mathbf{k_j}\}]'\Big) \text{(other factors)} \\
&= 0.
\end{aligned}$$

□

Given a column C_i and an element $\mathbf{k_j}$ in a column C_j to the right of i, we denote by $\alpha_{i,\mathbf{k_j}}$ the sum of all transpositions consisting of $\mathbf{k_j}$ and an element of C_i, *i.e.*,

$$\alpha_{i,\mathbf{k_j}} = \sum_{t=1}^{|C_i|} \big(\mathbf{t_i}, \mathbf{k_j}\big)$$

An important property of this element $\alpha_{i,\mathbf{k_j}}$ is the following:

Lemma 3. *The element $\alpha_{i,\mathbf{k_j}}$ leaves $\gamma(T)$ invariant, i.e.,*

$$\alpha_{i,\mathbf{k_j}} \gamma(T) = \gamma(T)$$

Proof. It suffices to show that $(1 - \alpha_{i,\mathbf{k_j}}) N(T) P(T) = 0$. The first step is to write $N(T)$ as $[C_i]' \prod_{r \neq i} [C_r]'$. We begin by noting that

$$(1 - \alpha_{i,\mathbf{k_j}}) N(T) P(T) = \Bigg(\prod_{t \neq i,j} [C_t]'\Bigg) (1 - \alpha_{i,\mathbf{k_j}}) [C_i]' [C_j]' P(T) \tag{4.2}$$

since the elements of C_t, for $t \notin \{i, j\}$ are disjoint from $C_i \cup \{\mathbf{k_j}\}$. By Lemma 1 we have

$$(1 - \alpha_{i,\mathbf{k_j}}) [C_i]' = \Bigg(1 - \sum_{r=1}^{|C_i|} (\mathbf{r_i}, \mathbf{k_j})\Bigg) [C_i]' \qquad = [C_i \cup \{\mathbf{k_j}\}]' \tag{4.3}$$

so substituting (4.3) into (4.2) and expanding $[C_j]'$ by Lemma 1 gives

$$\begin{aligned}
(1-\alpha_{i,\mathbf{k_j}})&N(T)P(T)\\
&=\Big(\prod_{t\neq i,j}[C_t]'\Big)[C_i\cup\{\mathbf{k_j}\}]'[C_j]'P(T)\\
&=\Big(\prod_{t\neq i,j}[C_t]'\Big)[C_i\cup\{\mathbf{k_j}\}]'\Big(1-(\mathbf{1_j},\mathbf{2_j})\Big)\cdots\\
&\quad\cdots\Big(1-(\mathbf{1_j},\mathbf{k_j})-(\mathbf{2_j},\mathbf{k_j})-\cdots-\widehat{(\mathbf{k_j},\mathbf{k_j})}-\cdots-(|\mathbf{C_j}|_\mathbf{j},\mathbf{k_j})\Big)P(T).
\end{aligned}$$

Moving the factors which do not involve $\mathbf{k_j}$ to the left and rewriting gives

$$\begin{aligned}
(1-\alpha_{i,\mathbf{k_j}})&N(T)P(T)\\
&=\Big(\text{other factors}\Big)([C_i\cup\{\mathbf{k_j}\}]')\\
&\quad\Big(1-(\mathbf{1_j},\mathbf{k_j})-(\mathbf{2_j},\mathbf{k_j})-\cdots-\widehat{(\mathbf{k_j},\mathbf{k_j})}-\cdots-(|\mathbf{C_j}|_\mathbf{j},\mathbf{k_j})\Big)(P(T))\\
&=\Big(\text{other factors}\Big)\\
&\quad\Big([C_i\cup\{\mathbf{k_j}\}]'P(T)-\sum_{\substack{t=1\\t\neq k}}^{|C_j|}[C_i\cup\{\mathbf{k_j}\}]'(\mathbf{t_j},\mathbf{k_j})P(T)\Big)
\end{aligned}$$

We now use the fact that $[C_i\cup\{\mathbf{k_j}\}]'(\mathbf{t_j},\mathbf{k_j})=(\mathbf{t_j},\mathbf{k_j})[C_i\cup\{\mathbf{t_j}\}]'$ to obtain

$$\begin{aligned}
(1-\alpha_{i,\mathbf{k_j}})N(T)P(T)=&\Big(\text{other factors}\Big)\\
&\Big([C_i\cup\{\mathbf{k_j}\}]'P(T)-\sum_{\substack{t=1\\t\neq k}}^{r}(\mathbf{t_j},\mathbf{k_j})[C_i\cup\{\mathbf{t_j}\}]'P(T)\Big)\\
=&0
\end{aligned}$$

where the last equality follows from Lemma 2. ☐

We now have the tools to prove the difficult containment of Theorem 1.

Lemma 4. *For all standard tableaux T and all $m\geq 0$, we have the following containment of vector spaces:*

$$\gamma_T\mathbf{R}\cap V_T^{2m+1}\mathbf{R}\subseteq\gamma_T\mathbf{QI}_m.$$

Proof. Since γ_T is an idempotent, it suffices to show that for any polynomial P in the ideal $V_T^{2m+1}\mathbf{R}$, $\gamma_T P=P$ implies that P is m-quasiinvariant.

Let P be such that $V_T^{2m+1}|P$ and $\gamma_T P = P$. We wish to show that $\Big(1 - (a,b)\Big)P$ is divisible by $(x_a - x_b)^{2m+1}$ for all transpositions (a,b). We first consider the case where a and b are in the same column of T. In this case we have

$$(a,b)N(T) = -N(T)$$

and so

$$(a,b)P = (a,b)\gamma_T P = -\gamma_T P = -P.$$

Thus

$$\Big(1 - (a,b)\Big)P = 2P \in V_T^{2m+1}\mathbf{R}$$

which is divisible by the required factor.

Now suppose without loss that $a = \mathbf{k_i}$ is to the left of b in column C_j. By Lemma 3 P, is preserved by $\alpha_{i,b}$:

$$\alpha_{i,b} P = \alpha_{i,b}\gamma_T P = \gamma_T P = P. \tag{4.4}$$

Equation (4.4) gives

$$\Big(1 - (a,b)\Big)P = P - (a,b)P \tag{4.5}$$

$$= \alpha_{i,b} P - (a,b)P \tag{4.6}$$

$$= \sum_{\substack{t=1\\ t\neq k}}^{|C_i|} (\mathbf{t_i}, b)P. \tag{4.7}$$

Since $P \in V_T^{2m+1}\mathbf{R}$, for any $t \in \{1, \ldots, |C_i|\}$ with $t \neq k$ we can rewrite P as

$$P = (x_{\mathbf{t_i}} - x_a)^{2m+1}(\text{other factors}).$$

Thus

$$(\mathbf{t_i}, b)P = (x_b - x_a)^{2m+1}(\text{other factors})$$

and we have

$$(x_b - x_a)^{2m+1} \text{ divides } (\mathbf{t_i}, b)P \text{ for every } t \in \{1, \ldots, |C_i|\} \text{ with } t \neq k.$$

Hence $(x_b - x_a)^{2m+1}$ divides the right-hand side of equation 4.7, which completes the proof. ☐

The proof of Theorem 1 now follows easily.

Proof of Theorem 1. Lemma 4 gives one containment. It remains to show that

$$\gamma_T \mathbf{QI}_m \subseteq \gamma_T \mathbf{R} \cap V_T^{2m+1}\mathbf{R}.$$

In particular, we must show that for $Q \in \mathbf{QI}_m$ we have

$$\gamma_T Q \in V_T^{2m+1}\mathbf{R}.$$

Let $P = \gamma_T Q = N(T)Q'$. P must be anti-symmetric with respect to all transpositions in $C(T)$ since it is in the image of $N(T)$. Thus, for any $(a, b) \in C(T)$, $\Big(1 - (a, b)\Big)P = 2P$. Hence $(x_a - x_b)^{2m+1}$ divides $2P$ (and also P) for all $(a, b) \in C(T)$. This establishes equation (4.1) and hence the theorem. □

5 A Basis For The Isotypic Component $\lambda(T) = [n-1, 1]$

In this section, we refer to the quotient $\mathbf{QI}_m/\langle e_1, \ldots, e_n\rangle$ by the symbol $\mathbf{QI}_m^*$. Our object here is to describe a basis for $\gamma_T \mathbf{QI}_m^*$ when T has a hook shape of the form $[n-1, 1]$. Until otherwise specified, let λ be the partition $[n-1, 1]$ and let T be one of the $(n-1)$ standard tableaux of shape λ. In fact T is uniquely defined by the lone entry of the second row. Suppose it's $j \in \{2, 3, \ldots, n\}$. We define

$$Q_T^{k,m} = \int_{x_1}^{x_j} t^k \prod_{i=1}^{n} (t - x_i)^m dt.$$

Our goal will be to show that the polynomials $\{Q_T^{k,m}\}_{k=0}^{n-2}$ are a set of representatives for a basis of $\gamma_T \mathbf{QI}_m^*$. Before we do this, we show that these polynomials satisfy a remarkable recursion. In what follows, e_i will denote the ith elementary symmetric function in the variables $x_1, \ldots, x_n$, with the convention that $e_0 = 1$.

We first state for reference a classical symmetric function identity:

$$\prod_{i=1}^{n} (t - x_i) = \sum_{i=0}^{n} (-1)^i e_i t^{n-i}. \tag{5.1}$$

We now state our recursion.

Proposition 6. *For $m > 1$ we have the identity*

$$Q_T^{k,m} = \sum_{i=0}^{n} (-1)^i e_i \ Q_T^{n-i+k, m-1}.$$

Proof. Unpacking the product in the definition of $Q_T^{k,m}$ we get

$$Q_T^{k,m} = \int_{x_1}^{x_j} \left(\prod_{i=1}^{n} (t - x_i) \right) t^k \prod_{l=1}^{n} (t - x_l)^{m-1} dt, \tag{5.2}$$

and substituting (5.1) into (5.2) and pulling out the factors not involving t gives

$$\begin{aligned} Q_T^{k,m} &= \int_{x_1}^{x_j} \left(\sum_{i=0}^{n} (-1)^i e_i t^{n-i} \right) t^k \prod_{l=1}^{n} (t - x_l)^{m-1} dt \\ &= \sum_{i=0}^{n} (-1)^i e_i \int_{x_1}^{x_j} t^{n-i+k} \prod_{l=1}^{n} (t - x_l)^{m-1} dt \\ &= \sum_{i=0}^{n} (-1)^i e_i \; Q_T^{n-i+k,m-1}. \end{aligned}$$

□

Felder and Veselov [5, Section 4.5] gave elements of the lowest non-trivial degree with the following function, defined for $m \geq 1$, and $2 \leq j \leq n$. These are given by:

$$\varphi_m^{(j)} = \frac{1}{1+mn} \int_{x_1}^{x_j} \sum_{i=1}^{n} \frac{x_i}{t - x_i} \prod_{k=1}^{n} (t - x_k)^m dt$$

There is a very simple relationship between the $\varphi_m^{(j)}$ and the $Q_T^{0,m}$, which we describe here.

Proposition 7. *For fixed $m \geq 1$, $j \in \{2, \ldots, n\}$, and tableaux T of shape $(n-1, 1)$ with j the element in the second row we have*

$$Q_T^{0,m} = -m\varphi_m^{(j)}.$$

Proof. Throughout this proof, $f(X)$ will mean we evaluate the symmetric function f on the variables $x_1, \ldots, x_n$, and $f(X_k)$ will mean we omit x_k; we evaluate f on the variables $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$. The following calculation establishes a simple identity which will be useful for expressing $\varphi_m^{(j)}$ in terms of the $Q_T^{k,m}$:

$$\sum_{k=1}^{n} x_k \prod_{\substack{l=1 \\ l \neq k}}^{n} (t - x_l) = \sum_{i=0}^{n-1} (-1)^i t^{n-1-i} \sum_{k=1}^{n} x_k e_i(X_k) \tag{5.3}$$

$$= \sum_{i=0}^{n-1} (-1)^i t^{n-1-i} (i+1) e_{i+1}(X) \tag{5.4}$$

With this in hand, we can write $\varphi_m^{(j)}$ in terms of the $Q_T^{k,m}$ as follows:

$$\begin{aligned}
\varphi_m^{(j)} &= \frac{1}{1+mn}\int_{x_1}^{x_j}\sum_{i=1}^{n}\frac{x_i}{t-x_i}\prod_{k=1}^{n}(t-x_k)^m dt \\
&= \frac{1}{1+mn}\int_{x_1}^{x_j}\sum_{i=1}^{n}x_i\prod_{\substack{k=1\\k\neq i}}^{n}(t-x_k)\prod_{k=1}^{n}(t-x_k)^{m-1}dt \\
&= \frac{1}{1+mn}\int_{x_1}^{x_j}\sum_{i=0}^{n-1}(-1)^i t^{n-1-i}(i+1)e_{i+1}(X)\prod_{l=1}^{n}(t-x_l)^{m-1}dt \\
&= \frac{1}{1+mn}\sum_{i=0}^{n-1}(-1)^i(i+1)e_{i+1}(X)\int_{x_1}^{x_j}\prod_{l=1}^{n}t^{n-1-i}(t-x_l)^{m-1}dt \\
&= \frac{1}{1+mn}\sum_{i=0}^{n-1}(-1)^i(i+1)e_{i+1}(X)Q_T^{n-1-i,m-1} \\
&= \frac{1}{1+mn}\sum_{i=0}^{n}(-1)^{i-1}(i)e_i(X)Q_T^{n-i,m-1} \qquad (5.5)
\end{aligned}$$

where the in last equality we have shifted the index i by 1, and then included the (trivial) $i=0$ term.

It remains to express the relationship between $Q_T^{0,m}$ and the previous sum. To do this, we have the following lemma:

Lemma 5. *For $m \geq 1$, we have*

$$\sum_{i=0}^{n}(-1)^i\left(m(n-i)+(k+1)\right)e_i(X)Q_T^{n-i+k,m-1}=0$$

Proof. We prove this by induction on m. When $m=1$, we have

$$\begin{aligned}
&\sum_{i=0}^{n}(-1)^i\left(n-i+k+1\right)e_i(X)\frac{1}{n-i+k+1}\left(x_j^{n-i+k+1}-x_1^{n-i+k+1}\right) \\
=&\left(x_j^{k+1}\sum_{i=0}^{n}(-1)^i e_i(X)x_j^{n-i}\right)-\left(x_1^{k+1}\sum_{i=0}^{n}(-1)^i e_i(X)x_1^{n-i}\right) \\
=&x_j^{k+1}\prod_{i=1}^{n}(x_j-x_i)-x_1^{k+1}\prod_{i=1}^{n}(x_1-x_i)=0
\end{aligned}$$

Proceeding with the induction we must evaluate

$$\sum_{i=0}^{n}(-1)^i\left((m+1)(n-i)+(k+1)\right)e_i(X)Q_T^{n-i+k,m} \qquad (5.6)$$

Substituting with Proposition 6 gives

$$
\begin{aligned}
&= \sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^{i+j}e_i(X)e_j(X)\left((m+1)(n-i)+(k+1)\right)Q_T^{n-i+(n-j+k),m-1} \\
&= \sum_{j=0}^{n}\left(\sum_{i=0}^{n}(-1)^i(m(n-i)+n-j+k+1)e_i(X)Q_T^{n-i+(n-j+k),m-1}\right)(-1)^j e_j(X)+ \qquad (5.7) \\
&\sum_{j=0}^{n}\sum_{i=0}^{n}(-1)^{i+j}e_i(X)e_j(X)(j-i)Q_T^{2n-(i+j)+k,m-1} \qquad (5.8)
\end{aligned}
$$

Now, (5.7) is 0 by the inductive hypothesis and (5.8) is 0 by pairing the terms with factor $(i-j)$ to those with factor $(j-i)$. □

Rewriting the statement of this lemma for the case $k = 0$ and combining with (5.5), we have

$$
\begin{aligned}
\sum_{i=0}^{n}(-1)^i e_i(X)(mn+1)Q_T^{n-i,m-1} &= \sum_{i=0}^{n}(-1)^i e_i(X)(mi)Q_T^{n-i,m-1} \\
&= -m\sum_{i=0}^{n}(-1)^{i-1}(i)e_i(X)Q_T^{n-i,m-1} \\
&= -m(mn+1)\varphi_m^{(j)} \qquad (5.9)
\end{aligned}
$$

By Proposition 6 we also have

$$
(mn+1)Q_T^{0,m} = \sum_{i=0}^{n}(-1)^i e_i(X)(mn+1)Q_T^{n-i,m-1}. \qquad (5.10)
$$

Combining (5.9) with (5.10), and dividing by $mn+1$ completes the proof. □

We now show that we have $Q_T^{k,m} \in \gamma_T \mathbf{QI}_m$. By Theorem 1 it is enough to show that we have $Q_T^{k,m} \in \gamma_T \mathbf{R}$ and $V_T^{2m+1} \mid Q_T^{k,m}$.

Proposition 8. *The polynomial $Q_T^{k,m}$ is invariant under the action of the group algebra element γ_T.*

Proof. We first show the statement is true in the case $m = 0$, and then proceed by induction. From the definition of $Q_T^{k,m}$ we have

$$
\begin{aligned}
Q_T^{k,0} &= \int_{x_1}^{x_j} t^k \prod_{i=1}^{n}(t-x_i)^0 dt \qquad (5.11) \\
&= \frac{x_j^{k+1}-x_1^{k+1}}{k+1}. \qquad (5.12)
\end{aligned}
$$

Thus $Q_T^{k,0}$ is invariant under the transposition (a,b) for $a,b \in \{2,\ldots,\hat{j},\ldots,n\}$. This immediately gives

$$[S_{\{2,\ldots,\hat{j},\ldots,n\}}]Q_T^{k,0} = (n-2)!Q_T^{k,0}. \tag{5.13}$$

Now, $P(T) = [S_{\{1,2,\ldots,\hat{j},\ldots,n\}}]$ and expanding this according to Lemma 1 yields

$$P(T) = \Big(1 + (1,2) + \cdots + \widehat{(1,j)} + \cdots (1,n)\Big)[S_{\{2,\ldots,\hat{j},\ldots,n\}}]. \tag{5.14}$$

Using (5.12), (5.13) and (5.14) and performing a simple calculation, we obtain

$$P(T)Q_T^{k,0} = [S_{\{1,2,\ldots,\hat{j},\ldots,n\}}]\frac{x_j^{k+1} - x_1^{k+1}}{k+1} \tag{5.15}$$

$$= \Big(1 + (1,2) + \cdots + \widehat{(1,j)} + \cdots (1,n)\Big)[S_{\{2,\ldots,\hat{j},\ldots,n\}}]\frac{x_j^{k+1} - x_1^{k+1}}{k+1} \tag{5.16}$$

$$= \frac{(n-2)!}{k+1}\Big(1 + (1,2) + \cdots + \widehat{(1,j)} + \cdots (1,n)\Big)(x_j^{k+1} - x_1^{k+1}) \tag{5.17}$$

$$= \frac{(n-2)!}{k+1}\Big((n-1)(x_j^{k+1}) - x_1^{k+1} - (x_2^{k+1} + \cdots + \widehat{x_j^{k+1}} + \cdots + x_n^{k+1})\Big). \tag{5.18}$$

Since $N(T) = \Big(1 - (1,j)\Big)$ we can use (5.18) to get

$$N(T)P(T)Q_T^{k,0} = \frac{(n-2)!}{k+1}\Big(1 - (1,j)\Big)\Big((n-1)x_j^{k+1} - x_1^{k+1} - (x_2^{k+1} + \cdots + \widehat{x_j^{k+1}} + \cdots + x_n^{k+1})\Big) \tag{5.19}$$

$$= \frac{(n-2)!}{k+1}\Big((n-1)(x_j^{k+1} - x_1^{k+1}) - (x_1^{k+1} - x_j^{k+1}) - 0\Big) \tag{5.20}$$

$$= \frac{n(n-2)!}{k+1}(x_j^{k+1} - x_1^{k+1}). \tag{5.21}$$

Finally, we use (5.21) and the fact $f_\lambda = (n-1) = \frac{n!}{n(n-2)!}$ to reach the desired conclusion

$$\begin{aligned}\gamma_T Q_T^{k,0} &= \frac{n!}{f_\lambda}N(T)P(T)Q_T^{k,0} \\ &= \frac{(x_j^{k+1} - x_1^{k+1})}{k+1} \\ &= Q_T^{k,0}.\end{aligned}$$

With this in hand, we use Proposition 6 to write

$$\gamma_T Q_T^{k,m} = \sum_{i=0}^{n} \gamma_T \ (-1)^i e_i \ Q_T^{n-i+k,m-1}.$$

Applying Proposition 5 and induction gives

$$\begin{aligned}\gamma_T Q_T^{k,m} &= \sum_{i=0}^{n} (-1)^i e_i \big(\gamma_T Q_T^{n-i+k,m-1}\big) \\ &= \sum_{i=0}^{n} (-1)^i e_i \big(Q_T^{n-i+k,m-1}\big) \\ &= Q_T^{k,m}.\end{aligned}$$

□

In order to complete our task of showing that $Q_T^{k,m} \in \gamma_T \mathbf{QI}_m$, we must show that $(x_j - x_1)^{2m+1} \mid Q_T^{k,m}$. We do so by proving the following stronger statement:

Proposition 9. *For all k,*

$$\lim_{x_1 \to x_j} \frac{Q_T^{k,m}}{(x_j - x_1)^{2m+1}} = \frac{(-1)^m m!^2}{(2m+1)!} x_j^k \prod_{\substack{i=2 \\ i \neq j}}^{n} (x_j - x_i)^m.$$

Proof. This proof will rely on Leibniz's integral formula, also known as the technique of differentiation underneath the integral sign. We state the rule here for the reader's convenience. For $f(x,y), u(x), v(x)$ continuous functions we have

$$\frac{d}{dx} \int_{u(x)}^{v(x)} f(x,y)dy = \left(f(x, v(x)) \cdot \frac{\partial v}{\partial x}\right) - \left(f(x, u(x)) \cdot \frac{\partial u}{\partial x}\right) + \int_{u(x)}^{v(x)} \frac{\partial f}{\partial x} \ dy \tag{5.22}$$

For example, we have

$$\frac{\partial}{\partial x_j}\left(\int_{x_1}^{x_j} t^k \prod_{i=1}^{n} (t - x_i)^m dt\right) = x_j^k \prod_{i=1}^{n} (x_j - x_i)^m - 0 + \int_{x_1}^{x_j} \frac{\partial}{\partial x_j}\left(t^k \prod_{i=1}^{n} (t - x_i)^m\right) \tag{5.23}$$

$$= 0 - 0 + \int_{x_1}^{x_j} (-m) t^k (t - x_j)^{m-1} \prod_{\substack{i=1 \\ i \neq j}}^{n} (t - x_i)^m dt. \tag{5.24}$$

A similar calculation of $\frac{\partial}{\partial x_l}$ for the cases $l = 1$ and $l \neq 1, j$ gives the more general rule

$$\frac{\partial}{\partial x_l}\left(\int_{x_1}^{x_j} t^k \prod_{i=1}^{n}(t-x_i)^m dt\right) = \int_{x_1}^{x_j} (-m)t^k(t-x_l)^{m-1} \prod_{\substack{i=1\\ i\neq l}}^{n}(t-x_i)^m dt. \quad (5.25)$$

Repeating this differentiation gives

$$\frac{\partial^p}{\partial {x_l}^p}\left(\int_{x_1}^{x_j} t^k \prod_{i=1}^{n}(t-x_i)^m dt\right) = \int_{x_1}^{x_j} (-1)^p (m)_p t^k (t-x_l)^{m-p} \prod_{\substack{i=1\\ i\neq l}}^{n}(t-x_i)^m dt \quad (5.26)$$

for $p \leq m$. Expanding $Q_T^{k,m}$ according to the definition gives

$$\lim_{x_j\to x_1} \frac{Q_T^{k,m}}{(x_j - x_1)^{2m+1}} = \lim_{x_j\to x_1} \frac{\int_{x_1}^{x_j} t^k \prod_{i=1}^{n}(t-x_i)^m dt}{(x_j-x_1)^{2m+1}} \quad (5.27)$$

which is an indeterminate expression of the form $\frac{0}{0}$. Applying L'Hopital's rule and evaluating the numerator with (5.24) gives that expression (5.27) equals

$$\lim_{x_j\to x_1} \frac{\frac{\partial}{\partial x_j}\left(\int_{x_1}^{x_j} t^k \prod_{i=1}^{n}(t-x_i)^m dt\right)}{\frac{\partial}{\partial x_j}(x_j-x_1)^{2m+1}}$$

$$= \lim_{x_j\to x_1} \frac{\int_{x_1}^{x_j} (-m)t^k(t-x_j)^{m-1} \prod_{\substack{i=1\\ i\neq j}}^{n}(t-x_i)^m dt}{(2m+1)(x_j-x_1)^{2m}}$$

which is still indeterminate. However, after m applications of L'Hopital's rule we obtain

$$\lim_{x_j\to x_1} \frac{(-1)^m \cdot m! \int_{x_1}^{x_j} t^k \prod_{\substack{i=1\\ i\neq j}}^{n}(t-x_i)^m dt}{(2m+1)(2m)(2m-1)\cdots(m+2)(x_j-x_1)^{m+1}}$$

and one more application of L'Hopital's rule, evaluated this time with the Fundamental Theorem of Calculus, yields

$$\lim_{x_j\to x_1} \frac{(-1)^m \cdot m! \cdot x_j^k \prod_{\substack{i=1\\ i\neq j}}^{n}(x_j-x_i)^m}{(2m+1)(2m)(2m-1)\cdots(m+1)(x_j-x_1)^m}. \quad (5.28)$$

We now cancel the term $(x_j - x_1)^m$ from both numerator and denominator and the Proposition is proven. □

The polynomiality of $\lim_{x_1\to x_j} \frac{Q_T^{k,m}}{(x_j-x_1)^{2m+1}}$ immediately gives that $(x_j - x_1)^{2m+1} \big| Q_T^{k,m}$.

Proposition 10. *The polynomial $Q_T^{k,m} \in \gamma_T \mathbf{QI}_m$ for all k, m.*

Proof. By Theorem 1 we have that $\gamma_T \mathbf{QI}_m = \gamma_T \mathbf{R} \cap V_T^{2m+1}\mathbf{R}$. Hence the result is proved by the previous two propositions. □

We now show that the polynomials $Q_T^{k,m}$ form a basis for the hook shape $[n-1,1]$. For this proof, we use Felder and Veselov's Hilbert series result, as stated in Equation (1.1). Furthermore, they show in [5] that $\mathbf{QI}_m^*$ affords the left-regular representation, so that one can break up a basis for $\mathbf{QI}_m^*$ into a set of bases for the various isotypic components. In particular, this shows for T of shape $[n-1,1]$ that the projection of the quotient $\gamma_T \mathbf{QI}_m^*$ has Hilbert series given by

$$\sum_{k=0}^{n-2} q^{mn+1+k}.$$

With this result in mind, we now prove the following main theorem.

Theorem 2. *The set*

$$\{Q_T^{0,m}, Q_T^{1,m}, Q_T^{2,m}, \ldots, Q_T^{n-2,m}\}$$

is a basis for $\gamma_T \mathbf{QI}_m^*$.

Proof. We first note that $Q_T^{k,m}$ has degree $mn+k+1$, and in particular, each of these elements are of different degrees, and matching that of the Hilbert series. Since the set $S = \{Q_T^{0,m}, Q_T^{1,m}, Q_T^{2,m}, \ldots, Q_T^{n-2,m}\}$ has size $n-1$, proving S is linearly independent in $\gamma_T \mathbf{QI}_m^*$ shows that S is a basis for $\gamma_T \mathbf{QI}_m^*$.

Since the quotient $\mathbf{QI}_m^*$ is graded and the polynomials $Q_T^{k,m}$ are of different degrees as k varies, it suffices to show that $Q_T^{k,m}$ is nonzero in the quotient for $0 \le k \le n-2$. Put another way, we must show that $Q_T^{k,m}$ is not in the ideal of $\gamma_T \mathbf{QI}_m$ generated by $\langle e_1, \ldots, e_n \rangle$. Equivalently we must show that polynomials of the form

$$P_k = Q_T^{k,m} + A_1 Q_T^{k-1,m} + \cdots + A_{k-1} Q_T^{1,m} + A_k Q_T^{0,m}$$

(where the A_i are symmetric functions of degree i) can only equal 0 if $k \ge n-1$.

In fact, we use the explicit formulas for $\lim_{x_j \to x_1} Q_T^{k,m}/V_T^{2m+1}$ given by Proposition 9 to show the stronger statement

$$\lim_{x_j \to x_1} P_k / V_T^{2m+1} = 0 \implies k \ge n-1$$

regardless of the choice of the symmetric functions. Letting $\widetilde{A_i}$ denote the limit $x_j \to x_1$ applied to the symmetric function A_i, and assuming w.l.o.g. that $j = 2$, we have

$$\begin{aligned} &\lim_{x_2 \to x_1} P_k / V_T^{2m+1} = 0 \\ \implies &\left(\frac{(-1)^m m!^2}{(2m+1)!} \prod_{i=3}^{n} (x_1 - x_i)^m \right) \left(x_1^k + \widetilde{A_1} x_1^{k-1} + \cdots + \widetilde{A_{k-1}} x_1 + \widetilde{A_k} \right) = 0 \\ &\implies x_1^k + \widetilde{A_1} x_1^{k-1} + \cdots + \widetilde{A_{k-1}} x_1 + \widetilde{A_k} = 0 \\ &\implies \lim_{x_2 \to x_1} \left(x_1^k + A_1 x_1^{k-1} + \cdots + A_k \right) = 0 \end{aligned}$$

Setting

$$Q(x_1, \ldots, x_n) = x_1^k + A_1 x_1^{k-1} + \cdots + A_k$$

we must have

$$Q(x_1, \ldots, x_n) = (x_2 - x_1) \cdot R(x_1, \ldots, x_n)$$

However, Q must be symmetric with respect to all pairs of variables not involving x_1. Thus, for any $\sigma \in S_{\{2,3,\ldots,n\}}$, $\sigma Q = Q$ and so

$$Q(x_1, \ldots, x_n) = \sigma Q(x_1, \ldots, x_n) = (x_{\sigma(2)} - x_1) \cdot \sigma R(x_1, \ldots, x_n),$$

and so $\prod_{i=2}^{n}(x_i - x_1)$ divides $Q(x_1, \ldots, x_n)$. Consequently, k, which is the degree of $Q(x_1, \ldots, x_n)$, must be greater than or equal to $n-1$. □

6 A More Explicit Description of $Q_T^{k,m}$

We now know that the set $\{Q_T^{k,m}\}_{k=0}^{n-2}$ is indeed a basis for $\gamma_T \mathbf{QI}_m^*$. In this section we show an even more explicit formula for the $Q_T^{k,m}$'s. Throughout, we shall assume without loss that the element in the second row of T (which we have been calling j) is 2. Since $V_T = (x_2 - x_1)$ divides $Q_T^{k,m}$, we change variables to understand $Q_T^{k,m}$ from a more combinatorial point of view. Namely, we expand with respect to the variables

$$Z = \{x_1, \widehat{x_2}, \ldots, x_n, z = x_2 - x_1\}.$$

This is in contrast to the usual set of variables

$$X = \{x_1, x_2, \ldots, x_n\}.$$

Theorem 3. *The coefficient of $(x_2 - x_1)^r = z^r$ in $Q_T^{k,m}$ (when expanded with respect to Z) is*

$$\frac{m!}{r(r-1)(r-2)\cdots(r-m)} \sum_{i_3=0}^{m} \sum_{i_4=0}^{m} \cdots \sum_{i_n=0}^{m} (-1)^{m+i_3+i_4+\cdots+i_n} \binom{m}{i_3}\binom{m}{i_4}\cdots\binom{m}{i_n} \times \binom{k+m(n-2)-i_3-i_4-\cdots-i_n}{r-(2m+1)} x_1^{(k+m(n-2)-i_3-i_4-\cdots-i_n)-(r-(2m+1))} x_3^{i_3} x_4^{i_4} \cdots x_n^{i_n}.$$

Proof. We begin by evaluating the integral

$$\int_{t_m=x_1}^{x_2} \int_{t_{m-1}=x_1}^{t_m} \cdots \int_{t_0=x_1}^{t_1} (-1)^m m!\; t_0^k \prod_{\substack{i=1 \\ i\neq 2}}^{n} (t_0 - x_i)^m \; dt_0 \cdots dt_m. \qquad (6.1)$$

We will then show that this integral is another way of writing $Q_T^{k,m}$. We begin our evaluation of (6.1) by expanding each of the $(t_0 - x_i)^m$ for $i \geq 3$ by the binomial theorem, thus obtaining

$$\int_{t_m=x_1}^{x_2} \int_{t_{m-1}=x_1}^{t_m} \cdots \int_{t_0=x_1}^{t_1} (-1)^m m!\; t_0^k (t_0 - x_1)^m \left(\sum_{i_3=0}^{m} (-1)^{i_3} \binom{m}{i_3} t_0^{m-i_3} x_3^{i_3} \right) \times \left(\sum_{i_4=0}^{m} (-1)^{i_4} \binom{m}{i_4} t_0^{m-i_4} x_4^{i_4} \right) \cdots \left(\sum_{i_n=0}^{m} (-1)^{i_n} \binom{m}{i_n} t_0^{m-i_n} x_n^{i_n} \right) dt_0 \cdots dt_m.$$

This quantity simplifies to

$$\int_{t_m=x_1}^{x_2} \int_{t_{m-1}=x_1}^{t_m} \cdots \int_{t_0=x_1}^{t_1} m!\; t_0^k (t_0 - x_1)^m \left(\sum_{i_3=0}^{m} \sum_{i_4=0}^{m} \cdots \sum_{i_n=0}^{m} (-1)^{m+i_3+i_4+\cdots+i_n} \times \binom{m}{i_3} \binom{m}{i_4} \cdots \binom{m}{i_n} x_3^{i_3} x_4^{i_4} \cdots x_n^{i_n} \quad \cdot \quad t_0^{m\cdot(n-2)-i_3-i_4-\cdots-i_n} \right) dt_0 \cdots dt_m,$$

and by rearranging we obtain

$$m! \sum_{i_3=0}^{m} \sum_{i_4=0}^{m} \cdots \sum_{i_n=0}^{m} (-1)^{m+i_3+i_4+\cdots+i_n} \binom{m}{i_3} \binom{m}{i_4} \cdots \binom{m}{i_n} x_3^{i_3} x_4^{i_4} \cdots x_n^{i_n} \times \left(\int_{t_m=x_1}^{x_2} \int_{t_{m-1}=x_1}^{t_m} \cdots \int_{t_0=x_1}^{t_1} t_0^{k+m(n-2)-i_3-i_4-\cdots-i_n} (t_0 - x_1)^m \; dt_0 \cdots dt_m \right).$$

For convenience of notation we let $K = k + m \cdot (n-2) - i_3 - i_4 - \cdots - i_n$, allowing us to write the above as

$$m! \sum_{i_3=0}^{m} \sum_{i_4=0}^{m} \cdots \sum_{i_n=0}^{m} (-1)^{m+i_3+i_4+\cdots+i_n} \binom{m}{i_3} \binom{m}{i_4} \cdots \binom{m}{i_n} x_3^{i_3} x_4^{i_4} \cdots x_n^{i_n} \times \left(\int_{t_m=x_1}^{x_2} \int_{t_{m-1}=x_1}^{t_m} \cdots \int_{t_0=x_1}^{t_1} t_0^{K} (t_0 - x_1)^m \; dt_0 \cdots dt_m \right).$$

At this point, we rewrite t_0^K as $(x_1 + (t_0 - x_1))^K$, which allows us to simplify $t_0^K (t_0 - x_1)^m$ as $\sum_{R=0}^{K} \binom{K}{R} x_1^{K-R} (t_0 - x_1)^{R+m}$, hence we conclude (6.1) equals

$$m! \sum_{i_3=0}^{m} \sum_{i_4=0}^{m} \cdots \sum_{i_n=0}^{m} (-1)^{m+i_3+i_4+\cdots+i_n} \binom{m}{i_3} \binom{m}{i_4} \cdots \binom{m}{i_n} \times \sum_{R=0}^{K} x_1^{K-R} \cdot x_3^{i_3} x_4^{i_4} \cdots x_n^{i_n} \left(\int_{t_m=x_1}^{x_2} \int_{t_{m-1}=x_1}^{t_m} \cdots \int_{t_0=x_1}^{t_1} (t_0 - x_1)^{m+R} \; dt_0 \cdots dt_m \right),$$

and the inside integral is easily seen to evaluate to

$$\frac{(x_2 - x_1)^{2m+1+R}}{(R + 2m + 1)(R + 2m) \cdots (R + m + 1)}.$$

Finally, we let $r = R + (2m+1)$, i.e. $R = r - (2m+1)$, so that r signifies the power of $z = (x_2 - x_1)$ in the expression. Thus the coefficient of z^r is as claimed in the statement of the theorem.

It remains to show that $Q_T^{k,m}$ is in fact equal to the quantity in (6.1). We note that the argument above shows that z^{2m} divides (6.1). We also know from Proposition 9 that z^{2m} divides $Q_T^{k,m}$. Thus, showing

$$\frac{\partial^m}{\partial z^m} Q_T^{k,m} = \frac{\partial^m}{\partial z^m}(6.1)$$

shows equality of $Q_T^{k,m}$ and (6.1). Furthermore, the operator $\frac{\partial}{\partial z}$ applied to a polynomial in the generating set Z is equivalent to the operator $\frac{\partial}{\partial x_2}$ applied to the same polynomial in the generating set X. Thus, we will show $Q_T^{k,m} = (6.1)$ by showing

$$\frac{\partial^m}{\partial x_2^m} Q_T^{k,m} = \frac{\partial^m}{\partial x_2^m}(6.1) \tag{6.2}$$

For the LHS, consider the function $f(t) = t^k \prod_{i=1}^n (t - x_i)^m$. As in the previous section, we use Leibniz's formula to obtain

$$\begin{aligned} \frac{\partial}{\partial x_2} \int_{t=x_1}^{x_2} f(t)\ dt &= f(x_2) + \int_{t=x_1}^{x_2} \left(\frac{\partial}{\partial x_2} f(t) \right) dt \\ &= \int_{t=x_1}^{x_2} \left(\frac{\partial}{\partial x_2} f(t) \right) dt. \end{aligned}$$

After iterating m times, we obtain

$$\frac{\partial^m}{\partial x_2^m} \int_{x_1}^{x_2} t^k \prod_{i=1}^n (t - x_i)^m dt = (-1)^m m! \int_{x_1}^{x_2} t^k \prod_{\substack{i=1 \\ i \neq 2}}^n (t - x_i)^m dt.$$

For the RHS, we let

$$g(t,m) = \int_{t_{m-1}=x_1}^{t} \cdots \int_{t_0=x_1}^{t_1} (-1)^m m!\ t_0^k \prod_{\substack{i=1 \\ i \neq 2}}^n (t_0 - x_i)^m\ dt_0 \cdots dt_{m-1},$$

and note by the Fundamental Theorem of Calculus that

$$\frac{\partial}{\partial x_2} \int_{t_m=x_1}^{x_2} g(t,m)\ dt_m = \int_{t_{m-1}=x_1}^{x_2} g(t,m-1)\ dt_{m-1}$$

since the integrand does not include variable x_2. Thus

$$\frac{\partial^m}{\partial x_2^m}\int_{t_m=x_1}^{x_2}\int_{t_{m-1}=x_1}^{t_m}\cdots\int_{t_0=x_1}^{t_1}(-1)^m m!\ t_0^k\prod_{\substack{i=1\\ i\neq 2}}^{n}(t_0-x_i)^m\ dt_0\cdots dt_m$$

$$=(-1)^m m!\ \int_{x_1}^{x_2} t^k\prod_{\substack{i=1\\ i\neq 2}}^{n}(t-x_i)^m dt$$

$$=\frac{\partial^m}{\partial x_2^m}\int_{x_1}^{x_2} t^k\prod_{i=1}^{n}(t-x_i)^m dt$$

which establishes (6.2). □

7 The Action of Calogero-Moser Operator L_m

In this section, we discuss a further property of our basis for $\gamma_T QI_m^*$ for T a standard Young tableau of shape $[n-1,1]$. In particular, as discussed in [4, 6] and elsewhere in the literature, there is a natural family of operators which act on the quasiinvariants. These are called the Calogero-Moser operators and we denote them by L_m. In particular these operators are defined, in the symmetric group case, as

$$L_m=\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}-2m\sum_{1\leq i<j\leq n}\frac{1}{x_i-x_j}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right).$$

The action of L_m on our basis is striking. In particular, we obtain the following formulas for this action:

Theorem 4. *$L_m(Q_T^{k,m})=k(k-1)Q_T^{k-2,m}$ for $k\geq 2$ and equals zero for $k=0$ or 1.*

The significance of this formula is how L_m acts as second differentiation with respect to the basis $\{Q_T^{0,m},Q_T^{1,m},\ldots,Q_T^{n-2,m}\}$. This action naturally generalizes the action of $L_0=\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ on the polynomial ring QI_0.

Proof. We now proceed with the proof of Theorem 4. For $m=0$, we have $Q_T^{k,0}=\frac{x_j^{k+1}-x_1^{k+1}}{k+1}$ by (5.12). Thus

$$\begin{aligned}L_0Q_T^{k,0}&=\left(\frac{\partial^2}{\partial {x_1}^2}+\frac{\partial^2}{\partial {x_j}^2}\right)\left(\frac{x_j^{k+1}-x_1^{k+1}}{k+1}\right)-0\\&=(k)(x_j^{k-1}-x_1^{k-1})\\&=(k)(k-1)Q_T^{k-2,0}.\end{aligned}$$

We state here some useful identities which are valid for $m \geq 1$. First, we have

$$\int_{x_1}^{x_j} \frac{\partial^2}{\partial t^2} \left(t^k \prod_{i=1}^{n} (t - x_i)^m \right) dt \tag{7.1}$$

$$= \int_{x_1}^{x_j} \left(\frac{\partial^2}{\partial t^2} t^k \right) \prod_{i=1}^{n} (t - x_i)^m dt \tag{7.2}$$

$$+ 2 \int_{x_1}^{x_j} \left(\frac{\partial}{\partial t} t^k \right) \left(\frac{\partial}{\partial t} \prod_{i=1}^{n} (t - x_i)^m \right) dt \tag{7.3}$$

$$+ \int_{x_1}^{x_j} t^k \left(\frac{\partial^2}{\partial t^2} \prod_{i=1}^{n} (t - x_i)^m \right) dt. \tag{7.4}$$

We also have

$$\int_{x_1}^{x_j} \frac{\partial}{\partial t} \left[t^k \frac{\partial}{\partial t} \left(\prod_{i=1}^{n} (t - x_i)^m \right) \right] dt \tag{7.5}$$

$$= \int_{x_1}^{x_j} \left(\frac{\partial}{\partial t} t^k \right) \left(\frac{\partial}{\partial t} \prod_{i=1}^{n} (t - x_i)^m \right) dt + \int_{x_1}^{x_j} t^k \left(\frac{\partial^2}{\partial t^2} \prod_{i=1}^{n} (t - x_i)^m \right) dt$$

$$= \frac{1}{2}(7.3) + (7.4).$$

Additionally, we can compute (7.2) as follows:

$$\int_{x_1}^{x_j} \left(\frac{\partial^2}{\partial t^2} t^k \right) \prod_{i=1}^{n} (t - x_i)^m dt = k(k-1) \int_{x_1}^{x_j} t^{k-2} \prod_{i=1}^{n} (t - x_i)^m dt$$

$$= k(k-1) Q_T^{k-2,m}.$$

Now, for $m \geq 2$ we recall equations (5.25) and (5.26) where we used Leibniz's integral formula to obtain

$$\frac{\partial}{\partial x_i} (Q_T^{k,m}) = \int_{x_1}^{x_j} (-m) t^k (t - x_i)^{m-1} \prod_{\substack{l=1 \\ l \neq i}}^{n} (t - x_l)^m dt$$

and

$$\frac{\partial^2}{\partial {x_i}^2} (Q_T^{k,m}) = \int_{x_1}^{x_j} m(m-1) t^k (t - x_i)^{m-2} \prod_{\substack{l=1 \\ l \neq i}}^{n} (t - x_l)^m dt.$$

Using these we can compute

$$\sum_{1\le i<l\le n} \frac{1}{x_i - x_l}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_l}\right) Q_T^{k,m}$$

$$=(-m)\int_{x_1}^{x_j} t^k \sum_{1\le i<l\le n} \frac{1}{x_i-x_l}\left(\left[(t-x_i)^{m-1}\prod_{\substack{p=1\\p\neq i}}^{n}(t-x_p)^m\right] - \left[(t-x_l)^{m-1}\prod_{\substack{p=1\\p\neq l}}^{n}(t-x_p)^m\right]\right)dt$$

$$=(-m)\int_{x_1}^{x_j} t^k \sum_{1\le i<l\le n} \frac{1}{x_i-x_l}\left[(t-x_i)^{m-1}(t-x_l)^m - (t-x_i)^m(t-x_l)^{m-1}\right]\prod_{\substack{p=1\\p\neq i,l}}^{n}(t-x_p)^m dt$$

$$=(-m)\int_{x_1}^{x_j} t^k \left[\prod_{p=1}^{n}(t-x_p)^{m-1}\right]\sum_{1\le i<l\le n}\left[\frac{(t-x_l)-(t-x_i)}{x_i-x_l}\prod_{\substack{p=1\\p\neq i,l}}^{n}(t-x_p)\right]dt$$

$$=(-m)\int_{x_1}^{x_j} t^k \sum_{1\le i<l\le n}(t-x_i)^{m-1}(t-x_l)^{m-1}\prod_{\substack{p=1\\l\neq i,l}}^{n}(t-x_l)^m dt$$

and hence

$$L_m(Q_T^{k,m}) = m(m-1)\int_{x_1}^{x_j} t^k \sum_{i=1}^{n}(t-x_i)^{m-2}\prod_{\substack{l=1\\l\neq i}}^{n}(t-x_l)^m dt$$
$$+ 2m^2\int_{x_1}^{x_j} t^k \sum_{1\le i<l\le n}(t-x_i)^{m-1}(t-x_l)^{m-1}\prod_{\substack{p=1\\p\neq i,j}}^{n}(t-x_l)^m dt.$$

We recognize this expression as being nothing more than

$$L_m(Q_T^{k,m}) = \int_{x_1}^{x_j} t^k \left(\frac{\partial^2}{\partial t^2}\prod_{i=1}^{n}(t-x_i)^m\right)dt. \tag{7.6}$$

Now, if we evaluate (7.1) by the fundamental theorem of calculus we get

$$\int_{x_1}^{x_j}\frac{\partial^2}{\partial t^2}\left(t^k\prod_{i=1}^{n}(t-x_i)^m\right)dt$$
$$= kt^{k-1}\prod_{i=1}^{n}(t-x_i)^m + t^k\sum_{i=1}^{n}(t-x_i)^{m-1}\prod_{\substack{l=1\\l\neq i}}(t-x_l)^m\Bigg|_{t=x_1}^{t=x_j}$$
$$= 0.$$

Thus we have $(7.2)+(7.3)+(7.4)=0$. Similarly we can evaluate (7.5) to obtain

$$\begin{aligned}
&\frac{1}{2}(7.3)+(7.4)\\
=&\int_{x_1}^{x_j}\frac{\partial}{\partial t}\left[t^k\frac{\partial}{\partial t}\left(\prod_{i=1}^{n}(t-x_i)^m\right)\right]dt\\
=&t^k\sum_{i=1}^{n}(t-x_i)^{m-1}\prod_{\substack{l=1\\ l\neq i}}(t-x_l)^m\Bigg|_{t=x_1}^{t=x_j}\\
=&0.
\end{aligned}$$

Using $(7.2)+(7.3)+(7.4)=0$ and $\frac{1}{2}(7.3)+(7.4)=0$, we obtain $(7.2)=(7.4)$. So by (7.6) we have

$$\begin{aligned}
L_m(Q_T^{k,m}) &= (7.4)\\
&= (7.2)\\
&= k(k-1)Q_T^{k-2,m}.
\end{aligned}$$

Thus we have proven the theorem for $m=0$ and $m\geq 2$. For $m=1$ similar logic works. We first compute

$$\begin{aligned}
\sum_{i=1}^{n}\frac{\partial^2}{\partial {x_i}^2}\left(\int_{x_1}^{x_j}t^k\prod_{l=1}^{n}(t-x_l)dt\right) &= \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(-\int_{x_1}^{x_j}t^k\prod_{\substack{l=1\\ l\neq i}}^{n}(t-x_l)dt\right)\\
&= -\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_j}\right)\left(\int_{x_1}^{x_j}t^k\prod_{\substack{l=1\\ l\neq i}}^{n}(t-x_l)dt\right)\\
&= x_1^k\left(\prod_{i=2}^{n}(x_1-x_i)\right)-x_j^k\left(\prod_{\substack{i=1\\ i\neq j}}^{n}(x_j-x_i)\right)
\end{aligned}$$

and we can easily verify that this quantity is also equal to

$$-\int_{x_1}^{x_j}\frac{\partial^2}{\partial t^2}\left(t^k\prod_{i=1}^{n}(t-x_i)\right)dt = -(7.1).$$

With that in hand, we also compute

$$\begin{aligned}
&\sum_{1\leq i<l\leq n} \frac{1}{x_i - x_l} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_l}\right) \int_{x_1}^{x_j} t^k \prod_{l=1}^{n}(t - x_l)dt \\
&= \sum_{1\leq i<l\leq n} \frac{1}{x_i - x_l} \int_{x_1}^{x_j} t^k \left(\prod_{\substack{p=1\\p\neq i}}^{n}(t - x_p) - \prod_{\substack{p=1\\p\neq l}}^{n}(t - x_p)\right) dt \\
&= \sum_{1\leq i<l\leq n} \int_{x_1}^{x_j} t^k \left(\prod_{\substack{p=1\\p\neq i,l}}^{n} (t - x_p) \left[\frac{(t - x_l) - (t - x_i)}{x_i - x_l}\right]\right) dt \\
&= \sum_{1\leq i<l\leq n} \int_{x_1}^{x_j} t^k \prod_{\substack{p=1\\p\neq i,l}}^{n} (t - x_p)dt
\end{aligned}$$

Combining this with the following:

$$\begin{aligned}
(7.4) &= \int_{x_1}^{x_j} t^k \left(\frac{\partial^2}{\partial t^2} \prod_{i=1}^{n}(t - x_i)\right) dt \\
&= 2 \sum_{1\leq i<l\leq n} \int_{x_1}^{x_j} t^k \prod_{\substack{p=1\\p\neq i,l}}^{n} (t - x_p)dt
\end{aligned}$$

shows that we have $L_1 Q_T^{k,1} = -(7.1) + (7.4)$. Further, we have

$$\begin{aligned}
(7.5) &= \int_{x_1}^{x_j} \frac{\partial}{\partial t}\left[t^k \frac{\partial}{\partial t}\left(\prod_{i=1}^{n}(t - x_i)\right)\right] dt \\
&= \int_{x_1}^{x_j} \frac{\partial}{\partial t}\left[t^k \sum_{i=1}^{n} \prod_{\substack{l=1\\l\neq i}}^{n}(t - x_l)\right] dt \\
&= x_j^k \left(\prod_{\substack{i=1\\i\neq j}}^{n}(x_j - x_i)\right) - x_1^k \left(\prod_{i=2}^{n}(x_1 - x_i)\right) \\
&= (7.1).
\end{aligned}$$

Hence we conclude

$$\begin{aligned}
(7.1) &= (7.2) + (7.3) + (7.4) \\
&= (7.2) + (-2(7.4) + 2(7.5)) + (7.4) \\
&= (7.2) - (7.4) + 2(7.1) \\
\Rightarrow (7.2) &= -(7.1) + (7.4) \\
&= L_1 Q_T^{k,1}
\end{aligned}$$

thus completing the proof. □

8 Change of Basis Matrix for Quasiinvariants

We now turn our attention to analyzing the relationship between the m-quasiinvariants and the $(m+1)$-quasiinvariants. In particular, recall that $\mathbf{QI}_m \supset \mathbf{QI}_{m+1} \supset \Lambda_n$ for all m, and so we can expand any basis for $\mathbf{QI}_{m+1}$ in terms of a basis for $\mathbf{QI}_m$ over the ring Λ_n of symmetric functions. Each of these bases has $n!$ elements, and thus we obtain a square change of basis matrix. Since the only invertible symmetric functions are the constants, any choice of bases for $\mathbf{QI}_m$ and $\mathbf{QI}_{m+1}$ will yield a change of basis matrix with the same determinant up to a scalar multiple. We in fact obtain the following explicit formula for these determinants:

Theorem 5. *For all n and m, any matrix expressing the expansion of a basis for $\mathbf{QI}_{m+1}$ in terms of a basis for $\mathbf{QI}_m$ with symmetric function coefficients will have a determinant equal to a scalar multiple of $(\Delta_n)^{n!}$, where Δ_n denotes the Vandermonde determinant $\prod_{1\le i<j\le n}(x_i - x_j)$.*

Lemma 6. *The ring $\Delta_n^2 \cdot \mathbf{QI}_m$ is a subring of $\mathbf{QI}_{m+1}$.*

Proof. Since Δ_n is antisymmetric, Δ_n^2 is a symmetric function and by Proposition 5, we have for polynomial $P \in \mathbf{QI}_m$,

$$(1-(i,j))(\Delta_n^2 P) = \Delta_n^2((1-(i,j))P) = \Delta_n^2(x_i - x_j)^{2m+1}P'$$

for all $1 \le i < j \le n$. In particular, for all $1 \le i < j \le n$, the polynomial $(1-(i,j))(\Delta_n^2 P)$ is divisible by $(x_i - x_j)^{2m+3}$ and thus $\Delta_n^2 P$ is $(m+1)$-quasiinvariant. □

Proof of Theorem 5. We begin picking a basis (over Λ_n) of homogeneous polynomials $\{\beta_{S,T}\}$ for $\mathbf{QI}_m$ where S and T range over all pairs of standard tableaux of the same shape and the degree of $\beta_{S,T}$ is $m\left(\binom{n}{2} - content(\lambda(T))\right) + cocharge(T)$. We know this is possible by the Hilbert series stated in (1.1). We similarly pick a basis $\{\alpha_{S,T}\}$ for $\mathbf{QI}_{m+1}$. Now, by Lemma 6 we have the following containments:

$$\Delta_n^2 \cdot \mathbf{QI}_m \subset \mathbf{QI}_{m+1} \subset \mathbf{QI}_m.$$

We label these modules M_1, M_2, and M_3 respectively and use the basis $\{\Delta_n^2 \beta_{S,T}\}$ for M_1. We set A to be the change of basis matrix between M_1 and M_2 and B to be the matrix from M_2 to M_3. We immediately obtain that $AB = diag(\Delta_n^2)$. Thus, in particular, $\det(B)$ divides $\Delta_n^{2(n!)}$.

We now consider the degree of an arbitrary non-zero term of $\det(B)$. By considering the difference in degrees of all basis elements, we must have

$degree(\det(B))$

$$= \left(\sum_{T \in ST(n)} f_{\lambda(T)} (m+1) \left(\binom{n}{2} - content(\lambda(T)) \right) + cocharge(T) \right)$$
$$- \left(\sum_{T' \in ST(n)} f_{\lambda(T')} m \left(\binom{n}{2} - content(\lambda(T')) \right) + cocharge(T') \right)$$
$$= \sum_{\lambda \vdash n} f_\lambda^2 \left(\binom{n}{2} - content(\lambda) \right)$$

However, it is easy to see that $f_\lambda = f_{\lambda'}$ and $content(\lambda) = -content(\lambda')$, where λ' is the conjugate (or transpose) of partition λ. Hence we have

$$degree(\det(B)) = \sum_{\lambda \vdash n} f_\lambda^2 \binom{n}{2} = \binom{n}{2} n!.$$

Since $\det(B)$ is a symmetric function of degree $\binom{n}{2}n!$ which divides $\Delta_n^{2(n!)}$, and Δ_n^2 has no nontrivial symmetric function factors, we conclude that $\det(B)$ equals $\Delta_n^{n!}$, up to a scalar multiple. □

9 Conclusions and Open Problems

In this paper, we provided a decomposition of the ring of m-quasiinvariants into isotypic components and gave two easy criteria for characterizing such elements. One application of this new characterization was an explicit description of a basis for the isotypic component corresponding to shape $[n-1, 1]$. In particular such basis elements can be written as either integrals or algebraically using polynomials with coefficients given as products of binomial coefficients.

One natural extension of this research involves further analysis of the representation theoretic aspects of m-quasiinvariants. In particular can one recharacterize quasiinvariants for other Coxeter groups using analogous criteria. Another direction is the computation of explicit bases for more isotypic components. It would be even better if the operator L_m respected these bases in a similar manner.

Acknowledgements.

The authors are grateful for Adriano Garsia's guidance in this project as well as the support of the NSF. We also would like to thank Vic Reiner for conversations which helped motivate the analysis of section 8.

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