Dimer Interpretations of toric cluster variables associated to del Pezzo quivers: From Dungeons to Dragons

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RIMS Cluster Algebra 19

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Based on Joint Work with Tri Lai.

Outline

1. Propp's Problems and Progress for Perfect Matchings
2. Aztec Diamonds, Dragons and Dungeons
3. Combinatorial Interpretation for dP_3, Model 1 (Lai-M, Leoni-M-Neel-Turner)
4. From Dungeons to Dragons for dP_3, Other Models (Lai-M)
5. Double Dimers and Progess towards the Conjecture
6. Epilogue: dP_2 (Somos 5, Gao-Li-Vuong-Yang)

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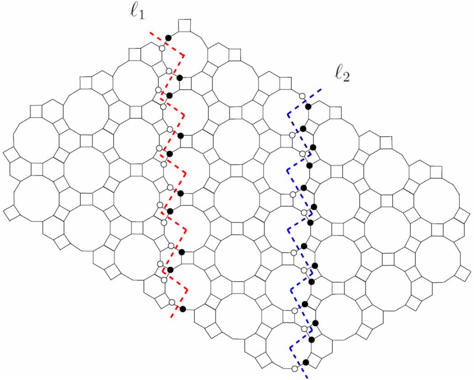
Part of this based on work done during the 2012, 2013 and 2016 REU (Research Experience for Undergraduates) in Combinatorics at University of Minnesota, Twin Cities.

Slides at `http://math.umn.edu/~musiker/RIMS19.pdf`

Combinatorial History

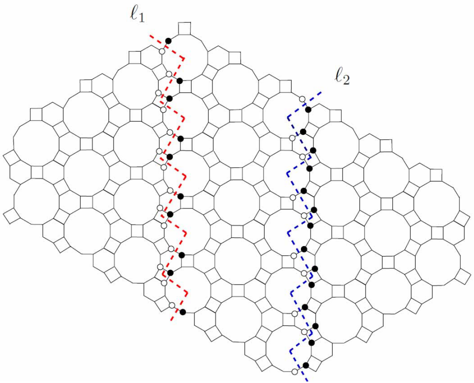
In 1999, Jim Propp published an article tracking the progress of 32 problems in the field of exact enumeration of perfect matchings.

One such problem therein was **Blum's conjecture** regarding the number of perfect matchings in a family of regions known as hexagonal dungeons.



Blum's Conjecture and Hexagonal Dungeons

Conjecture (Blum, Problem 25 of [Propp '99]) The number of perfect matchings of the hexagonal dungeon $HD_{a,2a,b}$ is $13^{2a^2}14^{\lfloor \frac{a^2}{2} \rfloor}$.



Theorem (Ciucu-Lai 2014) Blum's Conjecture is True.

Blum's Conjecture and Hexagonal Dungeons

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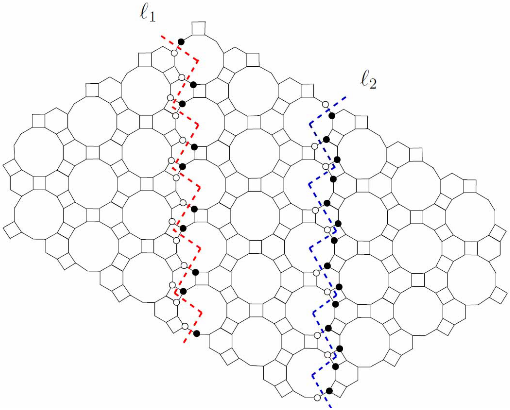
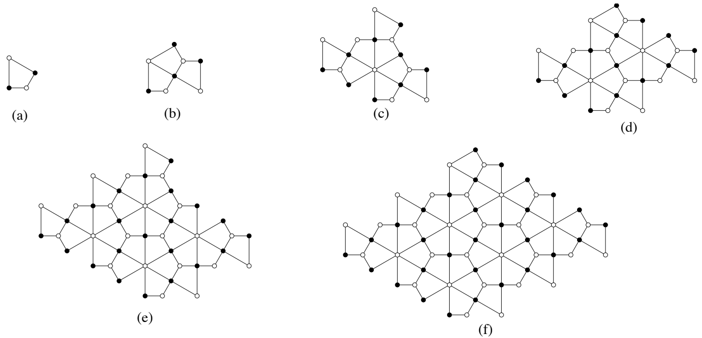


Proof Sketch: Follow the above graph cuts, introduce a new 3-parameter family of graphs, recursively enumerate their number of perfect matchings, and relate perfect matchings in the pieces to perfect matchings in the full graph. (All counts are products of powers of 13 and 14.)

Aztec Dragons

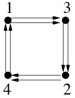
In the same article, Propp introduces another family of graphs.

Theorem (Wieland '99, Ciucu '03) The number of perfect matchings of Aztec Dragons is given by $2^{n(n+1)}$.



Theorem (Lai '15) Aztec Dragons sit inside a generalized 3-parameter family; each has $2^{(b-c+1)(2b-a-c)+(a-b)^2}3^{\frac{(a-b+c)(a-b+c-1)}{2}}$ perfect matchings.

Aztec Diamonds (F_0 / Hirzebruch Surface)

Let $Q =$ 

, and mutate periodically at $1, 2, 3, 4, 1, 2, 3, 4, \ldots$.

$$x_n x_{n-4} = \begin{cases} x_{n-1}^2 + x_{n-2}^2 & \text{when } n \text{ is odd, and} \\ x_{n-2}^2 + x_{n-3}^2 & \text{when } n \text{ is even.} \end{cases}$$

By letting $x_1 = x_2$ and $x_3 = x_4$, we get $x_{2n+1} = x_{2n}$ for all n.

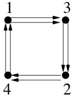
Letting $\{T_n\}$ be the sequence $\{x_{2n}\}_{n\in\mathbb{Z}}$, we obtain a single recurrence.

$$T_n T_{n-2} = 2T_{n-1}^2.$$

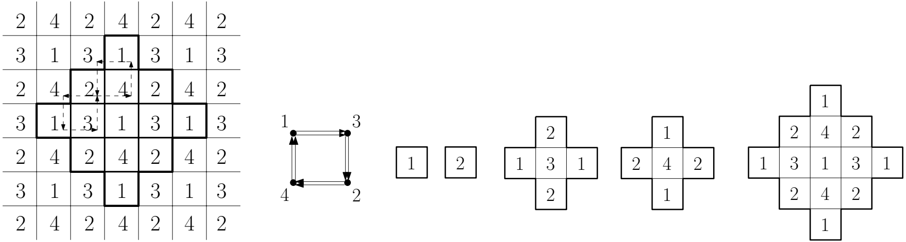
If $T_1 = T_2 = 1$, $\{T_n\} = \{1, 1, 2, 8, 64, 1024, 32768, \ldots\} = \left\{2^{\frac{(n-1)(n-2)}{2}}\right\}$.

For $n \geq 3$, $T_n = \#$ (perfect matchings of the $(n-2)$nd Aztec Diamond).

Aztec Diamonds (F_0 / Hirzebruch Surface)

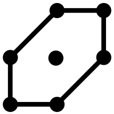
Let $Q =$ 

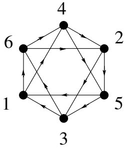
, and mutate periodically at $1, 2, 3, 4, 1, 2, 3, 4, \ldots$.

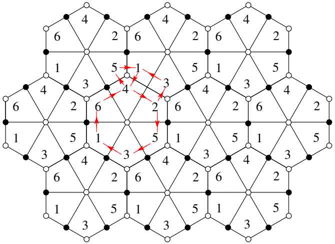


$$x_5 = \frac{x_3^2+x_4^2}{x_1},\ x_6 = \frac{x_3^2+x_4^2}{x_2},\ x_7 = \frac{(x_3^2+x_4^2)^2(x_1^2+x_2^2)}{x_1^2x_2^2x_3},\ \text{and } x_8 = \frac{(x_3^2+x_4^2)^2(x_1^2+x_2^2)}{x_1^2x_2^2x_4}.$$

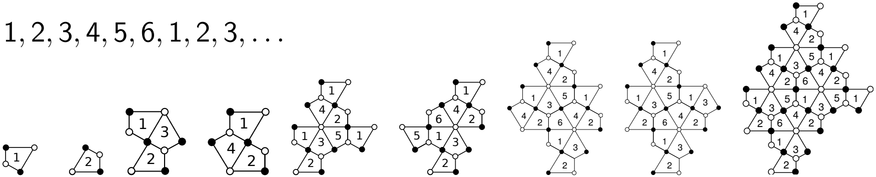
The Del Pezzo 3 Quiver and Aztec Dragons







Mutating $1, 2, 3, 4, 5, 6, 1, 2, 3, \ldots$



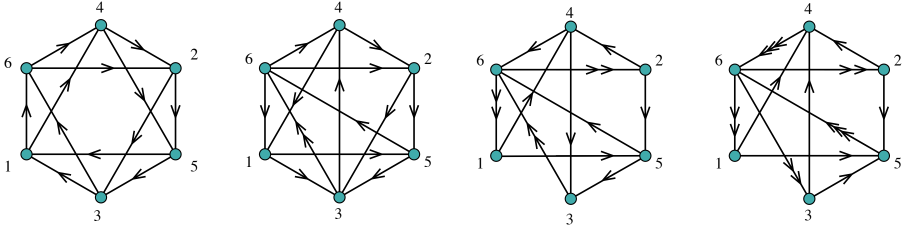
...

Introduced by Jim Propp, Ben Wieland, and Mihai Ciucu. Studied further by Cottrell-Young.

$$x_{2n+7}x_{2n+1} = x_{2n+3}x_{2n+5} + x_{2n+4}x_{2n+6} \text{ and}$$
$$x_{2n+8}x_{2n+2} = x_{2n+3}x_{2n+5} + x_{2n+4}x_{2n+6}.$$

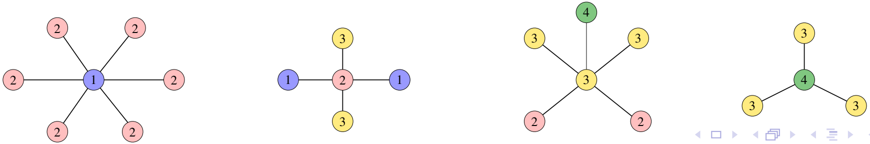
Toric Mutations and Toric Phases of dP_3

Toric mutations take place at vertices with in-degree and out-degree 2.



Starting with any of these four models of the dP_3 quiver, any sequence of toric mutations yields a quiver that is graph isomorphic to, or the opposite quiver of, one of these.

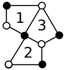
Figure 20 of Eager-Franco (Incidences betweeen these Models):



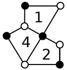
Example from S. Zhang (2012 REU): Periodic mutation $1, 2, 3, 4, 5, 6, 1, 2, \ldots$ yields partition functions for Aztec Dragons (as studied by Ciucu, Cottrell-Young, and Propp) under appropriate weighted enumeration of perfect matchings.

$$\frac{x_3x_5+x_4x_6}{x_1}$$

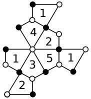
$$\frac{x_4x_6+x_3x_5}{x_2}$$



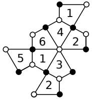
$$\frac{x_2x_3x_5^2+x_1x_3x_5x_6+x_2x_4x_5x_6+x_1x_4x_6^2}{x_1x_2x_3}$$



$$\frac{x_2x_3x_5^2+x_1x_3x_5x_6+x_2x_4x_5x_6+x_1x_4x_6^2}{x_1x_2x_4}$$



$$\frac{(x_2x_5+x_1x_6)(x_1x_3+x_2x_4)(x_3x_5+x_4x_6)^2}{x_1^2x_2^2x_3x_4x_5}$$



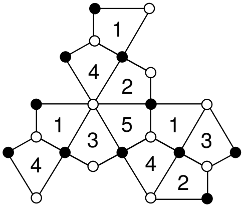
$$\frac{(x_2x_5+x_1x_6)(x_1x_3+x_2x_4)(x_3x_5+x_4x_6)^2}{x_1^2x_2^2x_3x_4x_6}$$

Goal: Combinatorial Formula for Toric Cluster Variables

Example from M. Leoni, S. Neel, and P. Turner (2013 REU): Mutations at antipodal vertices of dP_3 quiver yield τ-mutation sequences. Resulting Laurent polynomials correspond to Aztec Castles under appropriate weighted enumeration of perfect matchings.

e.g. $1, 2, 3, 4, 1, 2, 5, 6$ yields cluster variable (which is **not** an Aztec Dragon)

$$\begin{aligned}
&(x_1x_2^2x_3^3x_5^4 + x_2^3x_3^2x_4x_5^4 + 2x_1^2x_2x_3^3x_5^3x_6 + 4x_1x_2^2x_3^2x_4x_5^3x_6 + 2x_2^3x_3x_4^2x_5^3x_6 + x_1^3x_3^3x_5^2x_6^2 \\
+ \quad & 5x_1^2x_2x_3^2x_4x_5^2x_6^2 + 5x_1x_2^2x_3x_4^2x_5^2x_6^2 + x_2^3x_4^3x_5^2x_6^2 + 2x_1^3x_3^2x_4x_5x_6^3 + 4x_1^2x_2x_3x_4^2x_5x_6^3 \\
+ \quad & 2x_1x_2^2x_4^3x_5x_6^3 + x_1^3x_3x_4^2x_6^4 + x_1^2x_2x_4^3x_6^4)/x_1^2x_2^2x_3^2x_4^2x_6 = \frac{(x_1x_3+x_2x_4)(x_4x_6+x_3x_5)^2(x_1x_6+x_2x_5)^2}{x_1^2x_2^2x_3^2x_4^2x_6}
\end{aligned}$$

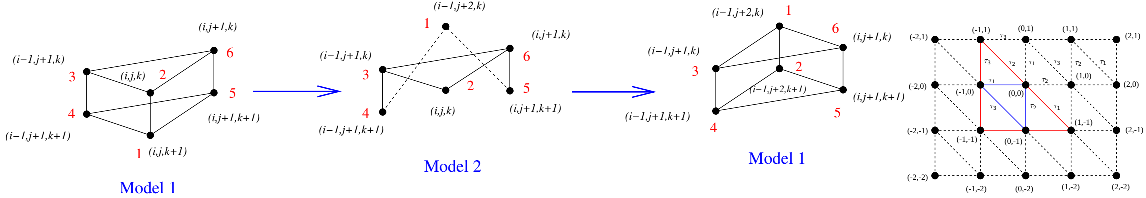


$\mathbb{Z}^3$ Parameterization for Toric Cluster Variables

Theorem 1 [Lai-M 2015] Starting from the initial cluster $\{x_1, x_2, \ldots, x_6\}$, the set of cluster variables reachable via toric mutations can be parameterized by $\mathbb{Z}^3$.

(Requires consideration of cycles in the toric exhange graph.)

Under this correspondence, the initial cluster bijects to the prism (i.e. zonotope) $[(0,-1,1),(0,-1,0),(-1,0,0),(-1,0,1),(0,0,1),(0,0,0)]$ and toric mutations transform the six-tuple in $\mathbb{Z}^3$ as we will illustrate.

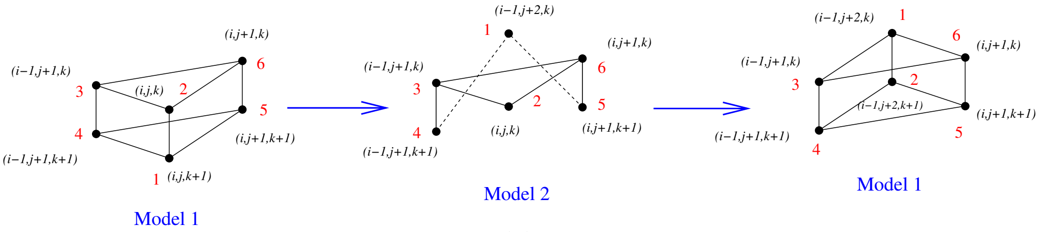


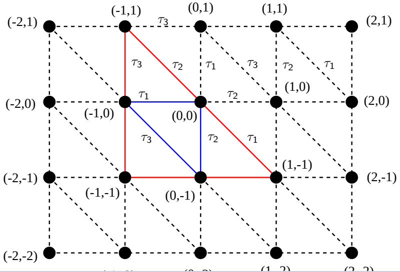
Up to symmetry, enough to consider $\mu_1\mu_2\mu_3\mu_4$, $\mu_3\mu_4\mu_5\mu_6$, $\mu_1\mu_4\mu_1\mu_5\mu_1$ and $\mu_2\mu_3\mu_2\mu_6\mu_2$.

Each induce affine translations of the prism. (Reflection induced by $\mu_1\mu_2$ illustrated.)

$\mathbb{Z}^3$ Parameterization for Toric Cluster Variables

Each induce affine translations of the prism. (Reflection induced by $\tau_1 = \mu_1\mu_2$ illustrated.)





Parameterizing Toric Exchange Graph for dP_2 (Gao-Li-Vuong-Yang 2016)

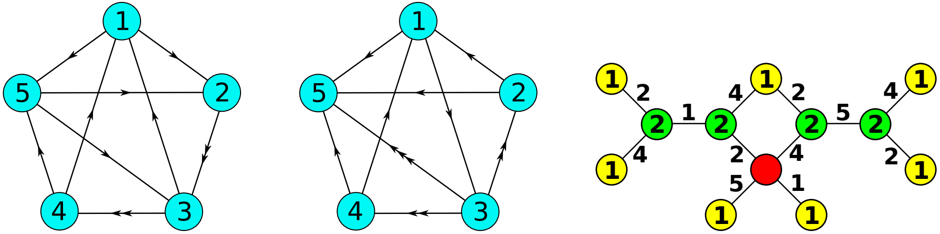


Figure: Toric mutation sequences that start from model 1 and return to model 1 the first time.

Definition (ρ-mutations, Building Blocks of Quiver Modular Group)

$$\rho_1 = \mu_1 \circ (54321), \quad \rho_2 = \mu_5 \circ (12345), \quad \rho_3 = \mu_2 \circ \mu_4 \circ (24),$$

$$\rho_4 = \mu_2 \circ \mu_1 \circ \mu_4 \circ (531), \quad \rho_5 = \mu_4 \circ \mu_5 \circ \mu_2 \circ (351),$$

$$\rho_6 = \mu_2 \circ \mu_1 \circ \mu_2 \circ (531)(24), \quad \rho_7 = \mu_4 \circ \mu_5 \circ \mu_4 \circ (135)(24).$$

Relations between $\rho-$mutations (Gao-Li-Vuong-Yang)

Definition (ρ-mutations, Building Blocks of Quiver Modular Group)

$$\rho_1 = \mu_1 \circ (54321), \quad \rho_2 = \mu_5 \circ (12345), \quad \rho_3 = \mu_2 \circ \mu_4 \circ (24),$$
$$\rho_4 = \mu_2 \circ \mu_1 \circ \mu_4 \circ (531), \quad \rho_5 = \mu_4 \circ \mu_5 \circ \mu_2 \circ (351),$$
$$\rho_6 = \mu_2 \circ \mu_1 \circ \mu_2 \circ (531)(24), \quad \rho_7 = \mu_4 \circ \mu_5 \circ \mu_4 \circ (135)(24).$$

Proposition for the Cluster Modular Group

$$\rho_4 = \rho_1^2\rho_3, \quad \rho_5 = \rho_2^2\rho_3, \quad \rho_6 = \rho_1^2, \quad \rho_7 = \rho_2^2.$$
$$\rho_1\rho_2 = \rho_2\rho_1 = \rho_3^2 = 1.$$
$$\rho_1^2\rho_3 = \rho_3\rho_1^2, \quad \rho_2^2\rho_3 = \rho_3\rho_2^2, \quad \rho_1\rho_3\rho_2 = \rho_2\rho_3\rho_1.$$

It suffices to consider ρ_1, ρ_2, ρ_3.

$\rho-$mutation sequence: visualization (Gao-Li-Vuong-Yang)

Proposition (Relations between ρ_1, ρ_2, ρ_3 on the level of clusters)

$$\rho_1\rho_2 = \rho_2\rho_1 = \rho_3^2 = 1.$$

$$\rho_1^2\rho_3 = \rho_3\rho_1^2, \quad \rho_2^2\rho_3 = \rho_3\rho_2^2, \quad \rho_1\rho_3\rho_2 = \rho_2\rho_3\rho_1.$$

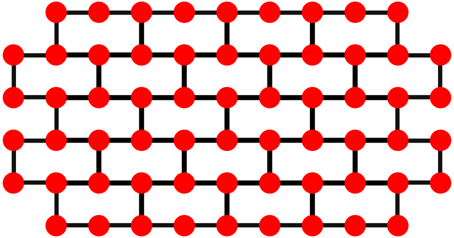
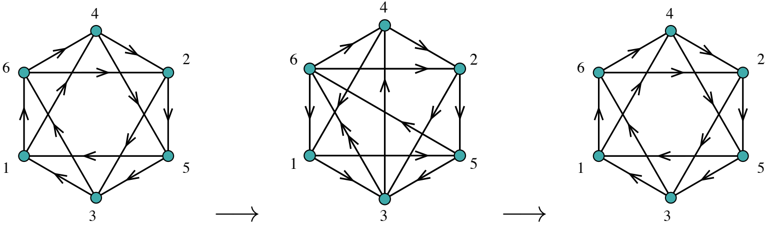


Figure: A visualization of $\rho-$mutations. Toric cluster variables in bijection with $\mathbb{Z}^2$ (faces) for dP_2.

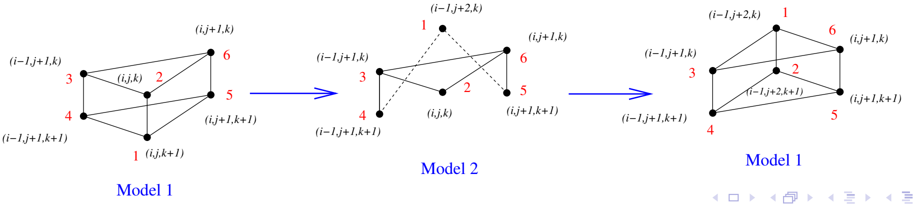
$\rho_1 :\rightarrow, \quad \rho_2 :\leftarrow, \quad \rho_3 :\uparrow / \downarrow.$ Note : Somos 5 cuts out a specific horizontal two layer slice.

Mutating Model I to Model II and Back (Reflection in (i,j)-plane)

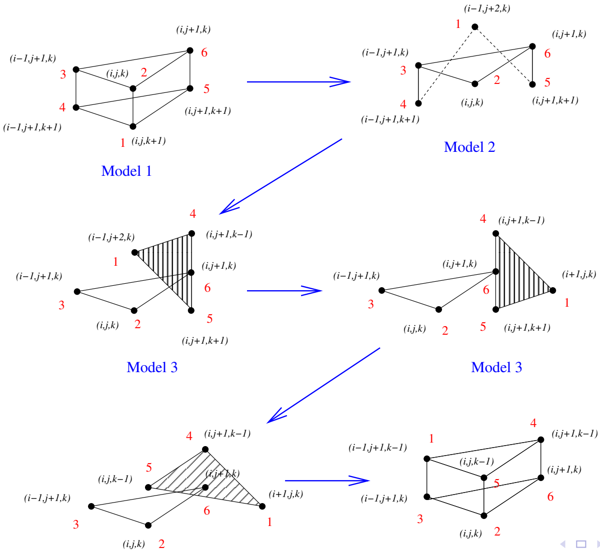
By applying $\mu_1 \circ \mu_2$, $\mu_3 \circ \mu_4$, or $\mu_5 \circ \mu_6$, we mutate the quiver (up to graph isomorphism):



Corresponding action in $\mathbb{Z}^3$ (on triangular prisms):



Illustrating the mutations $\mu_1\mu_4\mu_1\mu_5\mu_1$ (Translation in Coordinate k)



Segway: Algebraic Formula for Toric Cluster Variables

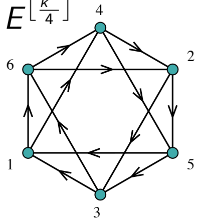
$$\text{Let} \quad A = \frac{x_3x_5 + x_4x_6}{x_1x_2}, \quad B = \frac{x_1x_6 + x_2x_5}{x_3x_4}, \quad C = \frac{x_1x_3 + x_2x_4}{x_5x_6},$$

$$D = \frac{x_1x_3x_6 + x_2x_3x_5 + x_2x_4x_6}{x_1x_4x_5}, \text{ and } E = \frac{x_2x_4x_5 + x_1x_3x_5 + x_1x_4x_6}{x_2x_3x_6}.$$

Let $z_{i,j,k}$ be the cluster variable corresponding to $(i,j,k) \in \mathbb{Z}^3$

Theorem 2 [Lai-M 2015] (Extension of [LMNT 2013] and [Lai 2014]):

$$z_{i,j,k} = x_r \; A^{\lfloor \frac{(i^2+ij+j^2+1)+i+2j}{3} \rfloor} B^{\lfloor \frac{(i^2+ij+j^2+1)+2i+j}{3} \rfloor} C^{\lfloor \frac{i^2+ij+j^2+1}{3} \rfloor} D^{\lfloor \frac{(k-1)^2}{4} \rfloor} E^{\lfloor \frac{k^2}{4} \rfloor}$$



where, working **modulo** 6, we have (cyclically around the dP_3 Quiver)

$r = 6$ if $2(i-j)+3k \equiv 0$, $\quad r = 4$ if $2(i-j)+3k \equiv 1$,
$r = 2$ if $2(i-j)+3k \equiv 2$, $\quad r = 5$ if $2(i-j)+3k \equiv 3$,
$r = 3$ if $2(i-j)+3k \equiv 4$, $\quad r = 1$ if $2(i-j)+3k \equiv 5$.

i.e. we **determine** x_r by looking at $(i-j)$ modulo 3 and k modulo 2.

Towards a Combinatorial Formula for Toric Cluster Variables

Theorem 3 [Lai-M 2015] (Extension of [Leoni-M-Neel-Turner 2014]):
Let $Z^S = [z_1, z_2, \ldots, z_6]$ be the cluster obtained after applying a toric mutation sequence S to the initial cluster $\{x_1, x_2, \ldots, x_6\}$.

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Then $\mathbf{Z^S} = [\mathbf{w}(\mathcal{G}(\mathcal{C}_\mathbf{1}^\mathbf{S}), \mathbf{w}(\mathcal{G}(\mathcal{C}_\mathbf{2}^\mathbf{S}), \ldots, \mathbf{w}(\mathcal{G}(\mathcal{C}_\mathbf{6}^\mathbf{S})]$ where we construct $\mathcal{G}(\mathcal{C}_i)$ as a subgraph cut out by the contour $\mathcal{C}_i$, $w(\mathcal{G})$ is the partition function $\sum_{M \text{ a perfect matching of } \mathcal{G}} x(M)$, and the contours $\mathcal{C}^{S_1}, \mathcal{C}^{S_2}, \ldots, \mathcal{C}^{S_6}$ are contours are defined as follows:

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1) Start with the six-tuple $[(0,-1,1),(0,-1,0),(-1,0,0),(-1,0,1),(0,0,1),(0,0,0)]$ in $\mathbb{Z}^3$.
2) Toric Mutations transform this six-tuple as illustrated earlier.
3) Map from $\mathbb{Z}^3$ to $\mathbb{Z}^6$:

$$(i,j,k) \to (a,b,c,d,e,f) = (j+k, -i-j-k, i+k, j-k+1, -i-j+k-1, i-k+1)$$

and use these six six-tuples to define contours $\mathcal{C}^{S_1}, \mathcal{C}^{S_2}, \ldots, \mathcal{C}^{S_6}$.

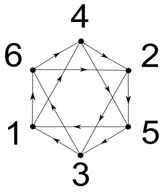
Note: The entries of the resulting 6-tuple sum to one, and satisfy the linear relations $a+b=d+e$, $b+c=e+f$, and $c+d=f+a$ (two out of three of these imply the third).

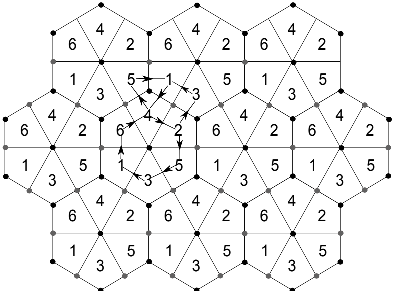
Towards a Combinatorial Formula for Toric Cluster Variables: Contours

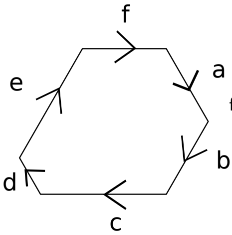
$\varphi : \mathbb{Z}^3 \to \mathbb{Z}^6$ by $\varphi(i,j,k) \to (a,b,c,d,e,f) = (j+k, -i-j-k, i+k, j-k+1, -i-j+k-1, i-k+1)$

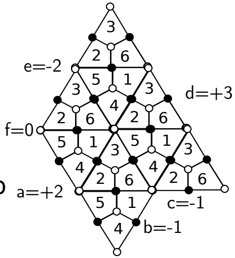
Note: The entries of the resulting 6-tuple sum to one, and satisfy the linear relations $a+b=d+e$, $b+c=e+f$, and $c+d=f+a$ (two out of three of these imply the third).

The contours $\mathcal{C}^{S_1}, \mathcal{C}^{S_2}, \ldots, \mathcal{C}^{S_6}$ are defined as: **(Linear relations ensure they close up.)**





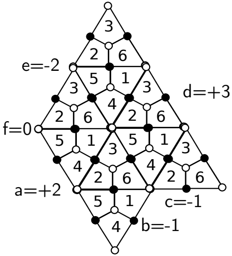




Sign determines direction of the sides. Magnitude determines length

Towards a Combinatorial Formula for Toric Cluster Variables: Subgraphs

1) Draw the contour $\mathcal{C}$ on top of the dP_3 lattice starting from a degree 6 white vertex.
2) For all sides of "positive" length, we erase all the **black** vertices.
3) For all sides of "negative" length, we erase all the **white** vertices.
For sides of "zero length" (between two sides of positive length), we erase the **white corner** or keep it depending on convexity.
4) After removing "dangling" edges and their incident faces, the remaining subgraph inside the contour is $\mathcal{G}(\mathcal{C})$.



e.g.

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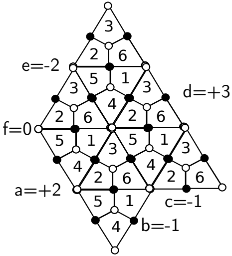

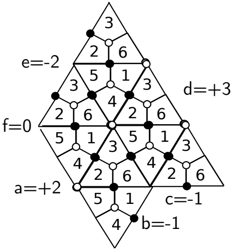

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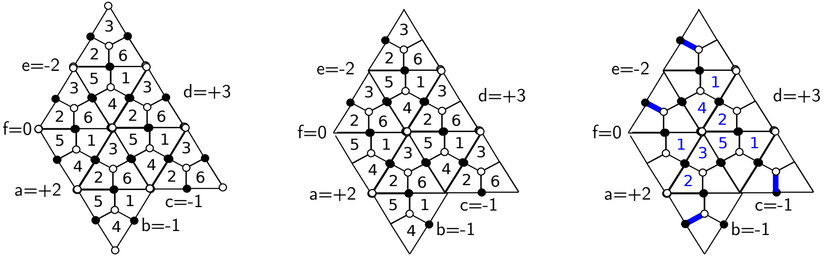


Towards a Combinatorial Formula for Toric Cluster Variables: $x(M)$-weight

$$\mathcal{G} \longrightarrow cm(\mathcal{G}) \sum_{M \,=\, \text{a perfect matching of } \mathcal{G}} \tilde{x}(M), \text{ where}$$

$\tilde{x}(M) = \prod_{\text{edge } e \in M} \frac{1}{x_i x_j}$ (for edge e straddling faces i and j),

$cm(\mathcal{G})$ = the covering monomial of the graph $\mathcal{G}$ (which records what face labels are contained in $\mathcal{G}$ and along its boundary). **(Note:** $x(M) = cm(\mathcal{G})\tilde{x}(M)$ **)**

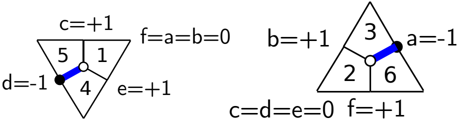
Remark: This is a reformulation of weighting schemes appearing in works such as Speyer ("Perfect Matchings and the Octahedron Recurrence"), Goncharov-Kenyon ("Dimers and cluster integrable systems"), and Di Francesco ("T-systems, networks and dimers").

Alternative definition of cm($\mathcal{G}$): We record all the face labels inside the contour and then divide by the face labels straddling dangling edges.

Initial cluster $\{x_1, x_2, \ldots, x_6\}$ in terms of contours

Consider the six special contours (initial prism $[(0,-1,1),(0,-1,0),(-1,0,0),(-1,0,1),(0,0,1),(0,0,0)]$)

$$C_1 = (0,0,1,-1,1,0), \quad C_2 = (-1,1,0,0,0,1), \quad C_3 = (0,1,-1,1,0,0),$$
$$C_4 = (1,0,0,0,1,-1), \quad C_5 = (1,-1,1,0,0,0), \quad C_6 = (0,0,0,1,-1,1).$$



Applying our general algorithm, $\mathcal{G}(C_i)$'s correspond to graphs consisting of a single edge and a triangle of faces.

Using $G \longrightarrow cm(G) \sum_{M\,=\,\text{a perfect matching of }G} x(M)$, we see

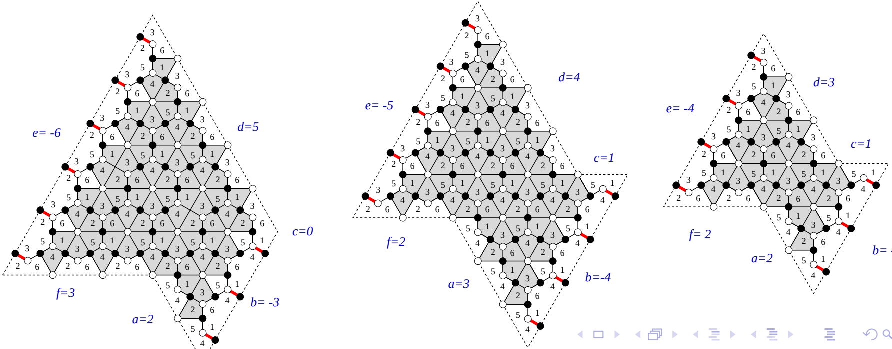
$$cm(\mathcal{G}(C_1)) = x_1x_4x_5 \text{ and } x(M) = \frac{1}{x_4x_5}, \text{ hence } G \longrightarrow \frac{x_1x_4x_5}{x_4x_5} = x_1$$

Similar calculations show $\mathcal{G}(C_i)) \longleftrightarrow x_i$ for $i \in \{1, 2, \ldots, 6\}$.

Example: $S = \tau_1\tau_2\tau_3\tau_1\tau_2\tau_3\tau_2\tau_1\tau_4$

We reach the prism $\{(1,3,1),(1,3,0),(1,2,0),(1,2,1),(0,3,1),(0,3,0)\}$ after applying $\tau_1\tau_2\tau_3\tau_1\tau_2\tau_3\tau_2\tau_1$ ($\tau_1 = \mu_1\mu_2$, $\tau_2 = \mu_3\mu_4$, and $\tau_3 = \mu_5\mu_6$) and then use $\tau_4 = \mu_1\mu_4\mu_1\mu_5\mu_1$ to reach $\{(1,3,-1),(1,3,0),(1,2,0),(1,2,-1),(0,3,-1),(0,3,0)\}$, which yields

$$\mathcal{C}^S = [(2,-3,0,5,-6,3),(3,-4,1,4,-5,2),(2,-3,1,3,-4,2),$$
$$(1,-2,0,4,-5,3),(2,-2,-1,5,-5,2),(3,-3,0,4,-4,1)].$$



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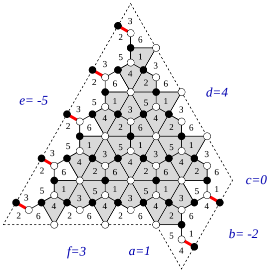
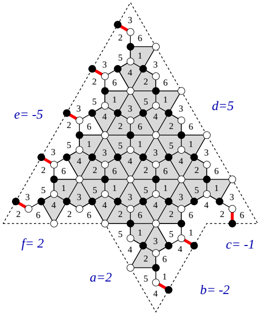
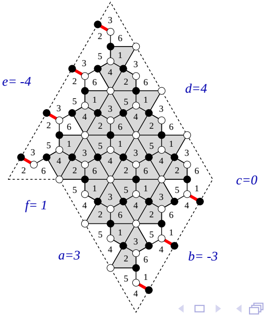
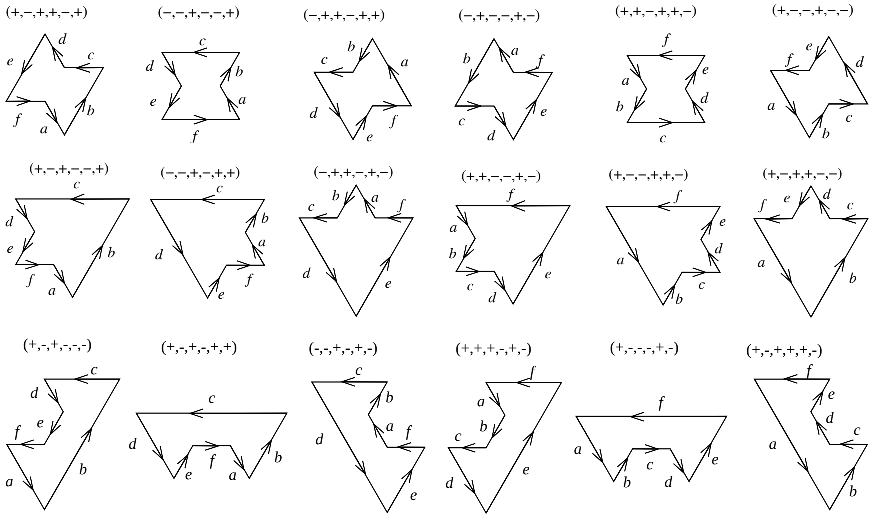


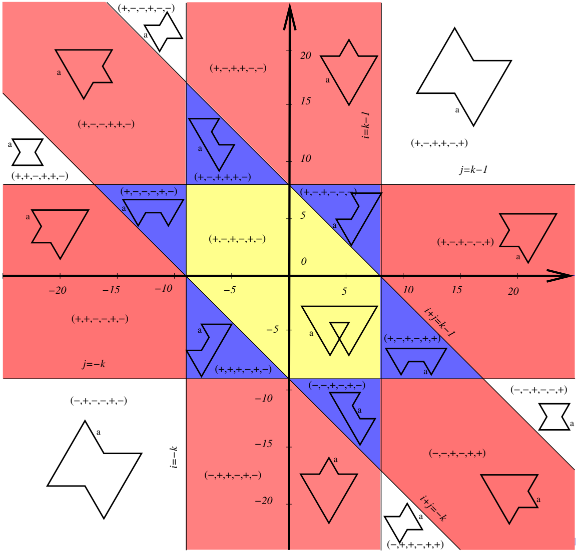




Possible Shapes of Aztec Castles



Cross-section when k positive (the case when k is negative is analogous)



Comparing Enumerative Formulas to Cluster Variables

Recall Lai's enumeration of perfect matchings in Generalized Aztec Dragon Regions:

$$2^{(b-c+1)(2b-a-c)+(a-b)^2}3^{\frac{(a-b+c)(a-b+c-1)}{2}}.$$

Comparing Enumerative Formulas to Cluster Variables

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Algebraic formula for Cluster Variables for dP_3 quiver:

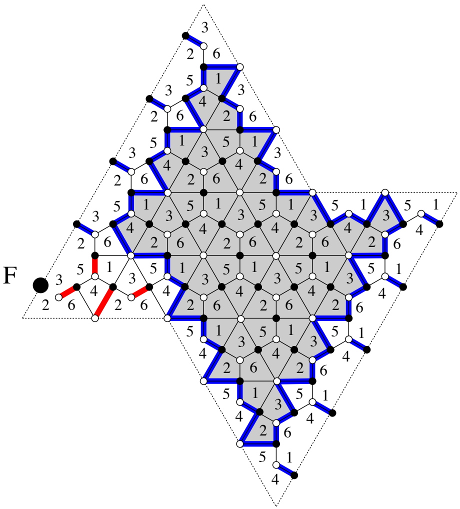
$$z_{i,j,k} = x_r \;\; A^{\lfloor\frac{(i^2+ij+j^2+1)+i+2j}{3}\rfloor}\; B^{\lfloor\frac{(i^2+ij+j^2+1)+2i+j}{3}\rfloor}\; C^{\lfloor\frac{i^2+ij+j^2+1}{3}\rfloor}\; D^{\lfloor\frac{(k-1)^2}{4}\rfloor}\; E^{\lfloor\frac{k^2}{4}\rfloor}$$

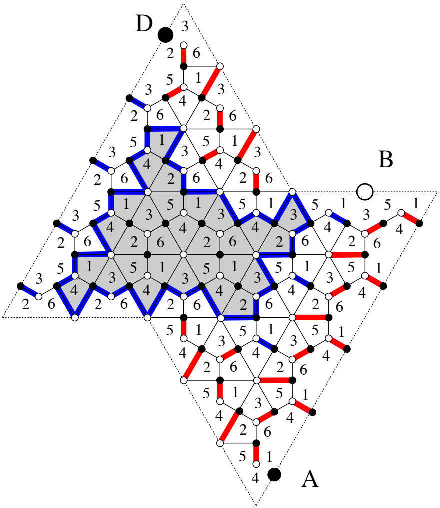
where

$$A = \frac{x_3x_5 + x_4x_6}{x_1x_2}, \quad B = \frac{x_1x_6 + x_2x_5}{x_3x_4}, \quad C = \frac{x_1x_3 + x_2x_4}{x_5x_6},$$

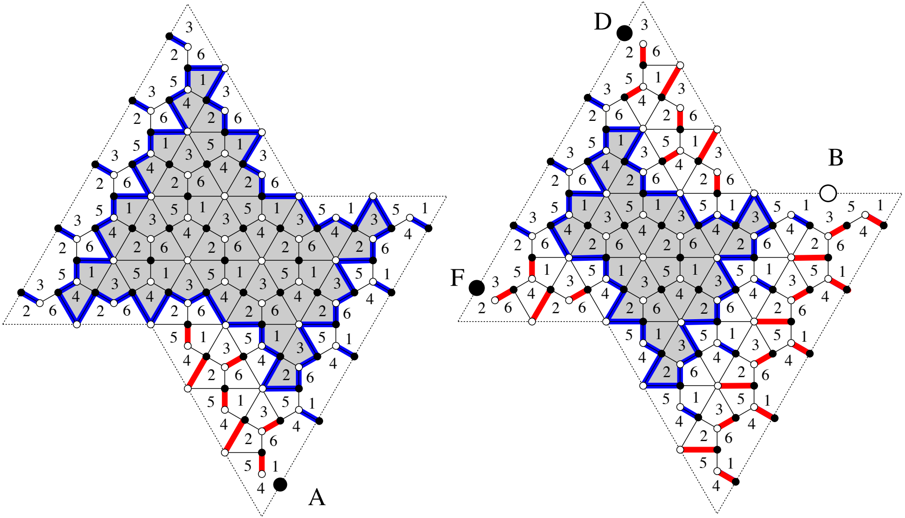
$$D = \frac{x_1x_3x_6 + x_2x_3x_5 + x_2x_4x_6}{x_1x_4x_5}, \text{ and } E = \frac{x_2x_4x_5 + x_1x_3x_5 + x_1x_4x_6}{x_2x_3x_6}.$$

Proof by Multiple Versions of Kuo Condensation: $z_{i,j,1}z_{i-1,j+2,0} = \cdots + \ldots$

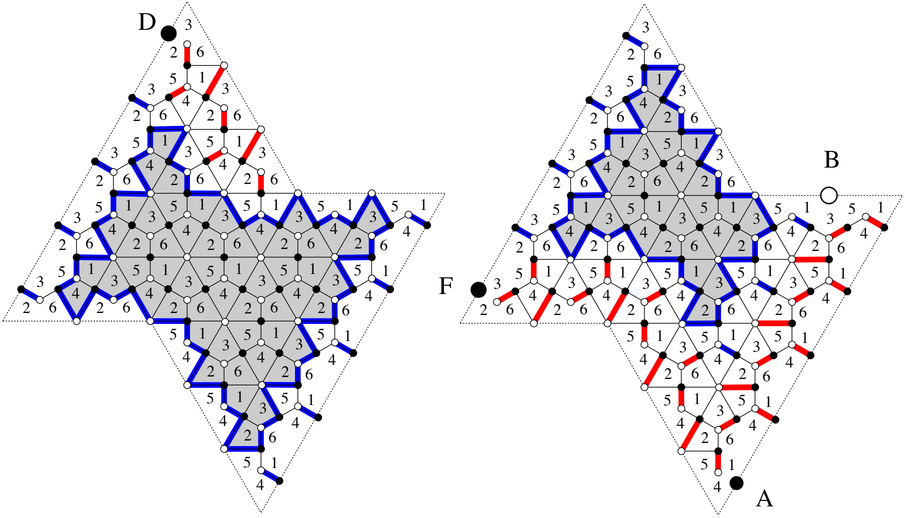




$$\cdots = z_{i-1,j+1,0} z_{i,j+1,1} + \cdots$$



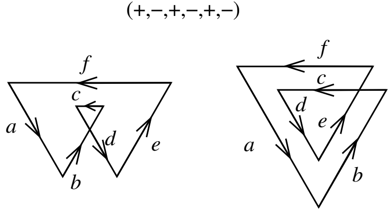
$$\cdots = \cdots + z_{i-1,j+1,1} z_{i,j+1,0}$$



Algebraic formula

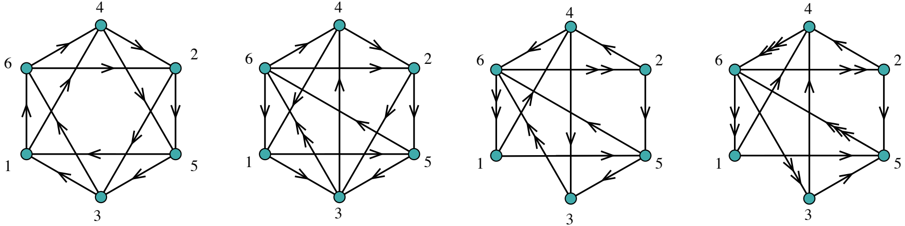
$$z_{i,j,k} = x_r \quad A^{\lfloor \frac{(i^2+ij+j^2+1)+i+2j}{3} \rfloor} \; B^{\lfloor \frac{(i^2+ij+j^2+1)+2i+j}{3} \rfloor} \; C^{\lfloor \frac{i^2+ij+j^2+1}{3} \rfloor} \; D^{\lfloor \frac{(k-1)^2}{4} \rfloor} \; E^{\lfloor \frac{k^2}{4} \rfloor}$$

still works for (a, b, c, d, e, f) when alternating in signs but combinatorial formula for such cases open.



Work in progress (with David Speyer): Promising calculations for conjectural Double-Dimer combinatorial interpretation for self-intersecting contours.

What if Initial Cluster corresponds to Model II, III, or IV?



By mutating according to the sequence $\mu_1 \circ \mu_4 \circ \mu_3$ to get from Model I to Model II, from Model II to Model III, and from Model III to Model IV, the initial cluster $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ also mutated accordingly.

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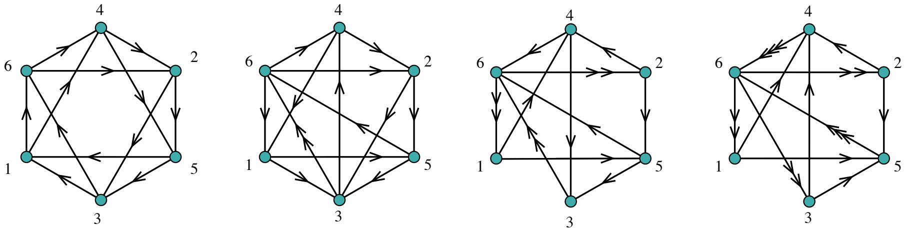

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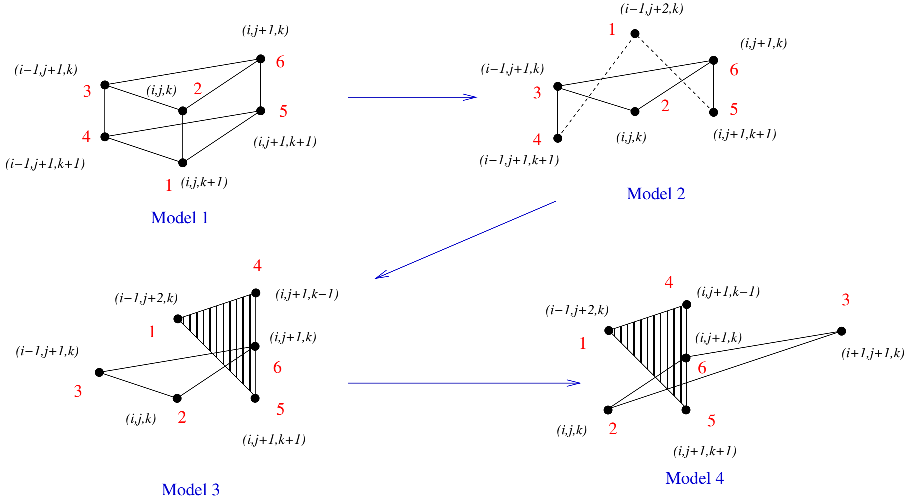
Let $z^{(2)}_{i,j,k}$, $z^{(3)}_{i,j,k}$, and $z^{(4)}_{i,j,k}$ denote the cluster variable parameterized by (i, j, k) starting from Model II, III, or IV respectively.

Note: In the following, Y_r (resp. Y'_r or Y''_r) is determined by the values of $(i - j)$ modulo 3 and k modulo 2 (i.e. cycles through $6, 4, 2, 5, 3, 1$ modulo 6).

Illustrating the mutation sequence $\mu_1\mu_4\mu_3$ (Reaching Model IV)



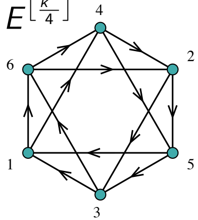
Resulting cluster variables still reachable by mutation sequences associated to $\mathbb{Z}^3$-geometry.

$$\text{Let} \quad A = \frac{x_3x_5 + x_4x_6}{x_1x_2}, \quad B = \frac{x_1x_6 + x_2x_5}{x_3x_4}, \quad C = \frac{x_1x_3 + x_2x_4}{x_5x_6},$$
$$D = \frac{x_1x_3x_6 + x_2x_3x_5 + x_2x_4x_6}{x_1x_4x_5}, \text{ and } E = \frac{x_2x_4x_5 + x_1x_3x_5 + x_1x_4x_6}{x_2x_3x_6}.$$

Let $z_{i,j,k}$ be the cluster variable corresponding to $(i,j,k) \in \mathbb{Z}^3$

Theorem 2 [Lai-M 2015] (Extension of [LMNT 2013] and [Lai 2014]):

$$z_{i,j,k} = x_r \; A^{\lfloor \frac{(i^2+ij+j^2+1)+i+2j}{3} \rfloor} B^{\lfloor \frac{(i^2+ij+j^2+1)+2i+j}{3} \rfloor} C^{\lfloor \frac{i^2+ij+j^2+1}{3} \rfloor} D^{\lfloor \frac{(k-1)^2}{4} \rfloor} E^{\lfloor \frac{k^2}{4} \rfloor}$$

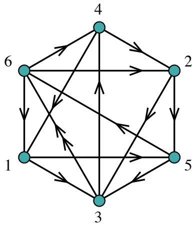


where, working **modulo** 6, we have (cyclically around the dP_3 Quiver)

$$r = 6 \text{ if } 2(i-j)+3k \equiv 0, \quad r = 4 \text{ if } 2(i-j)+3k \equiv 1,$$
$$r = 2 \text{ if } 2(i-j)+3k \equiv 2, \quad r = 5 \text{ if } 2(i-j)+3k \equiv 3,$$
$$r = 3 \text{ if } 2(i-j)+3k \equiv 4, \quad r = 1 \text{ if } 2(i-j)+3k \equiv 5.$$

i.e. we **determine** x_r by looking at $(i-j)$ modulo 3 and k modulo 2.

Formula for Model II

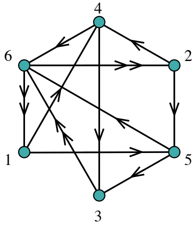


Theorem (Lai-M '17+) Let $A = \frac{x_1}{x_2}$, $B = \frac{x_4x_6^2+x_1x_2x_5+x_3x_4x_5}{x_1x_3x_4}$, $C = \frac{x_1x_2x_4+x_3x_4x_6+x_3^2x_5}{x_1x_5x_6}$, $D = \frac{x_1x_2+x_3x_6}{x_4x_5}$, $E = \frac{x_4^2x_6^2+x_1x_2x_4x_5+2x_3x_4x_5x_6+x_3^2x_5^2}{x_1x_2x_3x_6}$.

$$z^{(2)}_{i,j,k} = Y_r A^{\lfloor \frac{(i^2+ij+j^2+1)+i+2j}{3} \rfloor} B^{\lfloor \frac{(i^2+ij+j^2+1)+2i+j}{3} \rfloor} C^{\lfloor \frac{i^2+ij+j^2+1}{3} \rfloor} D^{\lfloor \frac{(k-1)^2}{4} \rfloor} E^{\lfloor \frac{k^2}{4} \rfloor}$$

where $Y_1 = \frac{x_4x_6+x_3x_5}{x_1}$, $Y_j = x_j$ for $2 \leq j \leq 6$. **(Powers of 2's, 3's and 5's)**

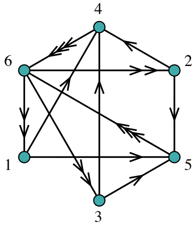
Formula for Model III



Theorem (Lai-M '17+) Let $A = \frac{x_1}{x_2}$, $B = \frac{x_4x_5 + x_6^2}{x_1x_3}$, $C = \frac{x_1^2x_2^2 + 2x_1x_2x_3x_6 + x_3^2x_6^2 + x_3^2x_4x_5}{x_1x_4x_5x_6}$, $D = \frac{x_4}{x_5}$,
(Powers of 2's, 5's and 11's) $E = \frac{x_1^2x_2^2x_6^2 + 2x_1x_2x_3x_6^3 + x_3^2x_6^4 + x_1^2x_2^2x_4x_5 + 3x_1x_2x_3x_4x_5x_6 + 2x_3^2x_4x_5x_6^2 + x_3^2x_4^2x_5^2}{x_1x_2x_3x_4^2x_6}$.

$$z_{i,j,k}^{(3)} = Y_r' A^{\lfloor \frac{(i^2+ij+j^2+1)+i+2j}{3} \rfloor} B^{\lfloor \frac{(i^2+ij+j^2+1)+2i+j}{3} \rfloor} C^{\lfloor \frac{i^2+ij+j^2+1}{3} \rfloor} D^{\lfloor \frac{(k-1)^2}{4} \rfloor} E^{\lfloor \frac{k^2}{4} \rfloor}$$

where $Y_1' = \frac{x_1x_2x_6 + x_3x_6^2 + x_3x_4x_5}{x_1x_4}$, $Y_4' = \frac{x_1x_2 + x_3x_6}{x_4}$, $Y_j' = x_j$ for $j \in \{2, 3, 5, 6\}$.



Theorem (Lai-M '17+) Let $A = \frac{x_1}{x_2}$, $B = \frac{x_3}{x_1}$,

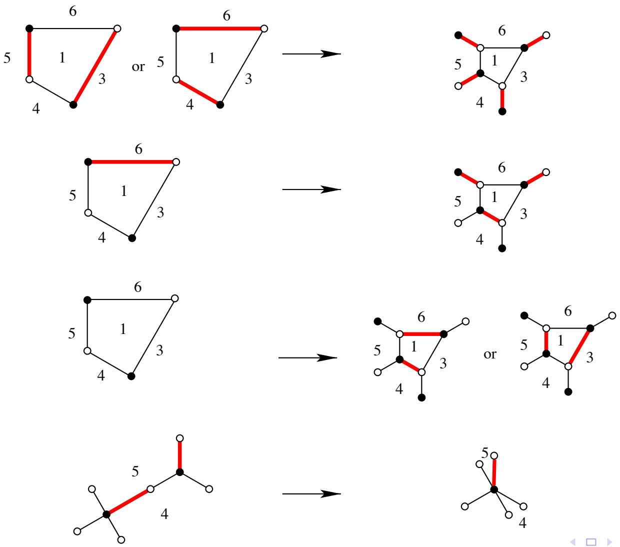
$C = \frac{x_6^6 + 2x_1x_2x_3x_6^3 + x_1^2x_2^2x_3^2 + 3x_4x_5x_6^4 + 2x_1x_2x_3x_4x_5x_6 + 3x_4^2x_5^2x_6^2 + x_4^3x_5^3}{x_1x_3^2x_4x_5x_6}$, $D = \frac{x_4}{x_5}$,

$E = \frac{x_6^6 + 2x_1x_2x_3x_6^3 + x_1^2x_2^2x_3^2 + 3x_4x_5x_6^4 + 3x_1x_2x_3x_4x_5x_6 + 3x_4^2x_5^2x_6^2 + x_4^3x_5^3}{x_1x_2x_3x_4^2x_6}$.

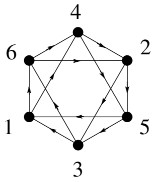
$$z_{i,j,k}^{(4)} = Y_r'' A^{\lfloor \frac{(i^2+ij+j^2+1)+i+2j}{3} \rfloor} B^{\lfloor \frac{(i^2+ij+j^2+1)+2i+j}{3} \rfloor} C^{\lfloor \frac{i^2+ij+j^2+1}{3} \rfloor} D^{\lfloor \frac{(k-1)^2}{4} \rfloor} E^{\lfloor \frac{k^2}{4} \rfloor}$$

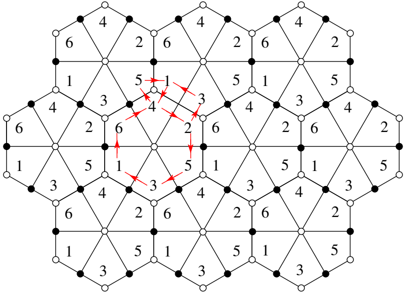
where $Y_1'' = \frac{x_6^4 + x_1x_2x_3x_6 + 2x_4x_5x_6^2 + x_4^2x_5^2}{x_1x_3x_4}$, $Y_3'' = \frac{x_4x_5 + x_6^2}{x_3}$, $Y_4'' = \frac{x_1x_2x_3 + x_4x_5x_6 + x_6^3}{x_3x_4}$, $Y_j'' = x_j$ for $j \in \{2, 5, 6\}$. **(Powers of 13's and 14's)**

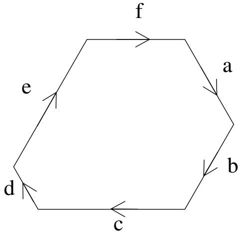
Back to Combinatorics: Applying Urban Renewal (as in [Speyer '08])



Review of Model I Combinatorics



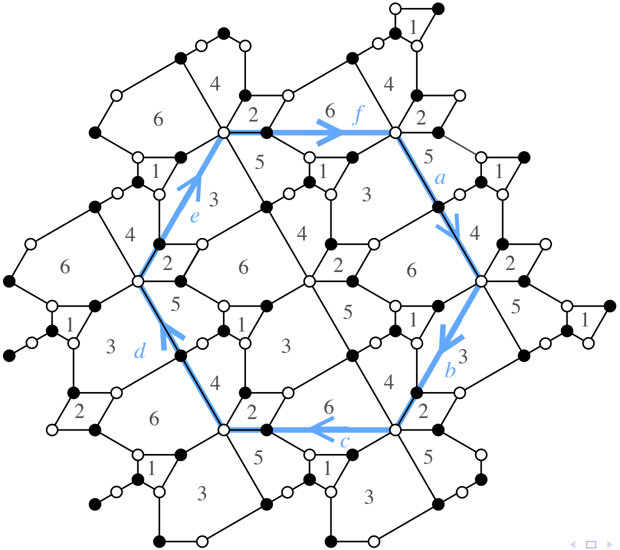


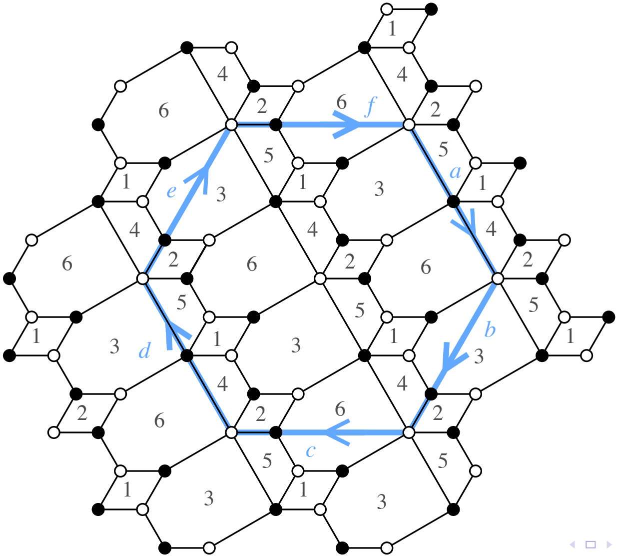


Note: Countour illustrated assuming all side lengths are positive.

We apply urban renewal to all quadrilaterals labeled with a 1.

Model I/II Combinatorics (after urban renewal at 1's)




1
4
6
2
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2
f
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5
5
a
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1
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e
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4
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4
6
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6
2
6
2
5
5
5
b
1
1
1
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c
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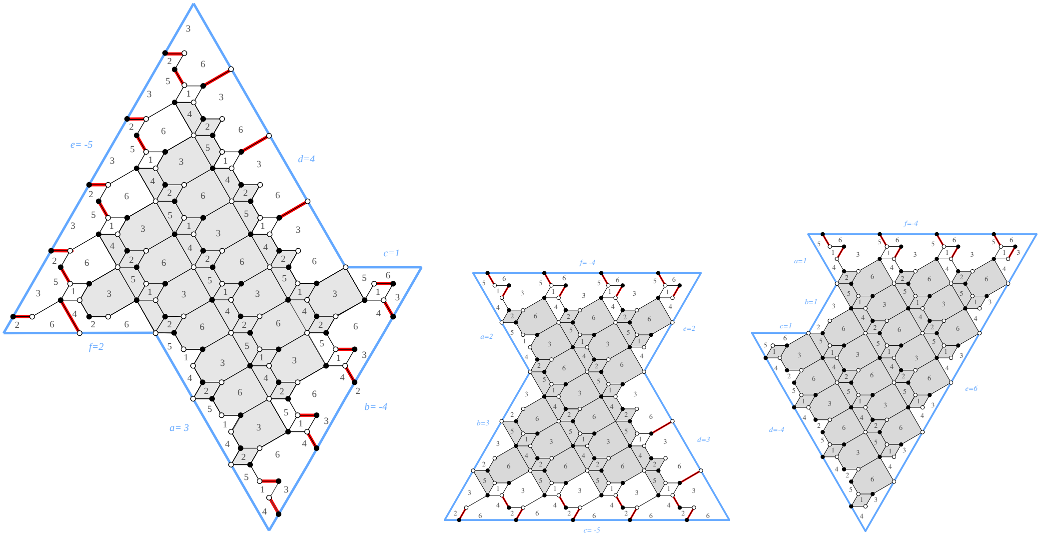
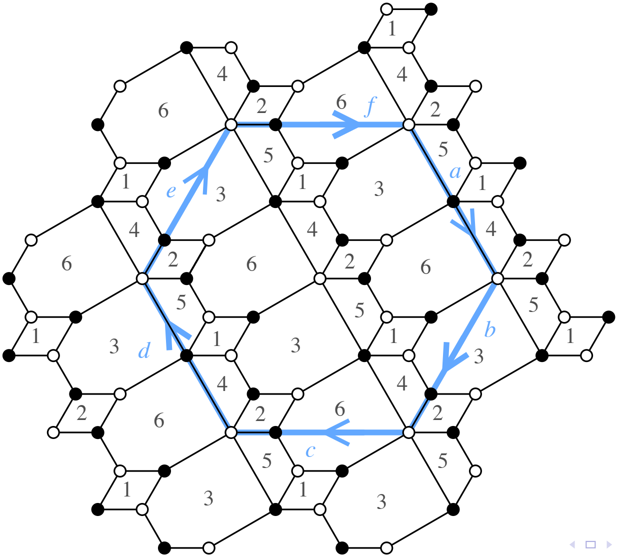
3

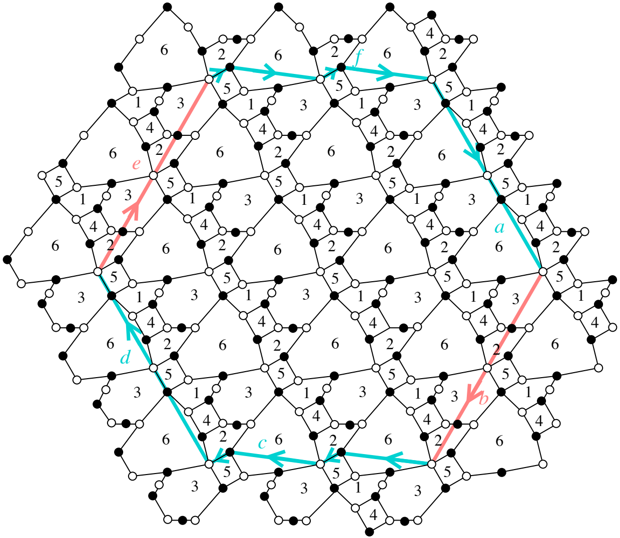
Model II Combinatorics



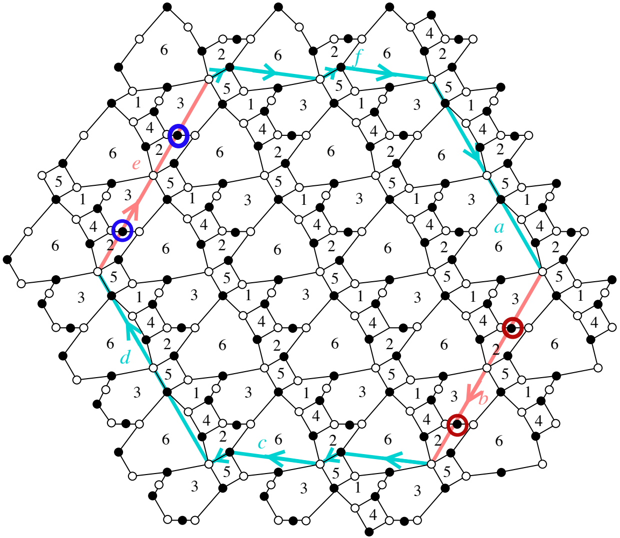
Model II Combinatorics (Revisited)



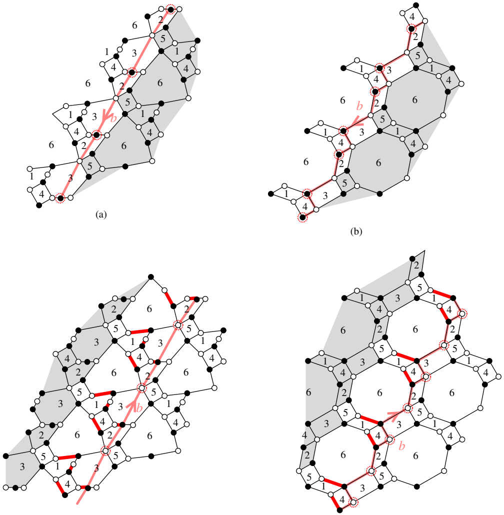
Model II/III (after urban renewal at 4's)



However, we have a problem: The contour runs through 2-valent vertices!



Model II/III (correcting contour as 2-valent vertices are collapsed)

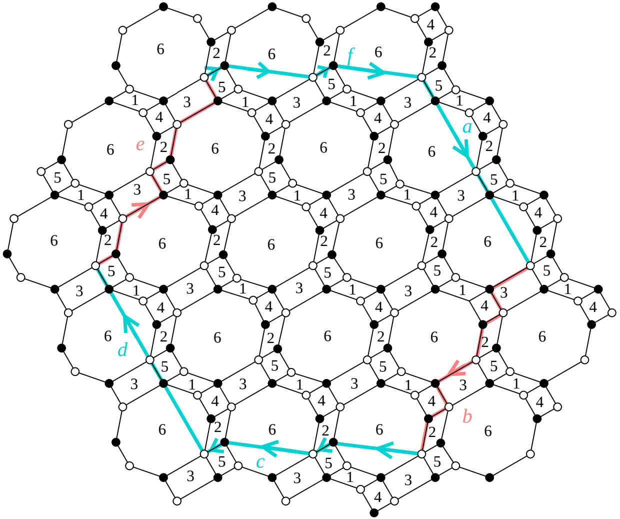


(a) (b) (c) (d)

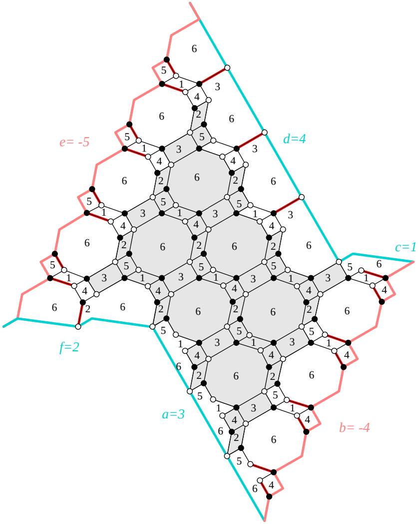
Side b positive or negative.

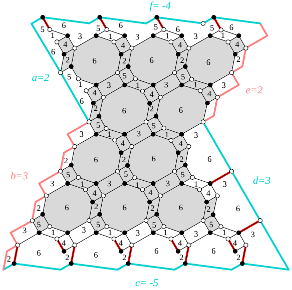
Side e is analogous.

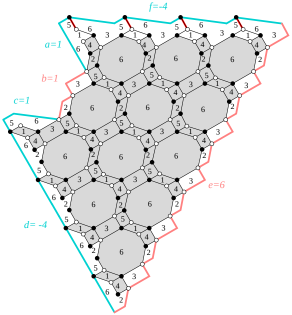
Model III Combinatorics (after collapsing 2-valent vertices)



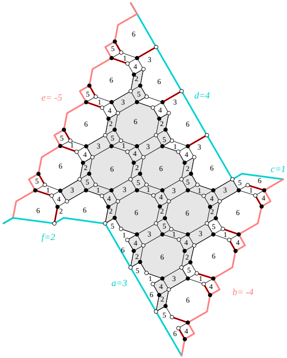
Model III Combinat. (Aztec Trimmed Rectangles, Lai '15)

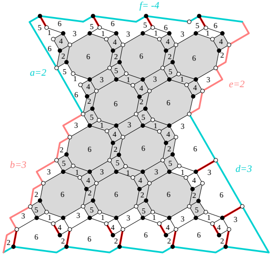


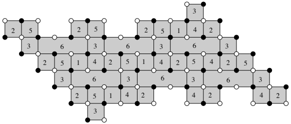


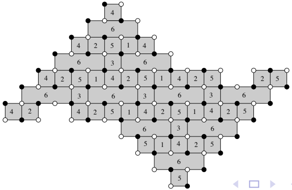


Aztec trimmed rectangles as Model III (Rotate and Straighten) $A^{(3)}_{3,4,1}$, $A^{(1)}_{3,5,2}$

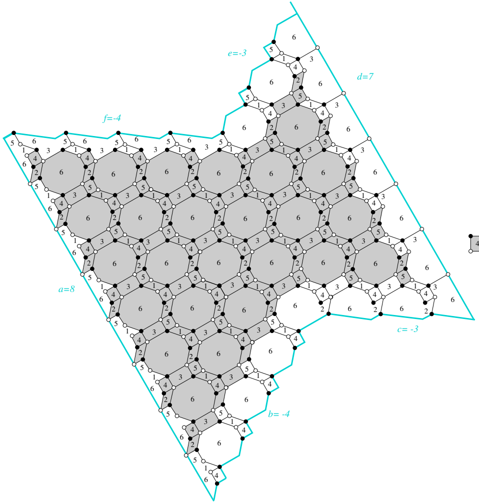


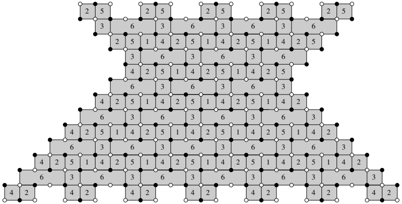




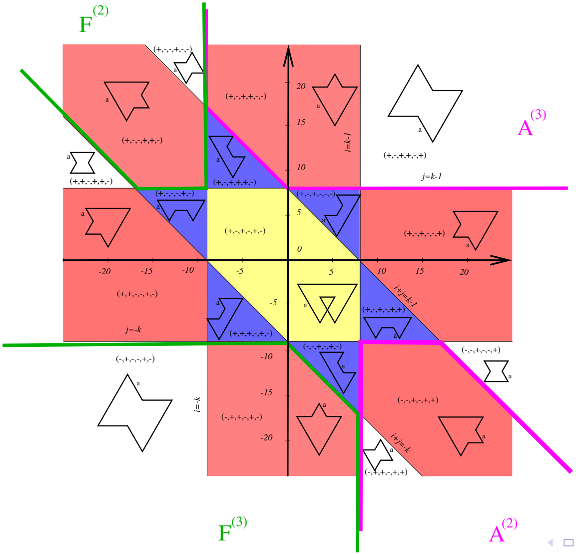


Aztec trimmed rectangles as Model III (Rotate and Straighten) $F^{(2)}_{5,8,4}$

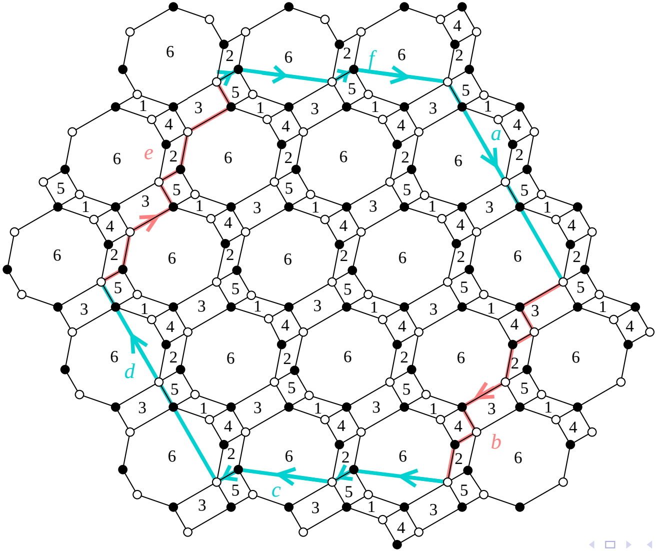




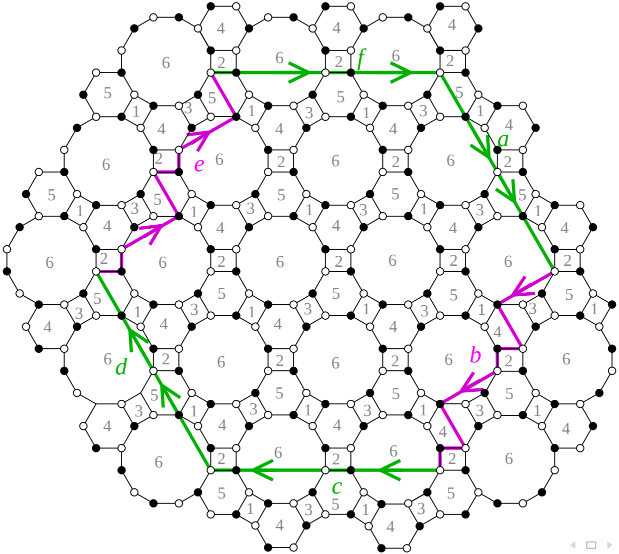
Aztec trimmed rectangles as Model III - Comparing Coordinate Systems



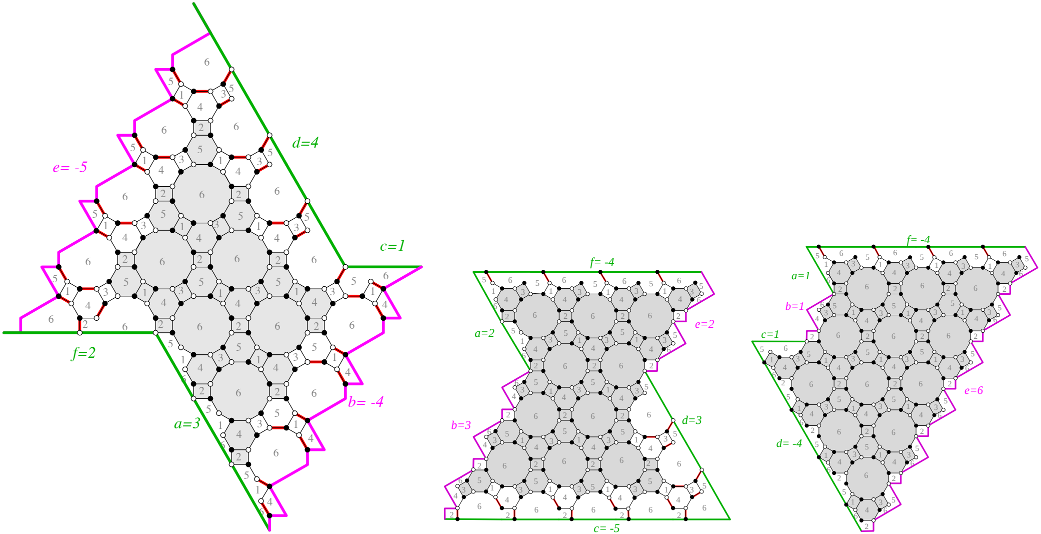
Model III Combinatorics (Revisited)



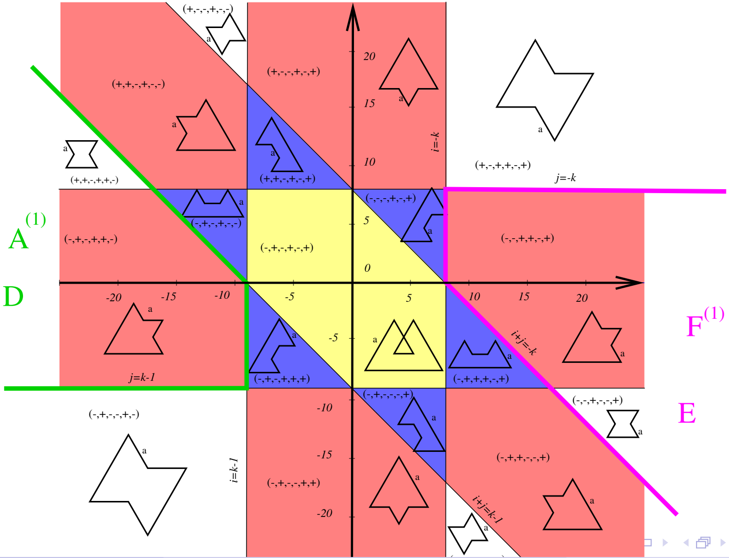
Model IV Combinatorics (after urban renewal at 3's & no 2-valent vertices)



Model IV Combinatorics (Hexagonal Dungeon Pieces) [Ciucu-Lai '14]



Hexagonal Dungeon Pieces (D- and E- regions) as Model IV Summary



Model IV Combinatorics (Hexahedron Recurrence)

We now compare our Combinatorial Formulas for Model IV to the work of Kenyon-Pemantle on the Hexahedron Recurrence. Starting with the infinite set of initial variables

$$\mathcal{I}_2 = \{A(i,j,k) : 0 \leq i+j+k \leq 2\} \ \cup \ \{A(i+\frac{1}{2}, j+\frac{1}{2}, k) : i+j+k = 0\}$$

$$\cup \ \{A(i+\frac{1}{2}, j, k+\frac{1}{2}) : i+j+k = 0\} \ \cup \ \{A(i, j+\frac{1}{2}, k+\frac{1}{2}) : i+j+k = 0\},$$

the hexahedron recurrence yields a family of Laurent polynomials

$$\{A(a,b,c) : a,b,c \in \mathbb{Z}/2 \text{ such that } a+b+c \in \mathbb{Z}\}$$

in terms of $\mathcal{I}_2$.

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$$\cup\; \{A(i+\frac{1}{2}, j, k+\frac{1}{2}) : i+j+k = 0\} \;\cup\; \{A(i, j+\frac{1}{2}, k+\frac{1}{2}) : i+j+k = 0\},$$

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$$\{A(a,b,c) : a,b,c \in \mathbb{Z}/2 \text{ such that } a+b+c \in \mathbb{Z}\}$$

in terms of $\mathcal{I}_2$. For $i,j,k \in \mathbb{Z}$, we use Kenyon-Pemantle's notation:

$$h = A(i,j,k),\, h^{(x)} = A(i, j+\frac{1}{2}, k+\frac{1}{2}),\, h^{(y)} = A(i+\frac{1}{2}, j, k+\frac{1}{2}), \text{ and } h^{(z)} = A(i+\frac{1}{2}, j+\frac{1}{2}, k),$$

$$h_{(1)} = A(i+1,j,k),\, h_{(12)} = A(i+1,j+1,k),\, h_{(123)} = A(i+1,j+1,k+1),$$

with $h_{(S)}$ defined analogously for $S \subset \{1,2,3\}$.

Hexahedron Recurrence [Kenyon-Pemantle '16]

For $i, j, k \in \mathbb{Z}$, we use Kenyon-Pemantle's notation:

$$h = A(i,j,k), h^{(x)} = A(i, j+\frac{1}{2}, k+\frac{1}{2}), h^{(y)} = A(i+\frac{1}{2}, j, k+\frac{1}{2}), \text{ and } h^{(z)} = A(i+\frac{1}{2}, j+\frac{1}{2}, k),$$

$$h_{(1)} = A(i+1, j, k), h_{(12)} = A(i+1, j+1, k), h_{(123)} = A(i+1, j+1, k+1).$$

Hexahedron Recurrence:

$$h^{(x)}_{(1)} h^{(x)} h = h^{(x)} h^{(y)} h^{(z)} + h_{(1)} h_{(2)} h_{(3)} + h h_{(1)} h_{(23)}$$

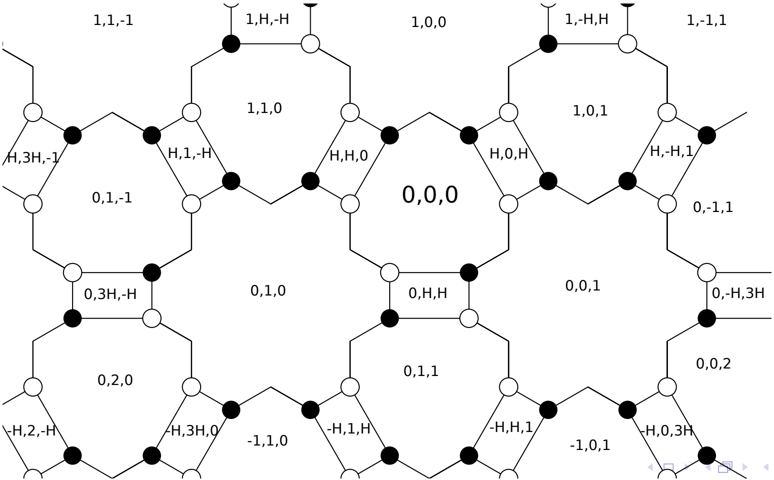
$$h^{(y)}_{(2)} h^{(y)} h = h^{(x)} h^{(y)} h^{(z)} + h_{(1)} h_{(2)} h_{(3)} + h h_{(2)} h_{(13)}$$

$$h^{(z)}_{(3)} h^{(z)} h = h^{(x)} h^{(y)} h^{(z)} + h_{(1)} h_{(2)} h_{(3)} + h h_{(3)} h_{(12)}$$

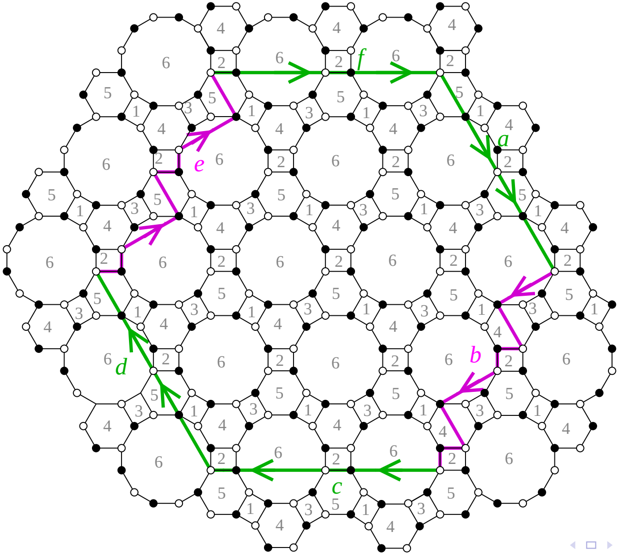
$$h_{(123)} h^2 h^{(x)} h^{(y)} h^{(z)} = (h^{(x)} h^{(y)} h^{(z)})^2 + (h_{(1)} h_{(2)} + h h_{(12)})(h_{(1)} h_{(3)} + h h_{(13)})(h_{(2)} h_{(3)} + h h_{(23)})$$
$$+ h^{(x)} h^{(y)} h^{(z)} [2 h_{(1)} h_{(2)} h_{(3)} + h h_{(1)} h_{(23)} + h h_{(2)} h_{(13)} + h h_{(3)} h_{(12)}].$$

Hexahedron Recurrence (Figure 7 of [Kenyon-Pemantle '16])

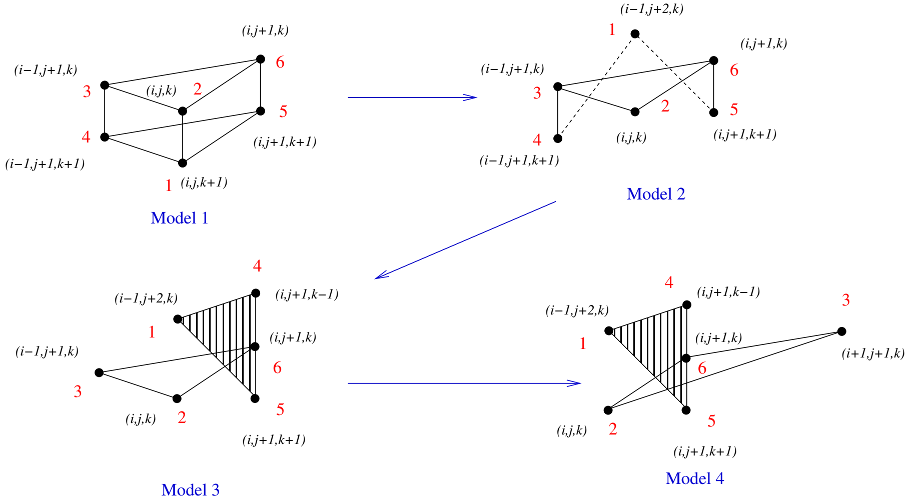
Initial Variables $\mathcal{I}_2$ can be arranged as $(H = \frac{1}{2})$. We can reach other $A(a,b,c)$'s by iterating **Super Urban Renewal** Transformations. (Corresponds to adding/removing cubes in dual.)



Project from $\mathcal{I}_2$ to $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ in Model IV Brane Tiling

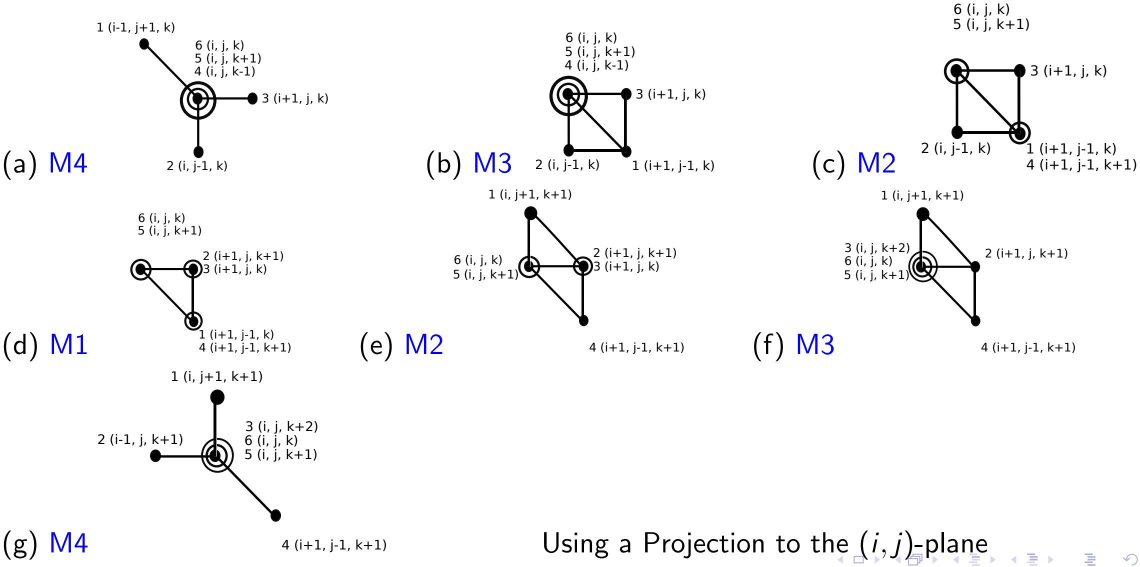


Illustrating the mutation sequence $\mu_1\mu_4\mu_3$ (Reaching Model IV)



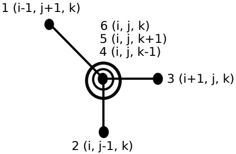
Resulting cluster variables still reachable by mutation sequences associated to $\mathbb{Z}^3$-geometry.

Illustrating the mutation sequence $\mu_1\mu_4\mu_2\mu_1\mu_3\mu_2$ starting from Model *IV*

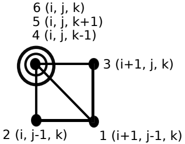


Using a Projection to the (i, j)-plane

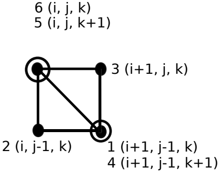
Model IV Combinatorics (Hexahedron Recurrence)



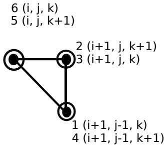
(a) M4



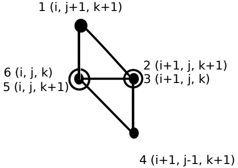
(b) M3



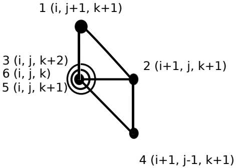
(c) M2



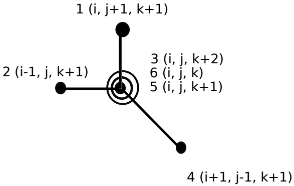
(d) M1



(e) M2



(f) M3

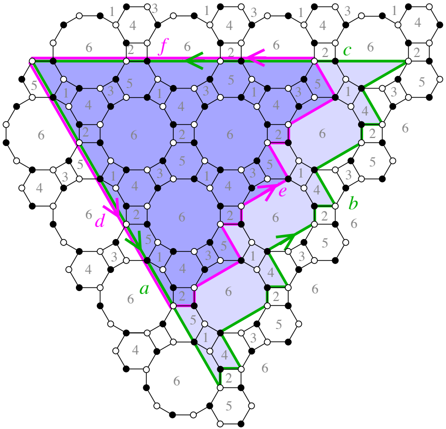


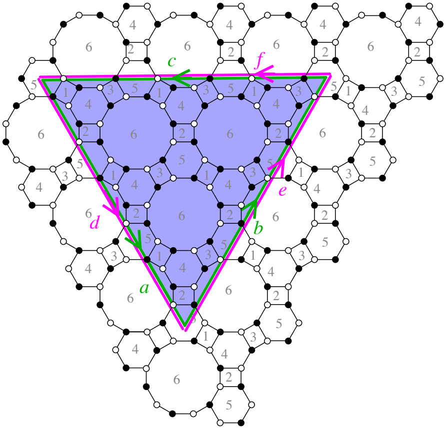
(g) M4

Super Urban Renewal $= \mu_1\mu_4\mu_2\mu_1\mu_3\mu_2.$

Model IV Combinatorics (Hexahedron Recurrence) $z_{0,0,4}^{(4)}$ or A_5

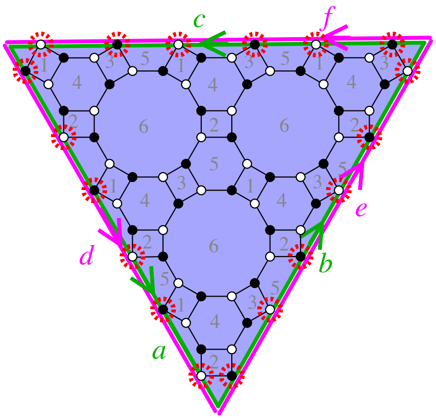
Self-intersecting contour $\mathcal{C}(4,-4,4,-3,3,-3)$ and modified contour $\mathcal{C}'(3,-3,3,-3,3,-3)$.

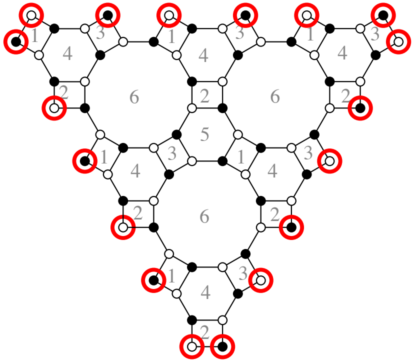




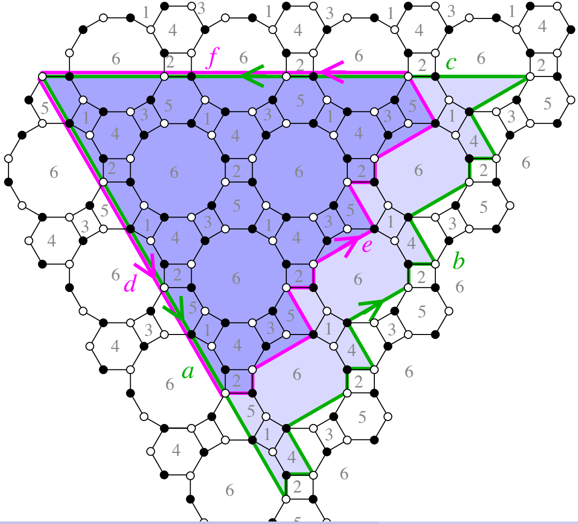
Model IV Combinatorics (Hexahedron Recurrence) $z^{(4)}_{0,0,4}$ or A_5

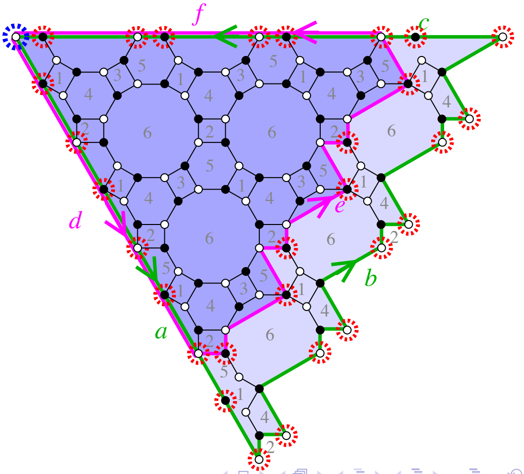
Modified Contour $(a-1, b+1, c-1, d, e, f)$ where the six-tuple sums to 0 (rather than 1) is more natural in Model IV case, as well as the Model III case.



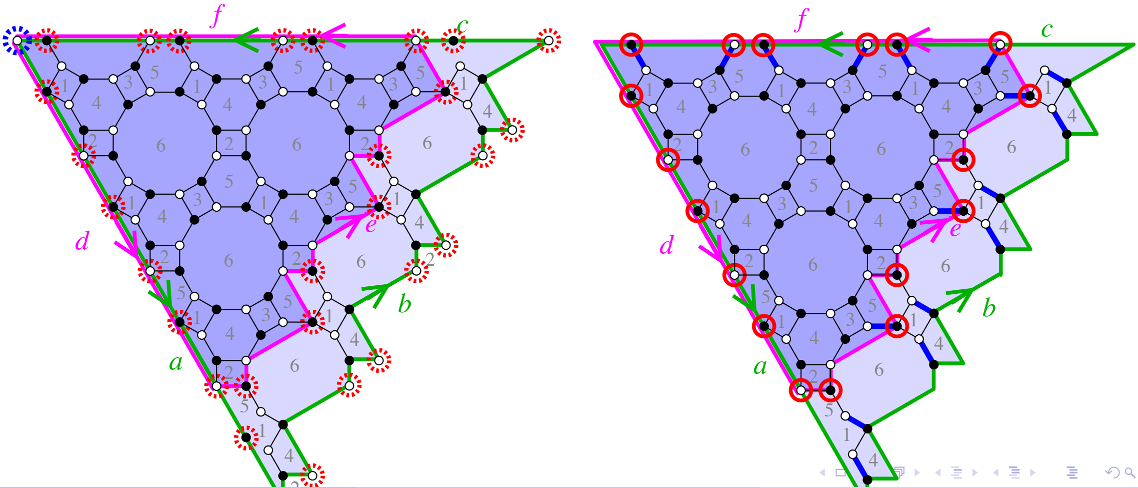


Model IV Combinatorics (Hexahedron Recurrence) $z^{(4)}_{0,0,4}$ or A_5

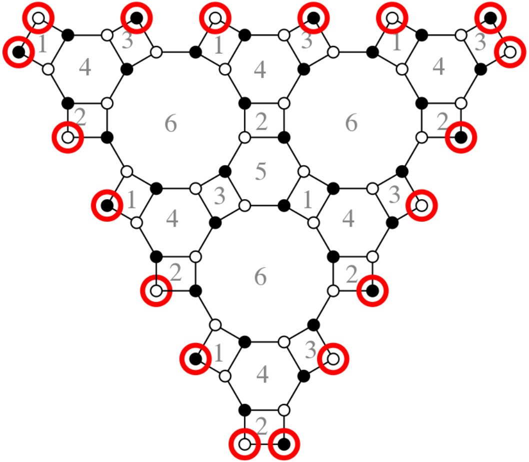


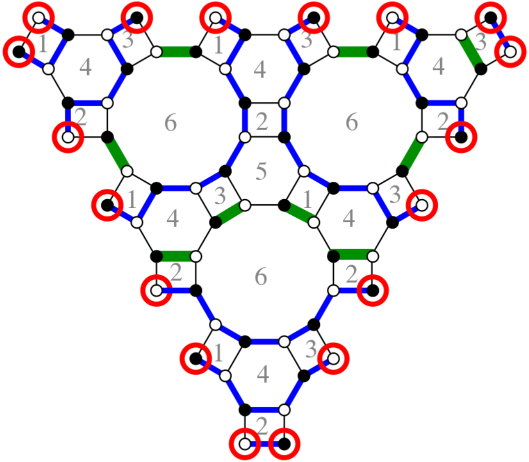


Model IV Combinatorics (Hexahedron Recurrence) $z^{(4)}_{0,0,4}$ or A_5



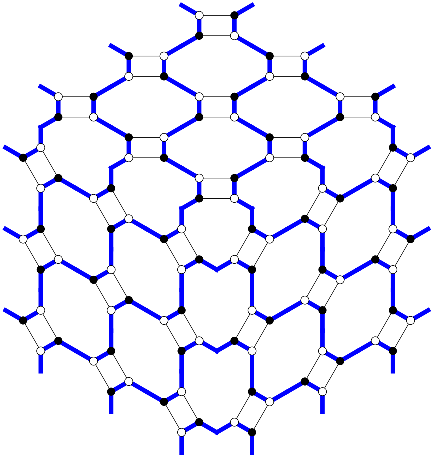
Model IV Combinatorics (Hexahedron Recurrence) $z_{0,0,4}^{(4)}$ or A_5





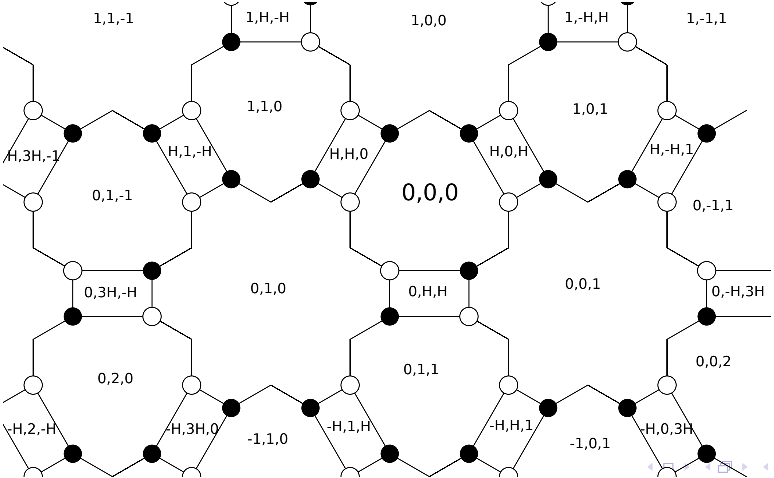
Taut Condition for Double Dimer Configurations [Kenyon-Pemantle '16])

Definition (Kenyon-Pemantle): A double dimer configuration on the infinite graph G_∞ is known as **Taut** if its connectivity looks like the below asymptotically outside of the center.



Taut Condition for Double Dimer Configurations [Kenyon-Pemantle '16])

By applying **Super Urban Renewal** to the center of G_∞, followed by its three neighbors, and so on, we get a new graph that looks like intersection of G_∞ and the $4-6-12$ graph.



Model IV Combinatorics (Hexahedron Recurrence)

Upshot: Hexahedron Recurrence Solutions $A(i, j, k)$ for $i, j, k \in \mathbb{Z}$ (A_{n+2} for $n + 2 = i + j + k$) are described via Taut Double Dimer Configurations as in Kenyon-Pemantle.

These can be re-interpreted as generalized subgraphs from self-intersecting contours on the Model IV brane tiling, corresponding to the **toric cluster variables** $z^{(4)}_{0,0,n+1}$.

Laurent monomials in expansion of $z^{(4)}_{0,0,n+1}$ correspond to the weight of double dimer configurations (where a closed cycle of size ≥ 4 come with coefficients of 2).

Model IV Combinatorics (Hexahedron Recurrence)

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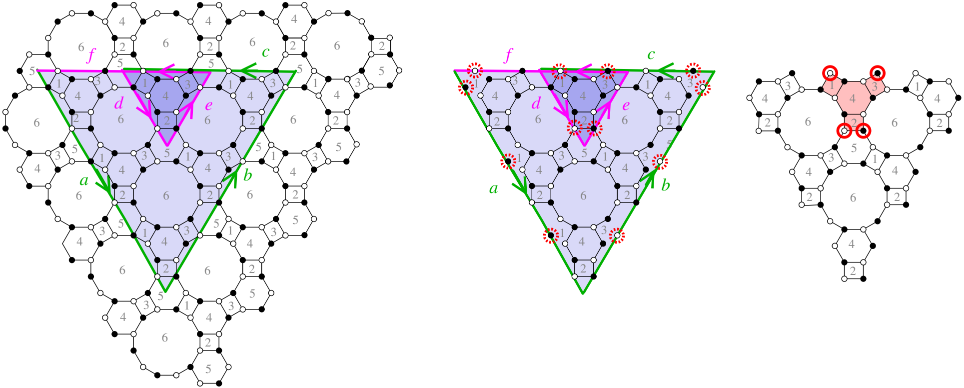
Laurent monomials in expansion of $z^{(4)}_{0,0,n+1}$ correspond to the weight of double dimer configurations (where a closed cycle of size ≥ 4 come with coefficients of 2).

Conjecture: Generalization of our method for building subgraphs from contours with self-intersections can yield combinatorial interpretations for other cluster variables using taut double dimer configurations (or possibly a mixed region of dimers and double-dimers).

Dovetails with promising calculations with David Speyer for self-intersecting contours for the Model I brane tiling.

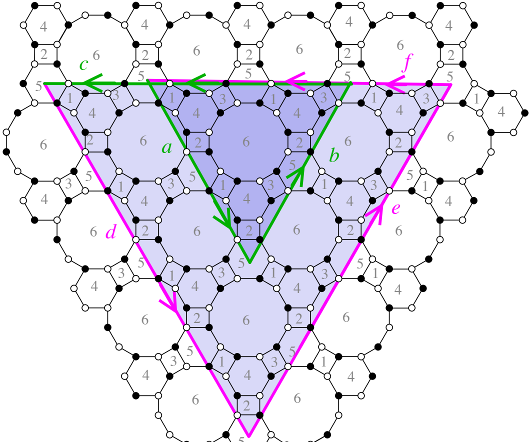
Model IV Combinatorics with Modified Contour $(a-1, b+1, c-1, d, e, f)$

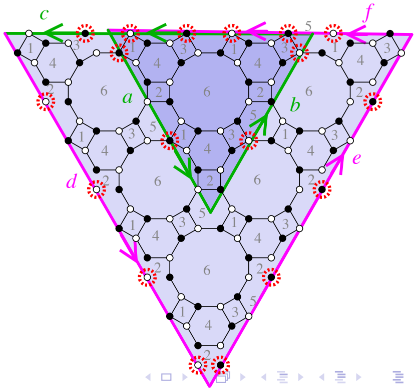
For B_3, using the Modified Contour $(\mathbf{3}, -\mathbf{3}, \mathbf{2}, -\mathbf{1}, \mathbf{1}, -\mathbf{2})$ rather than $(4, -4, 3, -1, 1, -2)$.



Cyclic symmetry visible in the Modified Contours: $(\mathbf{2}, -\mathbf{1}, \mathbf{1}, -\mathbf{2}, \mathbf{3}, -\mathbf{3})$ rather than $(3, -2, 2, -2, 3, -3)$ as well as $(\mathbf{1}, -\mathbf{2}, \mathbf{3}, -\mathbf{3}, \mathbf{2}, -\mathbf{1})$ rather than $(2, -3, 4, -3, 2, -1)$.

As an example, consider the half-integer solutions $A(0, 1/2, 1/2 + n)$, i.e. the B_n sequence, for the hexahedron recurrence from Kenyon-Pemantle. They give an algebraic formula but not a combinatorial model. **Modified Contour:** (3,-3,4,-4,4,-3)





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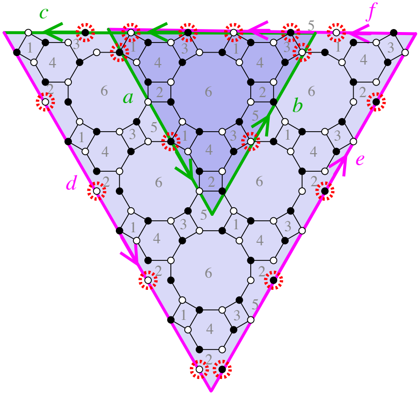
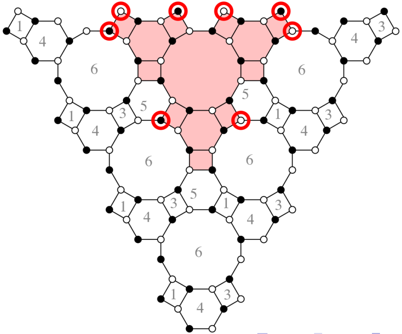
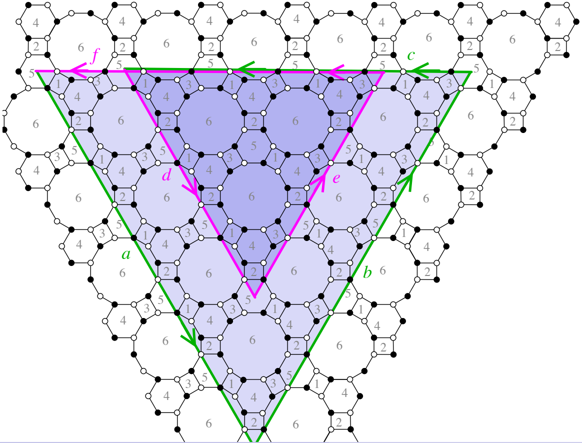
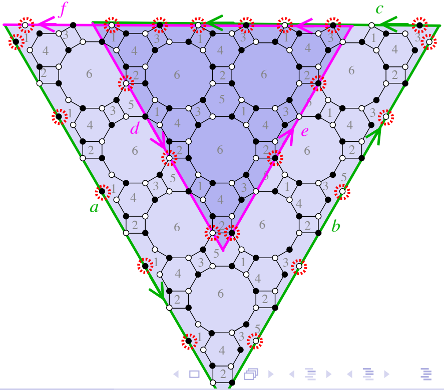




Model IV Combinatorics (Hexahedron Recurrence) $z^{(4)}_{0,1,5}$ or B_5

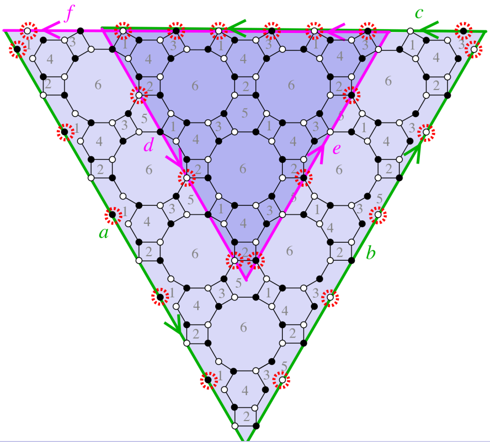
As an example, consider the half-integer solutions $A(0, 1/2, 1/2 + n)$, i.e. the B_n sequence, for the hexahedron recurrence from Kenyon-Pemantle. They give an algebraic formula but not a combinatorial model. **Modified Contour:** (5,-5,4,-3,3,-4)

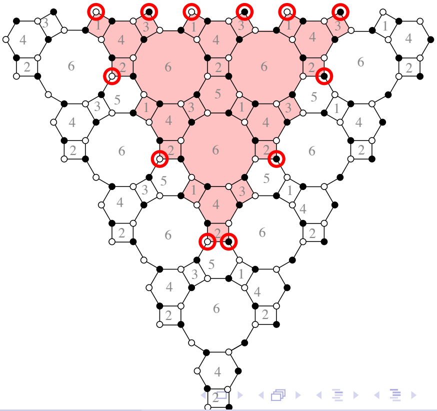




Model IV Combinatorics (Hexahedron Recurrence) $z^{(4)}_{0,1,5}$ or B_5

As an example, consider the half-integer solutions $A(0, 1/2, 1/2 + n)$, i.e. the B_n sequence, for the hexahedron recurrence from Kenyon-Pemantle. They give an algebraic formula but not a combinatorial model. **Modified Contour:** (5,-5,4,-3,3,-4)





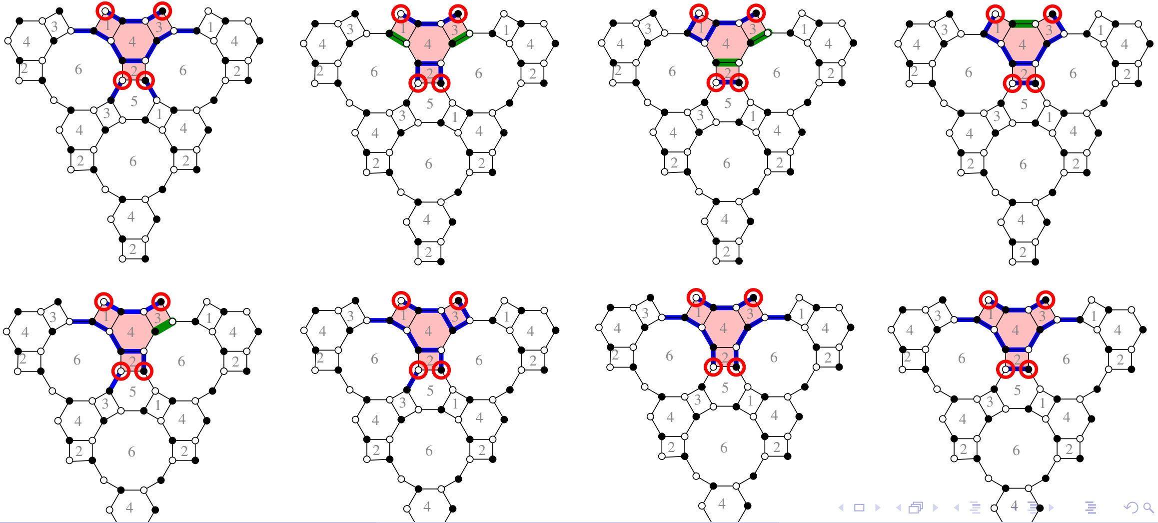
The Taut Condition restricting the subset of mixed dimer configurations

The modified contour can be built as two overlapping triangles, such that the innermost triangle resembles a graph in the family for A_n except that it may have vertices of multiplicity two on its boundary.

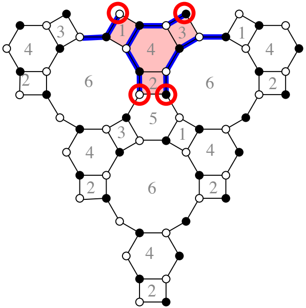
In fact, one of its three boundaries will again have all of its vertices of multiplicity one, and it follows from the construction that this boundary will have an even number of such vertices.

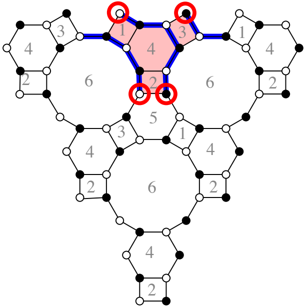
We define the **Taut Condition** analogously to the above for A_n so that the multiplicity one vertices along this special boundary are always connected to the multiplicity one vertices on the side nearer to it and in a non-crossing connectivity pattern.

Examples of Forbidden Mixed Dimer Configurations for B_3



Further Examples of Forbidden Mixed Dimer Configurations for B_3





These two configurations and those like it are (conjecturally) disallowed since the path **crossing from left to right** (or vice-versa).

Note the paths connect different sides of the innermost triangle, but this necessary but not sufficientcondition for a configuration to contribute to the partition function.

Thanks for Coming (Slides at http://math.umn.edu/∼musiker/RIMS19.pdf)

- Jim Propp, *Enumeration of matchings: problems and progress. New perspectives in algebraic combinatorics* (Berkeley, CA, 1996-97), 255-291, MSRI Publ., 38, Cambridge Univ. Press, Cambridge, 1999, http://faculty.uml.edu/jpropp/matchings.pdf.
- Richard Eager and Sebastian Franco, *Colored BPS Pyramid Partition Functions, Quivers and Cluster Transformations*, JHEP 2012, no. 9, 038, arXiv:1112.1132.
- Sicong Zhang, *Cluster Variables and Perfect Matchings of Subgraphs of the dP_3 Lattice*, 2012 REU Report, arXiv:1511.06055.
- Megan Leoni, Gregg Musiker, Seth Neel, and Paxton Turner, *Aztec Castles and the dP_3 Quiver*, *Journal of Physics A*: Math. Theor. 47 474011, arXiv:1308.3926.
- Tri Lai, *A Generalization of Aztec Dragons*, *Graphs and Combinatorics*, 32 (2016), no. 5, 1979-1999, arXiv:1504.00303.
- Tri Lai and Gregg Musiker, *Beyond Aztec Castles: Toric Cascades in the dP_3 Quiver*, Comm. Math. Phys. 356 (2017), no. 3, 823-881, arXiv:1512.00507.
- Yibo Gao, Zhaoqi Li, Thuy-Duong Vuong, and Lisa Yang, *Toric Mutations in the dP_2 Quiver and Subgraphs of the dP_2 Brane Tiling*, arXiv:1611.05320.
- Tri Lai and Gregg Musiker, *Dungeons and Dragons: Combinatorics for the dP_3 Quiver*, to appear in Annals of Comb., arXiv:1805.09280.

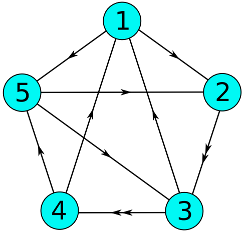
Epilogue: The Del Pezzo 2 (dP_2) Case

REU 2016: Yibo Gao, Zhaoqi Li, Thuy-Dong Vuong, and Lisa Yang

Toric Mutations in the dP_2 *Quiver and Subgraphs of the* dP_2 *Brane Tiling*
https://arxiv.org/abs/1611.05320

Won an MAA Outstanding Poster Award at 2017 JMM.

The second Del Pezzo Surface (dP_2) is first introduced in the physics literature.



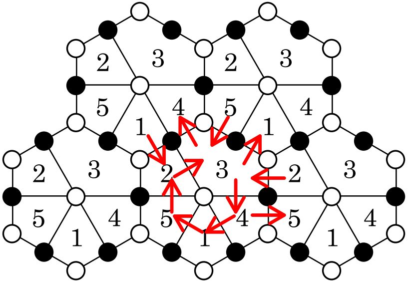


Figure: dP_2 quiver and its corresponding brane tiling from Hanany-Seong

Two Models of the dP_2 Quiver (Gao-Li-Vuong-Yang)

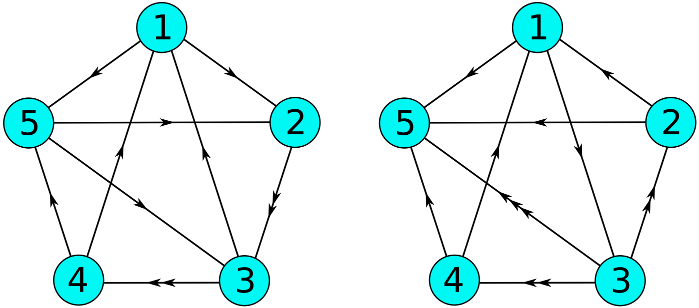
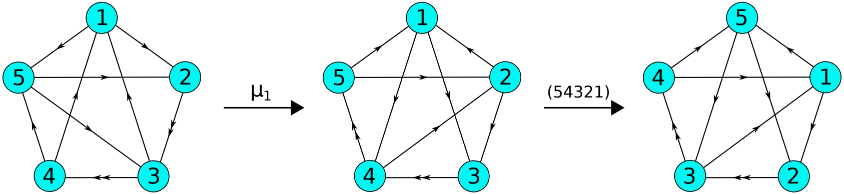


Figure: Model 1 (left) and Model 2 (right) of the dP_2 quiver

Periodicity in certain mutation directions



$$(x_1, x_2, x_3, x_4, x_5) \longrightarrow \left(\frac{x_2x_5 + x_3x_4}{x_1} = x_6, x_2, x_3, x_4, x_5\right) \longrightarrow (x_2, x_3, x_4, x_5, x_6)$$

Explicit Formula for Cluster Variables (Gao-Li-Vuong-Yang)

Definition (Laurent Polynomial for Somos-5 Sequence)

Let x_1, x_2, x_3, x_4, x_5 be our initial variables. Define x_n for each $n \in \mathbb{Z}$ by

$$x_n x_{n-5} = x_{n-1}x_{n-4} + x_{n-2}x_{n-3}.$$

Notice that $\{x_n\}_{n\geq 1}$ is the Somos-5 sequence if $x_1 = x_2 = x_3 = x_4 = x_5 = 1$.

Explicit Formula for Cluster Variables (Gao-Li-Vuong-Yang)

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Definition (Some Constants)

$$A := \frac{x_1 x_5 + x_3^2}{x_2 x_4}, \qquad B := \frac{x_2 x_6 + x_4^2}{x_3 x_5} \left(= \frac{x_1 x_4^2 + x_2 x_3 x_4 + x_2^2 x_5}{x_1 x_3 x_5} \right).$$

Explicit Formula for Cluster Variables (Gao-Li-Vuong-Yang)

Theorem

Define $g(s,k) := \left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{s+1}{2} \right\rfloor$ *if* k *is even and* $g(s,k) := \left\lfloor \frac{s-1}{2} \right\rfloor \left\lfloor \frac{s}{2} \right\rfloor$ *if* k *is odd. Then we have, for* $k \in \mathbb{Z}$ *and* $s \in \mathbb{Z}_{\geq 0}$,

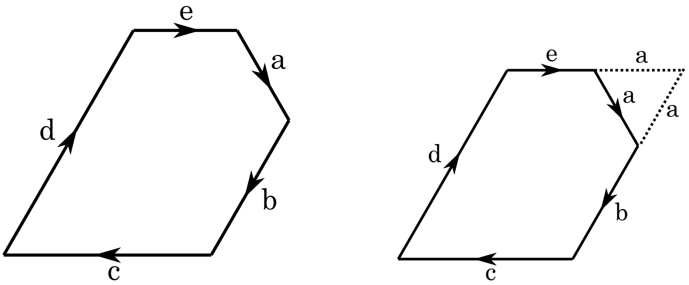
$$\begin{aligned}\rho_1^k(\rho_3\rho_1)^s\{x_1, x_2, x_3, x_4, x_5\} = \{&A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+1}, \quad A^{g(s,k)}B^{g(s,k+1)}x_{k+s+2}, \\ &A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+3}, \quad A^{g(s,k)}B^{g(s,k+1)}x_{k+s+4}, \\ &A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+5}\}.\end{aligned}$$

Corollary

All cluster variables generated by toric mutations can be written as either

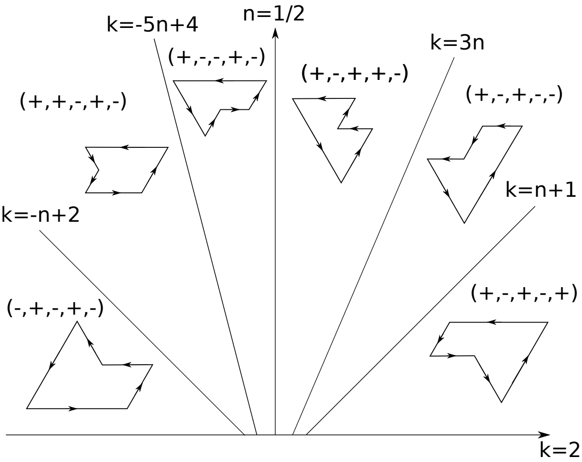
$$A^{n^2}B^{n(n-1)}x_{2m} \quad \text{or} \quad A^{n(n-1)}B^{n^2}x_{2m-1} \quad \text{for some } m, n \in \mathbb{Z}.$$

Contour: Fundamental Shape (Gao-Li-Vuong-Yang)



We get the relations $a + b = d$ and $a + e = c$.

Countour: Fundamental Shape (Gao-Li-Vuong-Yang)



From Contour to Subgraph (Gao-Li-Vuong-Yang)

Definition (Rules to Get Subgraph)

- positive length → keep black points; negative length → keep white points.
- $b \equiv d \pmod 2$, keep **special** point; $b \not\equiv d \pmod 2$, remove **special** point.

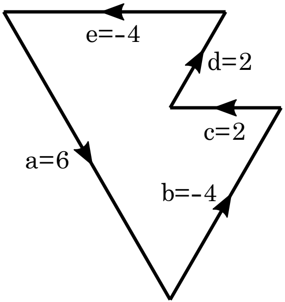


Figure: Length of Contour.

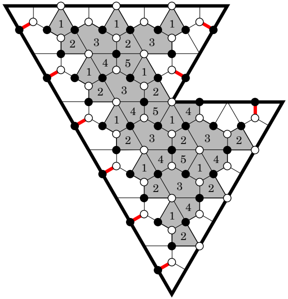


Figure: Example of Subgraph.

Main Result (Gao-Li-Vuong-Yang)

Theorem (Formula of Contours)

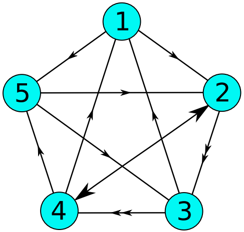
Define the contours as follows:

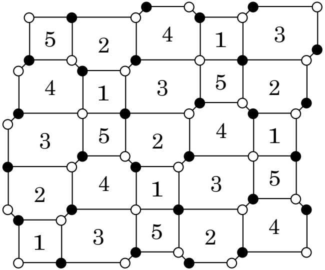
$$A^{n^2}B^{n^2-n}x_{2k} = \left(k-2+n, -\left\lceil \frac{k-4+5n}{2} \right\rceil, 2n-1, \left\lfloor \frac{k-3n}{2} \right\rfloor, 1+n-k\right)$$

$$A^{n^2+n}B^{n^2}x_{2k-1} = \left(k-2+n, -\left\lceil \frac{k-2+5n}{2} \right\rceil, 2n, \left\lfloor \frac{k-2-3n}{2} \right\rfloor, 2+n-k\right)$$

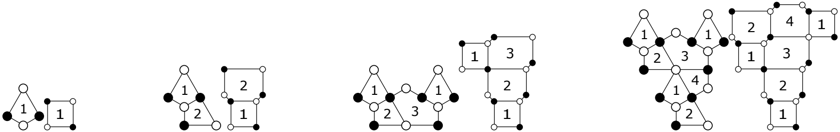
For any such cluster variable, if G is the subgraph of its corresponding contour, then $c(G)$ is the Laurent polynomial of the cluster variable.

Comparison with Somos 5 (Gao-Li-Vuong-Zhang)





The subgraphs corresponding to x_6, x_7, x_8, x_9 in the two different brane tilings.



Comparison with Somos 5 (Gao-Li-Vuong-Zhang)

The subgraphs corresponding to x_{10}, x_{11}, x_{12} in the two different brane tilings.