Linear Systems on Tropical Curves

Gregg Musiker (University of Minnesota)

Joint work with Christian Haase (U. Frankfurt)
and Josephine Yu (Georgia Tech)

Combinatexas 2011

April 16, 2011

Outline

1. Chip-firing, G-parking functions, and Riemann-Roch for graphs
2. Introduction to Tropical Arithmetic and Tropical Functions
3. Abstract Tropical Curves (Think Metric Graph)
4. Tropical Riemann-Roch and Linear Systems
5. Examples

The Laplacian Matrix and the Matrix-Tree Theorem

Our story begins with the Laplacian Matrix and the Matrix-Tree Theorem for graphs.

The Laplacian Matrix and the Matrix-Tree Theorem

Our story begins with the Laplacian Matrix and the Matrix-Tree Theorem for graphs.

Given a finite graph $G = (V, E)$, possibly with multiple edges, label the vertices as $V = \{v_1, v_2, \ldots, v_n\}$.

The Laplacian Matrix and the Matrix-Tree Theorem

Our story begins with the Laplacian Matrix and the Matrix-Tree Theorem for graphs.

Given a finite graph $G = (V, E)$, possibly with multiple edges, label the vertices as $V = \{v_1, v_2, \dots, v_n\}$.

We let d_{ij} denote the number of edges in E of the form (v_i, v_j) and $\mathrm{val}(v_i) = \sum_{j=1}^{n} d_{ij}$, i.e. the number of edges incident to v_i.

The Laplacian Matrix and the Matrix-Tree Theorem

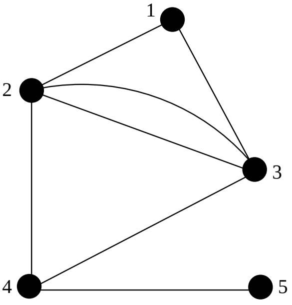
Our story begins with the Laplacian Matrix and the Matrix-Tree Theorem for graphs.

Given a finite graph $G = (V, E)$, possibly with multiple edges, label the vertices as $V = \{v_1, v_2, \ldots, v_n\}$.

We let d_{ij} denote the number of edges in E of the form (v_i, v_j) and $\mathrm{val}(v_i) = \sum_{j=1}^{n} d_{ij}$, i.e. the number of edges incident to v_i.

Define $L(G)$ to be the matrix whose diagonal entries are $\mathrm{val}(v_i)$ and whose off-diagonal entries are $-d_{ij}$.

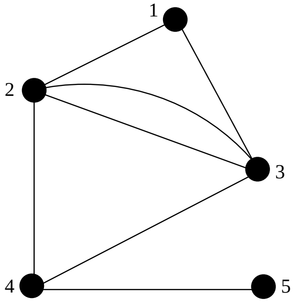
Example of a Laplacian Matrix



$$L(G) = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 4 & -2 & -1 & 0 \\ -1 & -2 & 4 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

The Reduced Laplacian matrix $L_0(G)$ is defined by deleting a row and column from $L(G)$. It is a theorem (the Matrix-Tree Theorem) that $\det L_0(G)$ does not depend on the choice of row and column deleted (as long as they are of the same index).

Example of a Laplacian Matrix



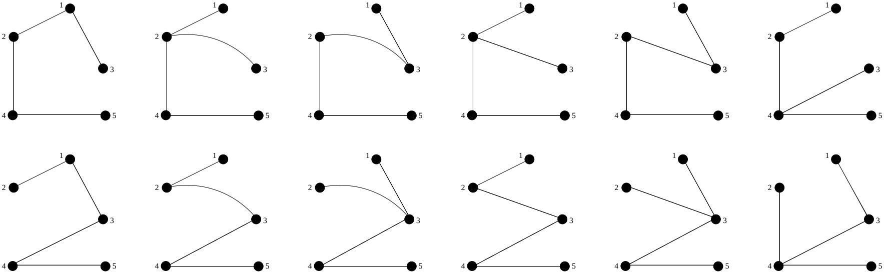
$$L(G) = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 4 & -2 & -1 & 0 \\ -1 & -2 & 4 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

The Reduced Laplacian matrix $L_0(G)$ is defined by deleting a row and column from $L(G)$. It is a theorem (the Matrix-Tree Theorem) that $\det L_0(G)$ does not depend on the choice of row and column deleted (as long as they are of the same index).

For example, in the above, $\det L_0(G) = 12$.

The Matrix-Tree Theorem

Theorem (The Matrix-Tree Theorem or Kirchoff's Theorem)
The determinant of the reduced Laplacian matrix $L_0(G)$ of a graph G is equal to the number of spanning trees of G.



For example, in the above, $\det L_0(G) = 12$.

Sandpiles, Chip-firing, and G-parking functions

We can get other families of objects in bijection with the set of spanning trees.

Sandpiles, Chip-firing, and G-parking functions

We can get other families of objects in bijection with the set of spanning trees.

We define a chip configuration or divisor on G to be an assignment of an integer to each vertex of G.

We say that two chip-configurations are equivalent if one can be reached from the other by a sequence of **chip-firing moves**.

Sandpiles, Chip-firing, and G-parking functions

We can get other families of objects in bijection with the set of spanning trees.

We define a chip configuration or divisor on G to be an assignment of an integer to each vertex of G.

We say that two chip-configurations are equivalent if one can be reached from the other by a sequence of **chip-firing moves**.

This means that we pick one vertex to share equally with all of its neighbors, sending one chip along each incident edge.

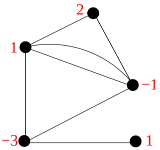
Sandpiles, Chip-firing, and G-parking functions

We can get other families of objects in bijection with the set of spanning trees.

We define a chip configuration or divisor on G to be an assignment of an integer to each vertex of G.

We say that two chip-configurations are equivalent if one can be reached from the other by a sequence of **chip-firing moves**.

This means that we pick one vertex to share equally with all of its neighbors, sending one chip along each incident edge.



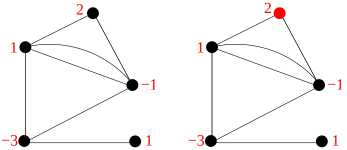
Sandpiles, Chip-firing, and G-parking functions

We can get other families of objects in bijection with the set of spanning trees.

We define a chip configuration or divisor on G to be an assignment of an integer to each vertex of G.

We say that two chip-configurations are equivalent if one can be reached from the other by a sequence of **chip-firing moves**.

This means that we pick one vertex to share equally with all of its neighbors, sending one chip along each incident edge.



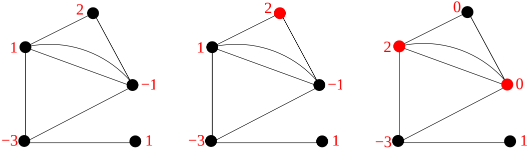
Sandpiles, Chip-firing, and G-parking functions

We can get other families of objects in bijection with the set of spanning trees.

We define a chip configuration or divisor on G to be an assignment of an integer to each vertex of G.

We say that two chip-configurations are equivalent if one can be reached from the other by a sequence of **chip-firing moves**.

This means that we pick one vertex to share equally with all of its neighbors, sending one chip along each incident edge.



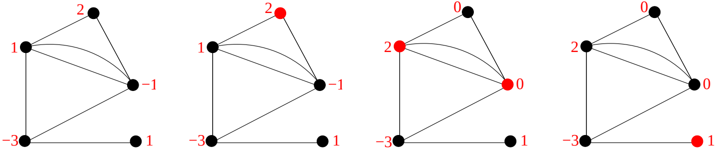
Sandpiles, Chip-firing, and G-parking functions

We can get other families of objects in bijection with the set of spanning trees.

We define a chip configuration or divisor on G to be an assignment of an integer to each vertex of G.

We say that two chip-configurations are equivalent if one can be reached from the other by a sequence of **chip-firing moves**.

This means that we pick one vertex to share equally with all of its neighbors, sending one chip along each incident edge.



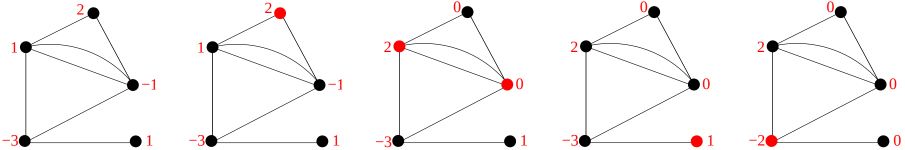
Sandpiles, Chip-firing, and G-parking functions

We can get other families of objects in bijection with the set of spanning trees.

We define a chip configuration or divisor on G to be an assignment of an integer to each vertex of G.

We say that two chip-configurations are equivalent if one can be reached from the other by a sequence of **chip-firing moves**.

This means that we pick one vertex to share equally with all of its neighbors, sending one chip along each incident edge.



Reduced Configurations or G-parking functions

If a configuration has a nonnegative number of chips on each vertex and no vertex can fire, we call such a configuration **stable**.

Reduced Configurations or G-parking functions

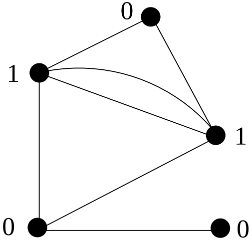
If a configuration has a nonnegative number of chips on each vertex and no vertex can fire, we call such a configuration **stable**.

However, in the case of a stable configuration, it might be possible for multiple vertices to simultaneously fire:

Reduced Configurations or G-parking functions

If a configuration has a nonnegative number of chips on each vertex and no vertex can fire, we call such a configuration **stable**.

However, in the case of a stable configuration, it might be possible for multiple vertices to simultaneously fire:

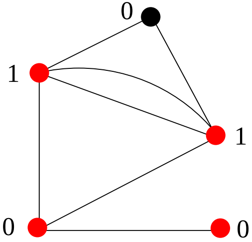


$\longrightarrow$

Reduced Configurations or G-parking functions

If a configuration has a nonnegative number of chips on each vertex and no vertex can fire, we call such a configuration **stable**.

However, in the case of a stable configuration, it might be possible for multiple vertices to simultaneously fire:

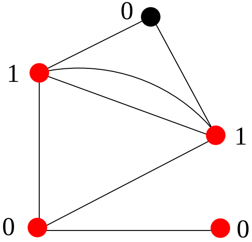


$\longrightarrow$

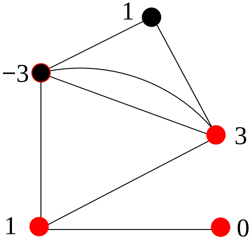
Reduced Configurations or G-parking functions

If a configuration has a nonnegative number of chips on each vertex and no vertex can fire, we call such a configuration **stable**.

However, in the case of a stable configuration, it might be possible for multiple vertices to simultaneously fire:



$\longrightarrow$

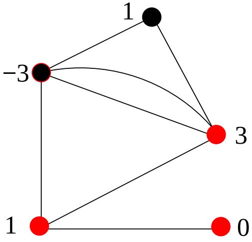


$\longrightarrow$

Reduced Configurations or G-parking functions

If a configuration has a nonnegative number of chips on each vertex and no vertex can fire, we call such a configuration **stable**.

However, in the case of a stable configuration, it might be possible for multiple vertices to simultaneously fire:

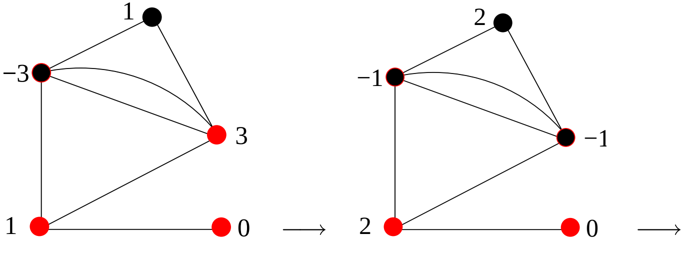


$\longrightarrow$

Reduced Configurations or G-parking functions

If a configuration has a nonnegative number of chips on each vertex and no vertex can fire, we call such a configuration **stable**.

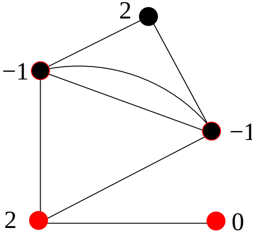
However, in the case of a stable configuration, it might be possible for multiple vertices to simultaneously fire:



Reduced Configurations or G-parking functions

If a configuration has a nonnegative number of chips on each vertex and no vertex can fire, we call such a configuration **stable**.

However, in the case of a stable configuration, it might be possible for multiple vertices to simultaneously fire:

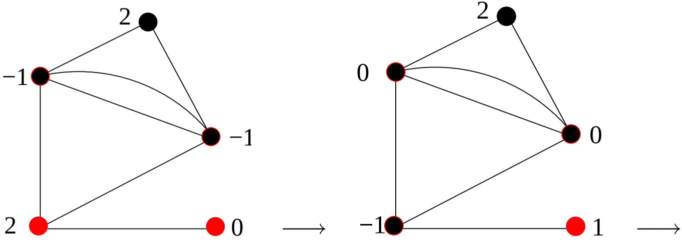


$\longrightarrow$

Reduced Configurations or G-parking functions

If a configuration has a nonnegative number of chips on each vertex and no vertex can fire, we call such a configuration **stable**.

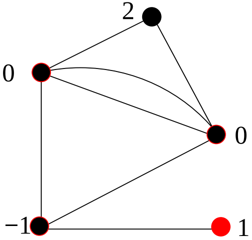
However, in the case of a stable configuration, it might be possible for multiple vertices to simultaneously fire:



Reduced Configurations or G-parking functions

If a configuration has a nonnegative number of chips on each vertex and no vertex can fire, we call such a configuration **stable**.

However, in the case of a stable configuration, it might be possible for multiple vertices to simultaneously fire:

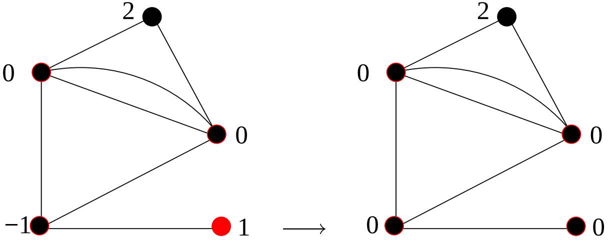


$\longrightarrow$

Reduced Configurations or G-parking functions

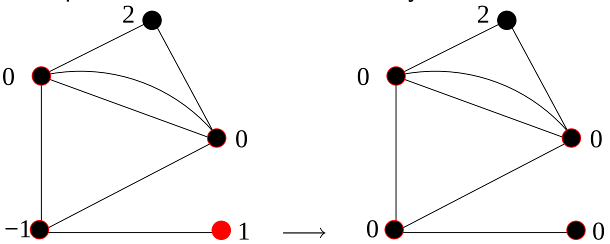
If a configuration has a nonnegative number of chips on each vertex and no vertex can fire, we call such a configuration **stable**.

However, in the case of a stable configuration, it might be possible for multiple vertices to simultaneously fire:



If a configuration has a nonnegative number of chips on each vertex and no vertex can fire, we call such a configuration **stable**.

However, in the case of a stable configuration, it might be possible for multiple vertices to simultaneously fire:

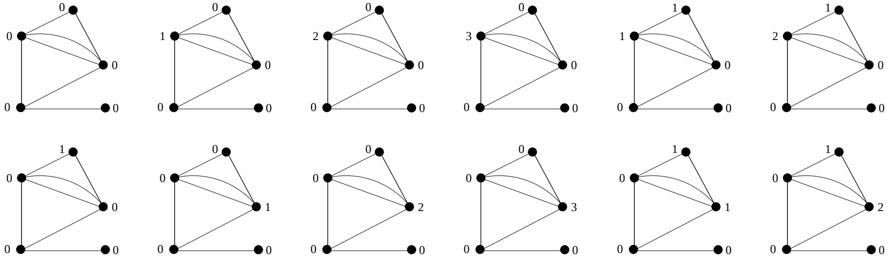


We call a configuration **super-stable** (with respect to v_0) if no subset of vertices $S \subseteq V \setminus \{v_0\}$ can fire.

These are also known as G-**parking functions** or v_0-**reduced divisors**.

Example of Super-stables/G-Parking Functions

In this example, we have 12 super-stable configurations (with respect to vertex v_5), which are also counted by $\det L_0(G)$.



We designate one vertex to be a sink and allow arbitrary addition or subtraction of chips to that vertex. Then up to equivalence by chip-firing moves, there is a unique super-stable configuration in each orbit.

Linear Systems on Graphs

In 2007, Matt Baker and Sergei Norine studied analogies between chip-firing dynamics on graphs and linear systems of divisors on curves.

Linear Systems on Graphs

In 2007, Matt Baker and Sergei Norine studied analogies between chip-firing dynamics on graphs and linear systems of divisors on curves.

We define a chip configuration (equivalently a divisor D on graph G) to be **effective** if the number of chips on v is nonnegative for each $v \in V$.

Two divisors are **linearly equivalent** if their chip-configuations differ by a sequence of chip-firing moves.

Given a divisor D, the **linear system** of D, denoted as $|D|$, is the set of all effective divisors that are linearly equivalent to D.

Linear Systems on Graphs

In 2007, Matt Baker and Sergei Norine studied analogies between chip-firing dynamics on graphs and linear systems of divisors on curves.

We define a chip configuration (equivalently a divisor D on graph G) to be **effective** if the number of chips on v is nonnegative for each $v \in V$.

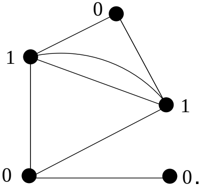
Two divisors are **linearly equivalent** if their chip-configuations differ by a sequence of chip-firing moves.

Given a divisor D, the **linear system** of D, denoted as $|D|$, is the set of all effective divisors that are linearly equivalent to D.

In other words, the set $\mathbb{Z}_{\geq 0}^{|V|}$ breaks up into equivalence classes via chip-firing. The linear systems are the orbits and each orbit has a representative which is of the form $S + dv_0$ where S is a super-stable configuration (with respect to sink v_0) and $d \in \mathbb{Z}_{\geq 0}$.

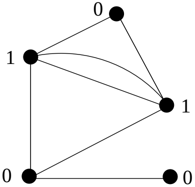
Example of a Linear System

Let G be as above and D be the divisor:

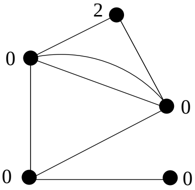


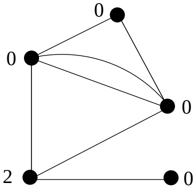
Example of a Linear System

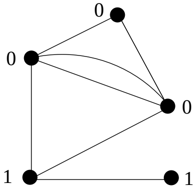
Let G be as above and D be the divisor:

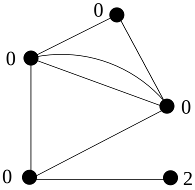


. Then the linear system $|D|$ consists of D and the following four divisors









Riemann-Roch Theorem for Graphs

Define the **degree** of a divisor to be the total number of chips in the configuration.

Let K_G (the canonical divisor) be the chip-configuration such that there are $\mathrm{val}(v) - 2$ chips on each vertex v.

The genus $g(G)$ of the graph is $|E| - |V| + 1 = b_1(G)$.

Riemann-Roch Theorem for Graphs

Define the **degree** of a divisor to be the total number of chips in the configuration.

Let K_G (the canonical divisor) be the chip-configuration such that there are $\mathrm{val}(v) - 2$ chips on each vertex v.

The genus $g(G)$ of the graph is $|E| - |V| + 1 = b_1(G)$.

We also have to define a rank function $r(D) = r(|D|)$ defined as follows:

1) If D is not effective nor linearly equivalent to an effective divisor, then $r(D) = -1$.

2) If D is linearly equivalent to an effective divisor, i.e. $|D| \neq \emptyset$, then $r(D) \geq 0$.

3) If $|D - E| \neq \emptyset$ for any effective divisor E of degree k, then $r(D) \geq k$.

Riemann-Roch Theorem for Graphs

We also have to define a rank function $r(D) = r(|D|)$ defined as follows:

1) If D is not effective nor linearly equivalent to an effective divisor, then $r(D) = -1$.

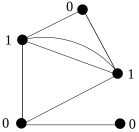
2) If D is linearly equivalent to an effective divisor, i.e. $|D| \neq \emptyset$, then $r(D) \geq 0$.

3) If $|D - E| \neq \emptyset$ for any effective divisor E of degree k, then $r(D) \geq k$.

Theorem (Baker-Norine 2007) We have the following equality for any graph G and any divisor D.

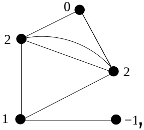
$$r(D) - r(K_G - D) = \deg(D) - g(G) + 1.$$

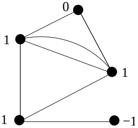
Example of Riemann-Roch

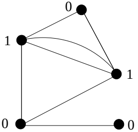
Example: Let D and G be as follows: 

.

Then the canonical divisor for this graph is

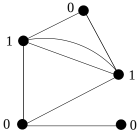
K_G is 

, and $K_G - D$ is 

$\sim$ 

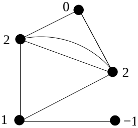
.

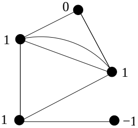
Example of Riemann-Roch

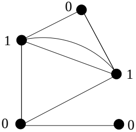
Example: Let D and G be as follows: 

.

Then the canonical divisor for this graph is

K_G is 

, and $K_G - D$ is 

$\sim$ 

.

Then $g(G) = 3$, $\deg(D) = 2$, $r(D) = r(K - D) = 1$, and the Riemann-Roch equality $1 - 1 = 2 - 3 + 1$ is satisfied.

(To see that $r(D) = 1$, note that we can subtract a chip from any vertex and we are still linearly equivalent to an effective divisor.

However, it is possible to subtract two chips and get a non-effective.)

And now for something completely different . . .

Tropical Arithmetic

We work over the tropical semi-ring

$$(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$$

where $a \oplus b = \max(a, b)$ and $a \odot b = a + b$.

Tropical Arithmetic

We work over the tropical semi-ring

$$(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$$

where $a \oplus b = \max(a, b)$ and $a \odot b = a + b$.

Notice that $a + \max(b, c) = \max(a + b, a + c)$, so we have the tropical distributive law

$$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c).$$

Tropical Arithmetic

We work over the tropical semi-ring

$$(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$$

where $a \oplus b = \max(a, b)$ and $a \odot b = a + b$.

Notice that $a + \max(b, c) = \max(a + b, a + c)$, so we have the tropical distributive law

$$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c).$$

We also have the tropical commutative and associative laws. Also,

$$a \oplus (-\infty) = a \quad \text{and} \quad b \odot 0 = b$$

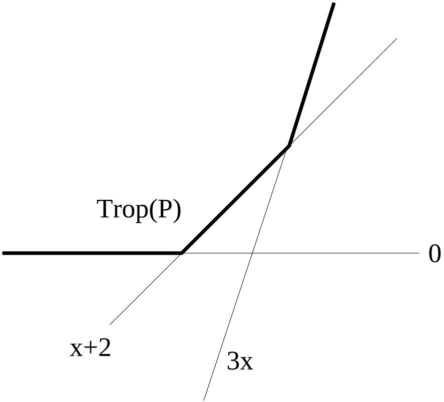
for any a and b, so we have additive and multiplicative identities.

Lastly, we have multiplicative inverses, but we do not have additive inverses.

Tropical Polynomials

We can form Tropical Polynomials such as

$$P = x^{\odot 3} \oplus 2 \odot x \oplus 0 = \max(3x, 2 + x, 0).$$

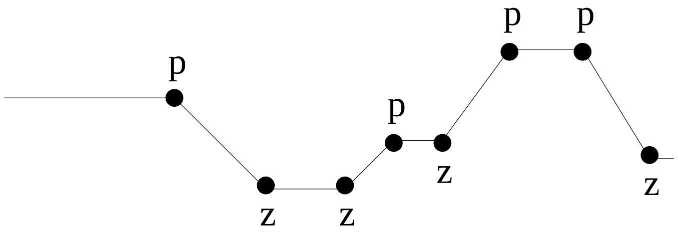


A tropical polynomial is a piecewise linear function with integer slopes, whose image is convex, and a finite number of linear pieces.

Tropical Rational Functions

A Tropical Rational Function is also a piecewise linear function of the same form, but the requirement of convexity is dropped.

The image of a Tropical Rational Function:



A zero of the Tropical Rational Function is a point where the slope increases, and a pole is a point where the slope decreases.

Notice that the image is convex at zeros, but is concave at poles.

Tropical Curves

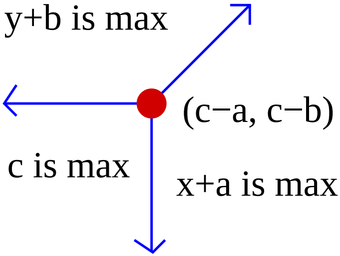
The Corner Locus of a Tropical Function is the set of all points where the slope changes (i.e. the maximum is achieved twice.)

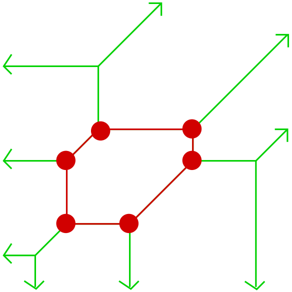
$1 - D$: the corner locus would be the set of zeros and poles.

$2 - D$: The corner locus looks like a Metric Graph (plus unbounded rays).

Tropical Line: $a \odot x \oplus b \odot y \oplus c$ and Tropical Cubic: $\bigoplus_{i+j \leq 3} x^i y^j$.

The Degree of the polynomial equals the $\#$ of rays in each direction.





Tropical Riemann-Roch

An Abstract Tropical Curve Γ is simply a Metric Graph, where we allow leaf edges to be of infinite length. The genus of Γ is $g(\Gamma) = |E| - |V| + 1$.

Examples (Finite portions of Genus 2):

Tropical Riemann-Roch

An Abstract Tropical Curve Γ is simply a Metric Graph, where we allow leaf edges to be of infinite length. The genus of Γ is $g(\Gamma) = |E| - |V| + 1$.

Examples (Finite portions of Genus 2):

A Chip Configuration C of Γ is a formal linear combination of points of Γ:

$$C = \sum_P c_P P \qquad \text{(only finitely many } c_P\text{'s are nonzero).}$$

The Canonical Chip Configuration $K = K(\Gamma) = \sum_{V \in \Gamma} (\mathrm{val}(V) - 2) V$.

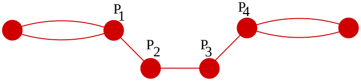
(Gathmann-Kerber, Mikhalkin-Zharkov): The Baker-Norine rank function $r(C)$ satisfies Riemann-Roch for Tropical Curves

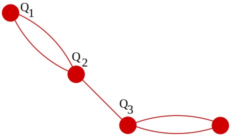
$$r(C) - r(K - C) = \deg C + 1 - g(\Gamma).$$

Tropical Linear Systems

Given a tropical rational function f, we let $ord_P(f)$ denote the sum of the outgoing slopes locally at point P with respect to the function f.

The Chip Configuration of f is defined as $(f) = \sum_{P \in \Gamma} ord_P(f) P$.

Examples: $g_1 =$ 

, $g_2 =$ 

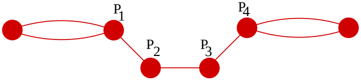
.

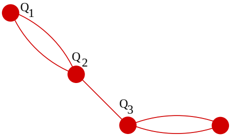
Then $(g_1) = -P_1 + P_2 + P_3 - P_4$. and $(g_2) = -2Q_1 + Q_2 + Q_3$.

Tropical Linear Systems

Given a tropical rational function f, we let $ord_P(f)$ denote the sum of the outgoing slopes locally at point P with respect to the function f.

The Chip Configuration of f is defined as $(f) = \sum_{P \in \Gamma} ord_P(f) P$.

Examples: $g_1 =$ 

, $g_2 =$ 

.

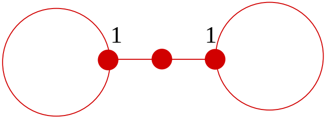
Then $(g_1) = -P_1 + P_2 + P_3 - P_4$. and $(g_2) = -2Q_1 + Q_2 + Q_3$.

Can also think of these transformations as weighted chip-firing moves.
(We can fire a subgraph of Γ in place of a subset of vertices.)

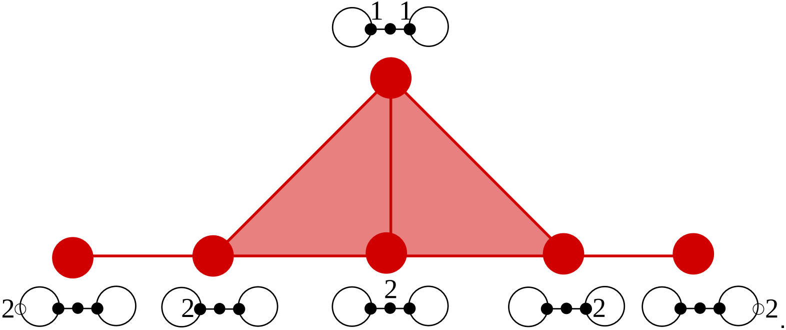
The Tropical Linear System of C (following Gathmann-Kerber):

$$|C| = \{C' \geq 0 : C' = C + (f) \text{ for some tropical rational funciton } f\}.$$

Tropical Linear Systems (Example Continued)

For $\Gamma =$ 

with C as specified, we have $|C|$ is



The Linear System $|C|$ contains six 0-cells, seven 1-cells and two 2-cells.

$|C|$ and $R(C)$ as polyhedral cell complexes

Recall $|C| = \{C' \geq 0 : C' = C + (f)$ for some tropical rational function $f\}$.

Let $R(C) = \{f : C + (f) \geq 0\}$. This is a tropical semi-module of functions.

$|C|$ and $R(C)$ as polyhedral cell complexes

Recall $|C| = \{C' \geq 0 : C' = C + (f)$ for some tropical rational function $f\}$.

Let $R(C) = \{f : C + (f) \geq 0\}$. This is a tropical semi-module of functions.

First observation: $R(C)$ is naturally embedded in $\mathbb{R}^\Gamma$ and $|C|$ is a subset of the dth symmetric product of Γ, where $d = \deg C$.

$|C|$ and $R(C)$ as polyhedral cell complexes

Recall $|C| = \{C' \geq 0 : C' = C + (f)$ for some tropical rational function $f\}$.

Let $R(C) = \{f : C + (f) \geq 0\}$. This is a tropical semi-module of functions.

First observation: $R(C)$ is naturally embedded in $\mathbb{R}^\Gamma$ and $|C|$ is a subset of the dth symmetric product of Γ, where $d = \deg C$.

Let $\mathbb{1}$ denote the set of constant functions on Γ. (Note that if f is constant, then the chip configuration $(f) = 0$.)

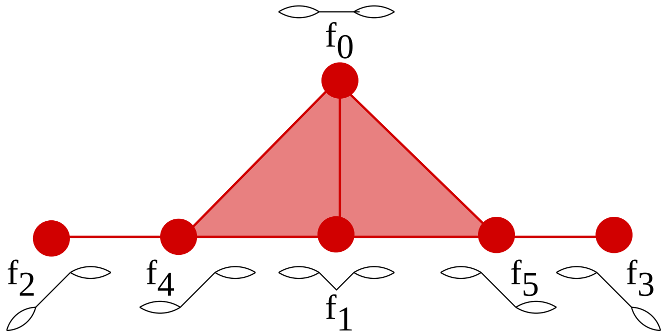
In fact, there is the natural homeomorphism:

$$\begin{array}{rcl} R(C)/\mathbb{1} & \longrightarrow & |C| \\ f & \mapsto & C + (f). \end{array}$$

So a linear system can be described also by tropical rational functions modulo tropical multiplication (i.e. translation by adding a a constant function). Only local slope changes matter, not the function values.

Back To Barbell Example

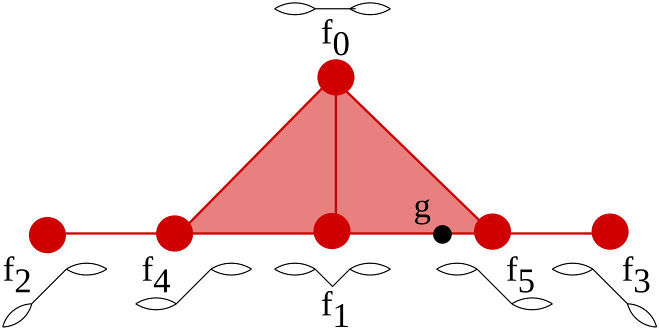
In terms of tropical rational functions, we obtain the following labeling of the polyhderal complex's vertices instead:



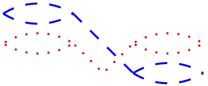
Each of the 1-cells and 2-cells are tropically convex.

Back To Barbell Example

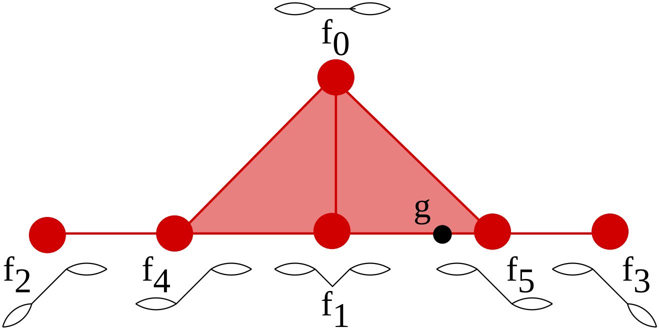
In terms of tropical rational functions, we obtain the following labeling of the polyhderal complex's vertices instead:



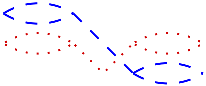
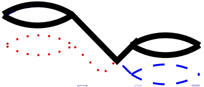
Each of the 1-cells and 2-cells are tropically convex. For example,

$g = f_1 \oplus (+1/4 \odot f_5) =$ 

In terms of tropical rational functions, we obtain the following labeling of the polyhderal complex's vertices instead:

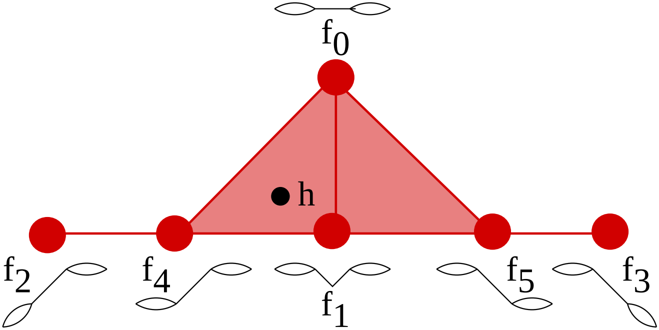


Each of the 1-cells and 2-cells are tropically convex. For example,

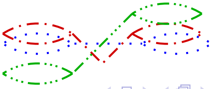
$g = f_1 \oplus (+1/4 \odot f_5) =$  .

Back To Barbell Example

In terms of tropical rational functions, we obtain the following labeling of the polyhderal complex's vertices instead:

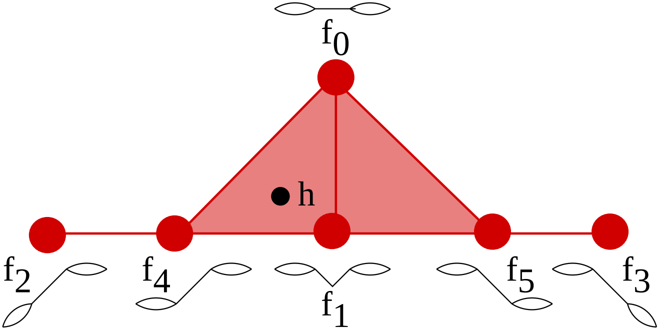


Each of the 1-cells and 2-cells are tropically convex. Second example,

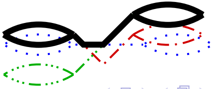
$h = f_0 \oplus (+1/4 \odot f_1) \oplus (+1/3 \odot f_4) =$ 

Back To Barbell Example

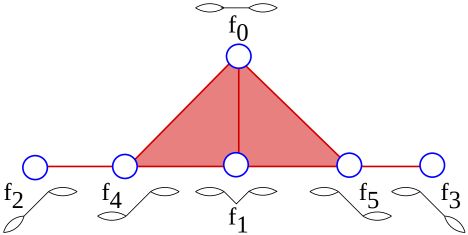
In terms of tropical rational functions, we obtain the following labeling of the polyhderal complex's vertices instead:



Each of the 1-cells and 2-cells are tropically convex. Second example,

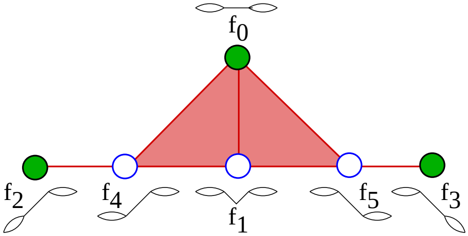
$h = f_0 \oplus (+1/4 \odot f_1) \oplus (+1/3 \odot f_4) =$  .

Back To Barbell Example (Continued)



In particular, every tropical rational function on Γ is the tropical convex hull of the 0-cells $\{f_0, f_1, \ldots, f_5\}$.

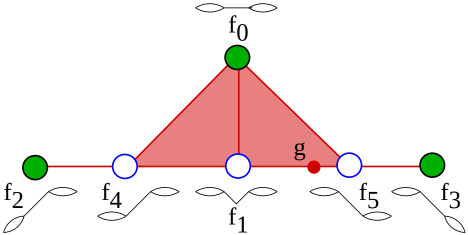
Back To Barbell Example (Continued)



In particular, every tropical rational function on Γ is the tropical convex hull of the 0-cells $\{f_0, f_1, \ldots, f_5\}$.

More strongly, every tropical rational function on Γ is tropical convex hull of $\{f_0, f_2, f_3\}$. Generators of this minimal set are called extremals.

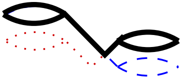
Back To Barbell Example (Continued)



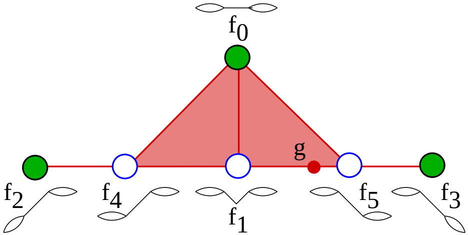
In particular, every tropical rational function on Γ is the tropical convex hull of the 0-cells $\{f_0, f_1, \ldots, f_5\}$.

More strongly, every tropical rational function on Γ is tropical convex hull of $\{f_0, f_2, f_3\}$. Generators of this minimal set are called extremals.

For example, $g = f_1 \oplus (+1/4 \odot f_5)$



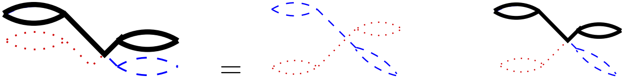
Back To Barbell Example (Continued)



In particular, every tropical rational function on Γ is the tropical convex hull of the 0-cells $\{f_0, f_1, \ldots, f_5\}$.

More strongly, every tropical rational function on Γ is tropical convex hull of $\{f_0, f_2, f_3\}$. Generators of this minimal set are called extremals.

For example, $g = f_1 \oplus (+1/4 \odot f_5) = f_2 \oplus (+1/4 \odot f_3) =$



Main Results

Theorem (HMY 2009) $R(C)$ is a finitely generated tropical semimodule.

If $C' \in |C|$, with $C' = C + (f)$, is in the cell with vertices $C_1, C_2, \ldots, C_k$ (with corresponding $f_1, f_2, \ldots, f_k$), then

$$f = (c_1 \odot f_1) \oplus (c_2 \odot f_2) \oplus \cdots \oplus (c_k \odot f_k),$$

i.e. the cells of $|C|$ are tropically convex.

Main Results

Theorem (HMY 2009) $R(C)$ is a finitely generated tropical semimodule.

If $C' \in |C|$, with $C' = C + (f)$, is in the cell with vertices $C_1, C_2, \ldots, C_k$ (with corresponding $f_1, f_2, \ldots, f_k$), then

$$f = (c_1 \odot f_1) \oplus (c_2 \odot f_2) \oplus \cdots \oplus (c_k \odot f_k),$$

i.e. the cells of $|C|$ are tropically convex.

In particular, $R(C)/\mathbb{1} \cong |C|$ is finitely generated by the 0-cells of $|C|$.

Theorem (HMY 2009) The 0-cells of $|C|$, as well as all other d-cells, can be described explicitly.

Dimension of a cell

Definition. A point $P \in \Gamma$ is **smooth** if it has valence two and is not a vertex (i.e. the interior of an edge).

Definition. The **support** of a chip configuration C is the set of points of Γ with nonzero coefficients in C.

Let $I(\Gamma, C') = \Gamma \setminus (\text{Supp } C' \cap \{\text{Smooth points}\})$.

Theorem (HMY 2009) The cell containing chip configuration C' is of Dimension $= \# \ (\text{Connected components of } I(\Gamma, C')) - 1$.

Dimension of a cell

Definition. A point $P \in \Gamma$ is **smooth** if it has valence two and is not a vertex (i.e. the interior of an edge).

Definition. The **support** of a chip configuration C is the set of points of Γ with nonzero coefficients in C.

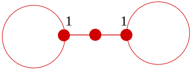
Let $I(\Gamma, C') = \Gamma \setminus (\mathrm{Supp}\ C' \cap \{\mathrm{Smooth\ points}\})$.

Theorem (HMY 2009) The cell containing chip configuration C' is of Dimension $= \#\ (\mathrm{Connected\ components\ of}\ I(\Gamma, C')) - 1.$

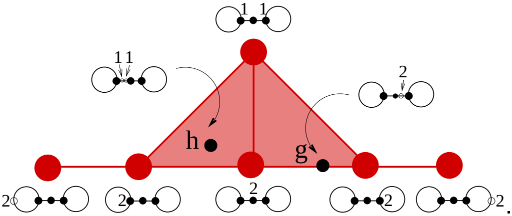
Corollary (HMY 2009) The 0-cells, i.e. a set of generators for $R(C)/\mathbb{1}$, correspond to the C''s whose smooth support does not disconnect Γ.

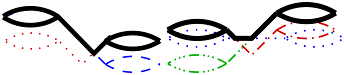
The extremals lie inside this set: They are the functions f precisely such that no two proper subgraphs Γ_1 and Γ_2 of Γ covering Γ (i.e. $\Gamma_1 \cup \Gamma_2 = \Gamma$) can both fire on the chip configuration $C + (f)$.

Another return to the barbell

For $\Gamma =$ 

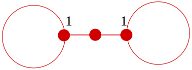
with C as specified, we have $|C|$ is



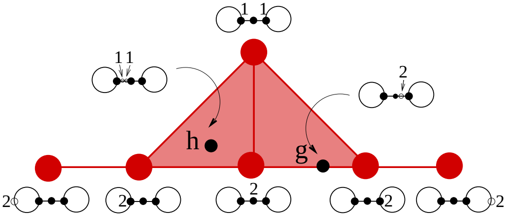


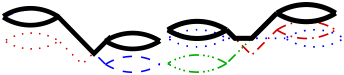
Notice that removal of the smooth support of C' (for C' a 0-cell) does not disconnect the graph Γ.

Another return to the barbell

For $\Gamma =$ 

with C as specified, we have $|C|$ is



.


Notice that removal of the smooth support of C' (for C' a 0-cell) does not disconnect the graph Γ.

Chip configurations corresponding to tropical rational functions g and h correspond to the interiors of 1-cells and 2-cells.

Removal of their breakpoints disconnects the graph into 2 and 3 pieces.

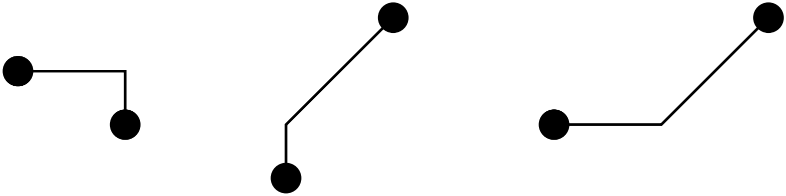
Other Results

Theorem (HMY) If $R(D) = tconv(f_0, f_1, \ldots, f_r)$, then

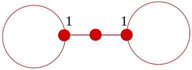
$$\begin{array}{rcl} \phi : \Gamma & \rightarrow & \mathbb{TP}^r \\ x & \mapsto & (f_0(x), \ldots, f_r(x)) \end{array}$$

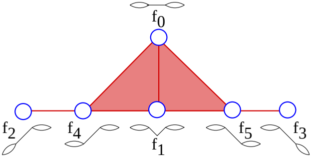
satisfies $|D| \cong tconv(\phi(\Gamma))$.

Recall that the tropical convex hull of two points is the tropical line segement between them.



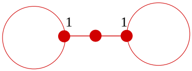
Embedding the Barbell

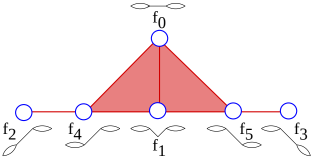
Letting $\Gamma =$ 

with D as specified, we note that the extremals of $|D|$ are f_0, f_2, and f_3 in the picture 

.

Embedding the Barbell

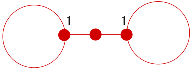
Letting $\Gamma =$ 

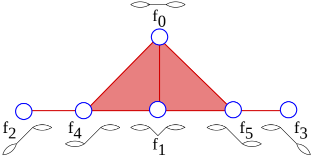
with D as specified, we note that the extremals of $|D|$ are f_0, f_2, and f_3 in the picture 

.

Letting P be the leftmost point of Γ, up to vertical translation (i.e. tropical projective scaling), we can assume that $f_0(P) = f_2(P) = f_3(P) = 0$.

Embedding the Barbell

Letting $\Gamma =$ 

with D as specified, we note that the extremals of $|D|$ are f_0, f_2, and f_3 in the picture 

.

Letting P be the leftmost point of Γ, up to vertical translation (i.e. tropical projective scaling), we can assume that $f_0(P) = f_2(P) = f_3(P) = 0$.

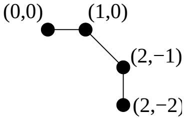
Graphing f_0, f_2, and f_3 along Γ, we get an infinite matrix with three rows and columns indexed by points of Γ.

$$\begin{bmatrix} 0 & \dots & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 1 & \dots & 3/2 & \dots & 2 & \dots & 2 \\ 0 & \dots & 0 & \dots & -1/2 & \dots & -1 & \dots & -2 \end{bmatrix}$$

Embedding the Barbell

We then plot the columns as projective points (ignoring the zeroes in the first row)

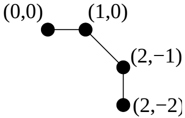
$$\begin{bmatrix} 0 & \dots & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 1 & \dots & 3/2 & \dots & 2 & \dots & 2 \\ 0 & \dots & 0 & \dots & -1/2 & \dots & -1 & \dots & -2 \end{bmatrix}$$

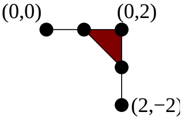


Embedding the Barbell

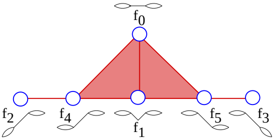
We then plot the columns as projective points (ignoring the zeroes in the first row)

$$\begin{bmatrix} 0 & \dots & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 1 & \dots & 3/2 & \dots & 2 & \dots & 2 \\ 0 & \dots & 0 & \dots & -1/2 & \dots & -1 & \dots & -2 \end{bmatrix}$$



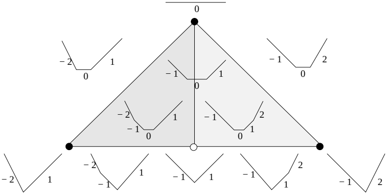


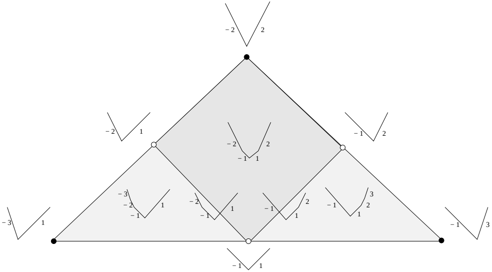
The second plot is the tropical convex hull of the points in the first.
Observe that $tconv(f_0, f_2, f_3)$ in $\mathbb{TP}^2 \cong$ the linear system $|D|$.



Final Examples: Genus One Circle Graph

Take the circle $\Gamma = S^1$ on one vertex and a chip configuration of degree d. E.g. $d = 3$ or 4:

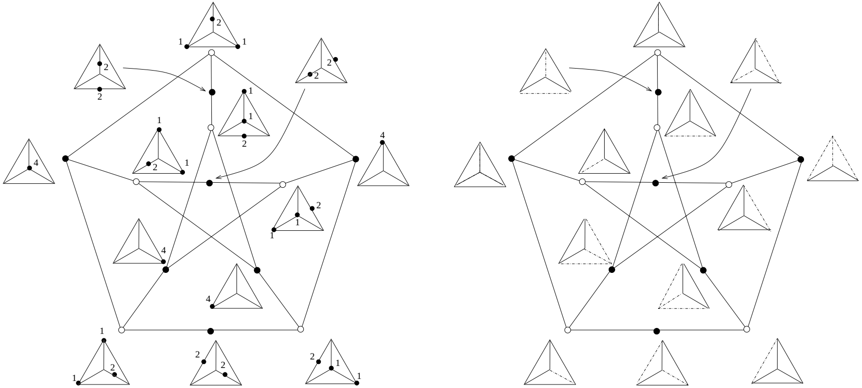




Black Vertices correspond to Extremals. $|C|$ is a subdivision of a $(d-1)$-simplex.

In the case of $d = 4$, $|C|$ is a cone over the triangle that is shown. The cone point is the constant function, and is another extremal.

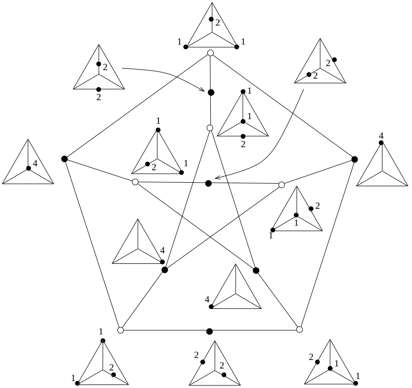
For $\Gamma = K_4$ with edges of equal length and K the canonical chip configuration with 1 at all four vertices: $|K|$ is a cone over the Petersen graph from point K.

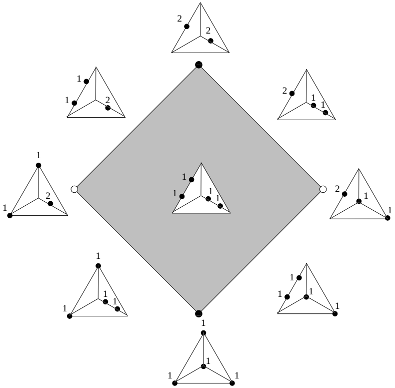


Theorem (HMY) For any Γ, the fine subdivision of $\text{link}(K, |K|)$ contains the fine subdivision of the Bergman complex $B(M^*(\Gamma))$ as a subcomplex.

Final Examples: Complete Graph on 4 Vertices (Continued)

Fourteen 0-cells, seven (black vertices) of which (not K) are extremal.





This is a 2-dimensional cell complex: including K (at the bottom), here is a close-up of a quadrilateral cell. In particular, $|K|$ is not simplicial.

Open Questions

Question: Is there a relationship between geometric properties of the polyhedral cell complex $|C|$ and the Baker-Norine rank function satisfying Tropical Riemann-Roch?

Open Questions

Question: Is there a relationship between geometric properties of the polyhedral cell complex $|C|$ and the Baker-Norine rank function satisfying Tropical Riemann-Roch?

Question: Can we identify geometrically for a given $|C|$ which of the 0-cells are extremals?

Open Questions

Question: Is there a relationship between geometric properties of the polyhedral cell complex $|C|$ and the Baker-Norine rank function satisfying Tropical Riemann-Roch?

Question: Can we identify geometrically for a given $|C|$ which of the 0-cells are extremals?

Question: What happens to $|C|$ as either C changes, the combinatorial type of Γ changes in a small way, or if the edge lengths of Γ change?

Open Questions

Question: Is there a relationship between geometric properties of the polyhedral cell complex $|C|$ and the Baker-Norine rank function satisfying Tropical Riemann-Roch?

Question: Can we identify geometrically for a given $|C|$ which of the 0-cells are extremals?

Question: What happens to $|C|$ as either C changes, the combinatorial type of Γ changes in a small way, or if the edge lengths of Γ change?

Thanks for Listening!

Linear Systems on Tropical Curves (with Christian Haase and Josephine Yu), `arXiv:math.AG/0909.3685`. To appear in Math. Zeitschrift

Slides at `http://www.math.umn.edu/~musiker/TropTalk.pdf`.