The m-Quasiinvariants of S_n

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The symmetric group S_n acts on the ring of polynomials $\mathbb{Q}[x_1, \ldots, x_n]$ by permuting indices. For example, if we let $P = x_1^2 x_5 + x_4 x_6 + x_3^3 + x_2 x_3$, then

$$\Big((132)(5)(46)\Big)P = x_3^2 x_5 + x_4 x_6 + x_2^3 + x_1 x_2$$

A polynomial P is symmetric (an invariant of S_n) if and only if

$$\sigma(P) = P \quad \text{for all} \quad \sigma \in S_n.$$

For example, in S_3 some invariants are:

$$\begin{aligned} e_1 &= x_1 + x_2 + x_3 \\ e_2 &= x_1 x_2 + x_1 x_3 + x_2 x_3 \\ e_3 &= x_1 x_2 x_3 \end{aligned}$$

Any invariant of S_n can be written as a polynomial in $\{e_1, e_2, \ldots, e_n\}$ where

$$e_k = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}$$

A classic result states that $\mathbb{Q}[x_1, \ldots, x_n]$ is a free module of rank $n!$ over the ideal $(e_1, \ldots, e_n)$.

For example any polynomial in $\mathbb{Q}[x_1, x_2, x_3]$ can be written uniquely as

$$A_1 + A_2 x_2 + A_3 x_3 + A_4 x_2 x_3 + A_5 x_3^2 + A_6 x_2 x_3^2$$

where $A_1, \ldots, A_6$ are symmetric polynomials.

A polynomial P is m-quasiinvariant if and only if $P - (i,j)(P)$ is divisible by $(x_i - x_j)^{2m+1}$ for all transpositions (i,j) in S_n.

Lemma 1 *The m-quasiinvariants of S_n, which we will denote as QI_m, form a sequence of nested rings.*

$$\mathbb{Q}[x_1, \ldots, x_n] = QI_0 \supset QI_1 \supset QI_2 \supset \cdots \supset QI_\infty = \Lambda_n$$

where Λ_n is the ring of n-variable symmetric polynomials.

The group S_n acts on the rings QI_m just as it acts on the polynomial ring.

Theorem 1 (Etingof and Ginzburg) *Just like in the $\mathbb{Q}[x_1, \ldots, x_n]$ case, we can write any element of QI_m as a unique sum*

$$\sum_{i=1}^{n!} A_i(e_1, \ldots, e_n) \cdot \eta_i$$

where the A_i's are polynomials and the η_i's are elements of QI_m.

These η_i's are therefore a basis for $QI_m / \big\langle (e_1, e_2, \ldots e_n) \big\rangle$, a space which has the following Hilbert Series [Felder and Veselov]:

$$\sum_{i=1}^{n!} q^{degree(\eta_i)} = \sum_{T \in ST(n)} q^{m\left(\binom{n}{2} - content(\lambda(T))\right) + cocharge(T)}$$

In the case that $n = 3$ the Hilbert Series of $QI_m / \langle (e_1, e_2, e_3) \rangle$ is

$$q^0 + 2q^{3m+1} + 2q^{3m+2} + q^{6m+3}.$$

Proposition 1 *$QI_m / \langle (e_1, e_2, e_3) \rangle$ has basis*

$$\{1, B_{3m+1}, (13)B_{3m+1}, B_{3m+2}, (13)B_{3m+2}, \Delta_3^{2m+1}\}$$

where $\Delta_3 = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$ and the polynomials B_{3m+1} and B_{3m+2} have the form

$$Q_d = \sum_{0 \leq i \leq j \leq m} C_{[i,j]} x_1^{d-i-j} m_{[i,j]}(x_2, x_3)$$

for $d = 3m + 1$ and $3m + 2$ respectively where $m_{[i,i]}(x_2, x_3) = x_2^i x_3^i$ and $m_{[i,j]}(x_2, x_3) = x_2^i x_3^j + x_2^j x_3^i$.

How to solve for the coefficients $C_{[i,j]}$

Lemma 2 *Q_d is m-quasiinvariant if and only if the coefficients $C_{[i,j]}$ satisfy the linear equations*

$$\sum_{0 \le j \le i \le m} A_{i,j,k,l} C_{[i,j]} = 0$$

for $k \in \{0, \ldots, m\}$ and $l \in \{1, 3, 5, \ldots, 2m-1\}$. Here we set

$$A_{i,j,k,l} = \begin{cases} \binom{i}{k}\binom{d-i-k}{l} - \binom{i}{k}\binom{2i-k}{l} & \text{if } i = j, \\ \binom{i}{k}\binom{d-j-k}{l} + \binom{j}{k}\binom{d-i-k}{l} - \left(\binom{i}{k} + \binom{j}{k}\right)\binom{i+j-k}{l} & \text{if } i \neq j. \end{cases}$$

Lemma 3 *In the cases $d = 3m+1$ or $3m+2$, solving these equations yields an m-quasiinvariant of degree d (unique up to scalar multiplication).*

To prove this Lemma, we show that the $m(m+1) \times \binom{m+2}{2}$ matrix of entries $A_{i,j,k,l}$ has a nullspace of dimension one.

We restrict to the $\left(\binom{m+2}{2} - 1\right) \times \left(\binom{m+2}{2} - 1\right)$ submatrix B_m of entries $A_{i,j,k,l}$ where $[i,j] \neq [m,m]$, $0 \leq k \leq m-1$ and $l \in \{2m-2k-1, \ldots, 2m-3, 2m-1\}$ or $k = m$ and $l \in \{1, 3, 5, \ldots, 2m-1\}$.

Proving that the matrix B_m is nonsingular will show that the rank of the full matrix is $\binom{m+2}{2} - 1$ and thus the nullspace has dimension ≤ 1. We conclude that the dimension is exactly one since by proposition 1, nonzero m-quasiinvariants of form Q_d exist.

Proving nonsingularity of B_m.

Matrix B_m is block diagonal with one block of size 1, one of size 2, and so on except that there will be two blocks of size m.

We let $B^{k,m}$ be the kth block for $k \in \{1, \ldots, m\}$ and denote the last block as B^m.

For $B^{k,m}$, $0 \leq j \leq k-1$, $l \in \{2m-2k+1, 2m-2k-1, \ldots, 2m-1\}$

and for B^m, $0 \leq j \leq m-1$ and $l \in \{1, 3, \ldots, 2m-1\}$.

$$B^{k,m} = \left[\binom{d-j-k-1}{l} - \binom{j}{l} \right]_{l,j}, \quad B^m = \left[\binom{d-j-m}{l} - \binom{j}{l} \right]_{l,j}.$$

For example, when $m = 3$ and $d = 10$, matrix B_m is

$$\begin{bmatrix} 252 & 378 & 126 & 308 & 182 & 56 & 273 & 147 & 75 \\ 0 & 126 & 56 & 252 & 133 & 42 & 378 & 174 & 75 \\ 0 & 84 & 56 & 168 & 147 & 68 & 252 & 184 & 125 \\ 0 & 0 & 0 & 56 & 21 & 6 & 168 & 63 & 19 \\ 0 & 0 & 0 & 56 & 35 & 20 & 168 & 105 & 66 \\ 0 & 0 & 0 & 8 & 6 & 4 & 21 & 15 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 & 21 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 35 & 20 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 5 & 3 \end{bmatrix}.$$

We will show that each of the matrices $B^{k,m}$ is nonsingular for the case $d = 3m + 1$. (A similar argument will hold for the other cases). Re-indexing gives

$$B^{k,m} = \left| \binom{3m+3-k-i}{2m+1-2j} - \binom{i-1}{2m+1-2j} \right|_{i,j=1}^{k}.$$

A literature search found no determinantal results for a matrix of differences of binomial coefficients where the tops were different and the bottoms the same.

We begin by considering a general form of the matrix, and factoring it. This factorization was suggested by an argument of Gessel and Viennot:

$$\det\left|\binom{a_i}{b_j} - \binom{c-a_i}{b_j}\right|_{i,j=1}^{k} =$$

$$\det\left|\frac{\binom{c}{b_{k-i+1}}}{\binom{c}{a_{k-j+1}}} \cdot \left(\binom{c-b_{k-i+1}}{c-a_{k-j+1}} - \binom{c-b_{k-i+1}}{a_{k-j+1}}\right)\right|_{i,j=1}^{k} =$$

$$\frac{\binom{c}{b_1}\cdots\binom{c}{b_k}}{\binom{c}{a_1}\cdots\binom{c}{a_k}} \cdot \det\left|\binom{c-b_{k-i+1}}{c-a_{k-j+1}} - \binom{c-b_{k-i+1}}{a_{k-j+1}}\right|_{i,j=1}^{k}$$

Notice that now the tops of the binomial coefficients are the same and the bottoms are different. This allows us to apply a generalization of a technique of Krattenthaler.

Lemma 4 *For any integers a, b, c, d, e, the determinant*

$$\det\left|\binom{a+bi}{c+dj} - \binom{a+bi}{e-dj}\right|_{i,j=1}^{n}$$

is the number of families of non-intersecting lattice paths with NORTH and WEST steps only from the points $\{(c+d, c+d), (c+2d, c+2d), \ldots (c+nd, c+nd)\}$ *to the points* $\{(0, a+b), (0, a+2b), \ldots, (0, a+nb)\}$ *which avoid the line* $y = -x + (c+e)$.

To show this, we first show that the number of paths from $(c+jd, c+jd)$ to $(0, a+ib)$ which avoid the line $y = -x + (c+e)$ is $\binom{a+bi}{c+dj} - \binom{a+bi}{e-dj}$.

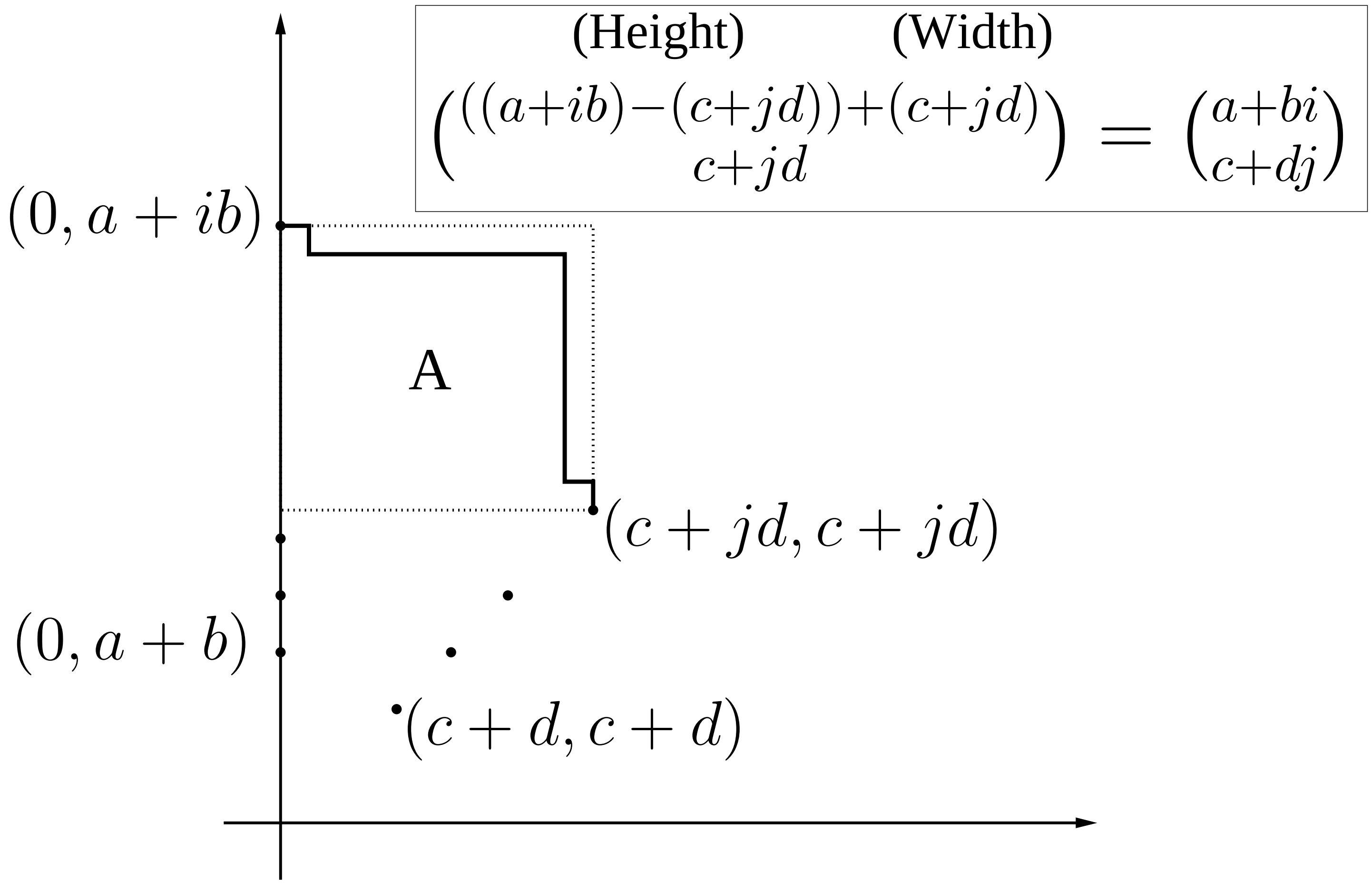

Figure 1: Counting paths from $(c+jd, c+jd)$ to $(0, a+ib)$.

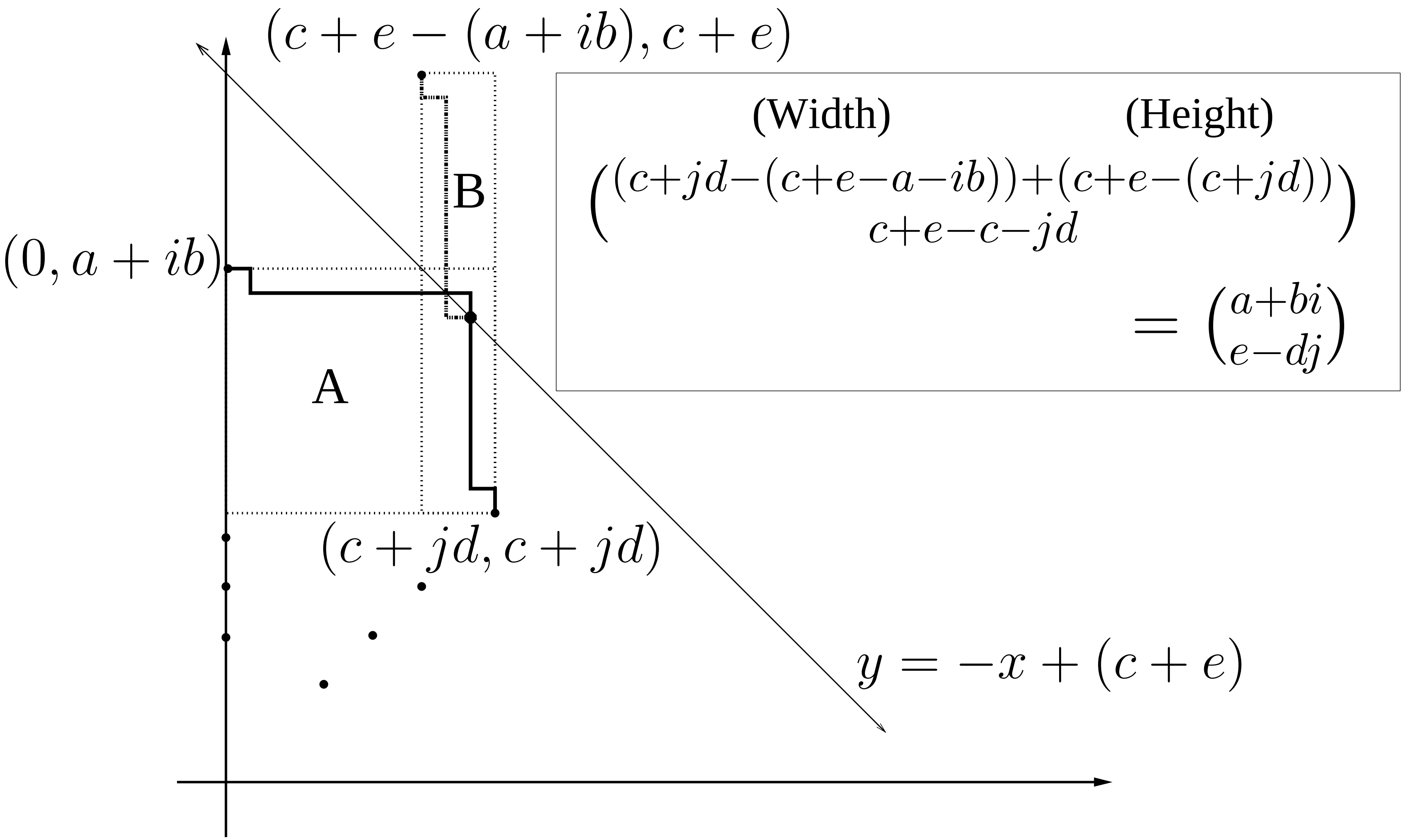

Figure 2: 'Bad' paths from $(c+jd, c+jd)$ to $(0, a+ib)$.

This shown, the classical involution of Gessel and Viennot shows that when the entries of a matrix count paths, the determinant counts families of non-intersecting paths. Applying this involution to our determinant gives

$$\det\left|\binom{3m+3-k-i}{2m+1-2j} - \binom{i-1}{2m+1-2j}\right|_{i,j=1}^{k} =$$

$$\frac{\binom{3m+2-k}{2m-1}\binom{3m+2-k}{2m-3}\cdots\binom{3m+2-k}{2m-2k+1}}{\binom{3m+2-k}{3m+2-k}\binom{3m+2-k}{3m+1-k}\cdots\binom{3m+2-k}{3m-2k+3}} \cdot |\mathcal{F}|$$

where $\mathcal{F}$ is the set of families of non-intersecting paths from $\{(0,0),(1,1),\ldots,(k-1,k-1)\}$ to $\{(0,m-k+3),(0,m-k+5),\ldots,(0,m+k+1)\}$ which stay below the line $y=-x+3m+2-k$.

This is easily seen to be positive while $m \geq k$.

Thus for $d = 3m + 1$ or $3m + 2$, if we fix $C_{[0,0]} = 1$ and solve the equations

$$\sum_{0 \le j \le i \le m} A_{i,j,k,l} C_{[i,j]} = 0$$

for appropriate k and l we will construct a unique element of QI_m of the form Q_d.

$$Q_d = \sum_{0 \le i \le j \le m} C_{[i,j]} x_1^{d-i-j} m_{[i,j]}(x_2, x_3)$$

Therefore we have an explicit basis

$$\{1, B_{3m+1}, (13)B_{3m+1}, B_{3m+2}, (13)B_{3m+2}, \Delta_3^{2m+1}\}$$

for $QI_m/\big\langle(e_1, e_2, e_3)\big\rangle$.

For $m = 1$,

$$\begin{aligned} B_4 &= x_1^4 - 2x_1^3(x_2+x_3) + 6x_1^2x_2x_3 \\ B_5 &= x_1^5 - \frac{5}{3}x_1^4(x_2+x_3) + \frac{10}{3}x_1^3x_2x_3. \end{aligned}$$

For $m = 2$,

$$\begin{aligned} B_7 &= x_1^7 - \frac{7}{2}x_1^6(x_2+x_3) + 14x_1^5x_2x_3 + \frac{7}{2}x_1^5(x_2^2+x_3^2) - \frac{35}{2}x_1^4(x_2^2x_3+x_2x_3^2) + 35x_1^3x_2^2x_3^2 \\ B_8 &= x_1^8 - \frac{16}{5}x_1^7(x_2+x_3) + \frac{56}{5}x_1^6x_2x_3 + \frac{14}{5}x_1^6(x_2^2+x_3^2) - \frac{56}{5}x_1^5(x_2^2x_3+x_2x_3^2) + 14x_1^4x_2^2x_3^2. \end{aligned}$$