

# A Simplified Suspension Calculus and its Relationship to Other Explicit Substitution Calculi

Andrew Gacek and Gopalan Nadathur

Digital Technology Center and Department of Computer Science and Engineering  
University of Minnesota

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This paper concerns the explicit treatment of substitutions in the lambda calculus. One of its contributions is the simplification and rationalization of the suspension calculus that embodies such a treatment. The earlier version of this calculus provides a cumbersome encoding of substitution composition, an operation that is important to the efficient realization of reduction. This encoding is simplified here, resulting in a treatment that is easy to use directly in applications. The rationalization consists of the elimination of a practically inconsequential flexibility in the unravelling of substitutions that has the inadvertent side effect of losing contextual information in terms; the modified calculus now has a structure that naturally supports logical analyses, such as ones related to the assignment of types, over lambda terms. The overall calculus is shown to have pleasing theoretical properties such as a strongly terminating sub-calculus for substitution and confluence even in the presence of term meta variables that are accorded a grafting interpretation. Another contribution of the paper is the identification of a broad set of properties that are desirable for explicit substitution calculi to support and a classification of a variety of proposed systems based on these. The suspension calculus is used as a tool in this study. In particular, mappings are described between it and the other calculi towards understanding the characteristics of the latter.

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## 1. INTRODUCTION

This paper concerns the explicit treatment of substitution in the lambda calculus. It has a twofold purpose within this context. First, it simplifies and rationalizes a particular calculus known as the *suspension calculus* that provides such a treatment [Nadathur and Wilson 1998]. Second, using the resulting system as a basis, it attempts to explicate the nuances of and differences between an array of explicit substitution calculi that have been proposed in recent years.

The desire to treat substitution directly in the syntax and rewrite rules of the lambda calculus has had a variety of motivations. The suspension calculus was developed originally with the intention of supporting a higher-order view of syntax, now commonly referred to as *higher-order abstract syntax* [Pfenning and Elliott

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Authors' addresses:

A. Gacek, University of Minnesota, 4-192 EE/CS Building, 200 Union Street SE, Minneapolis, MN 55455, USA, Email: [andrew.gacek@gmail.com](mailto:andrew.gacek@gmail.com)

G. Nadathur, University of Minnesota, 4-192 EE/CS Building, 200 Union Street SE, Minneapolis, MN 55455, USA, Email: [gopalan@cs.umn.edu](mailto:gopalan@cs.umn.edu)

1988] or *lambda tree syntax* [Miller 2000]. Success has been encountered in this endeavour: amongst other applications, the notation has been employed in the reasoning system called *Bedwyr* [Baelde et al. 2007], in the abstract machine for  $\lambda$ Prolog [Nadathur and Mitchell 1999] and in the implementation of the *FLINT* typed intermediate language [Shao et al. 1998]. Despite its use in practical systems, the original suspension calculus manifests some deficiencies. One problem is the building in of excessive flexibility in the unravelling of substitutions that leads inadvertently to the loss of certain kinds of context information. This added flexibility does not really enhance the efficiency of reduction and has unpleasant side effects such as the loss of the ability to associate a typing calculus with lambda terms. Another problem relates to the encoding of the composition of substitutions. Although the notation includes such a capability, its treatment is complicated and has led to the description of a derived calculus [Nadathur 1999] that is the one usually employed in applications. A drawback with this derived calculus is that it does not possess the property of confluence when meta variables are added to the syntax under the so-called *grafting* interpretation<sup>1</sup>. At a practical level, this has the impact that new approaches to higher-order unification based on using graftable meta variables [Dowek et al. 2000] cannot be exploited relative to it.

One contribution of this paper is the redressing of this situation. In particular, it describes a modified treatment of substitution composition that is simultaneously natural, easy to use directly in implementations and consistent with contextual properties.

The last fifteen years has seen the description of a large number of explicit substitution calculi, often without a clear enunciation of the goals underlying their design. A consequence of this phenomenon is that it has been difficult to evaluate the different calculi or even to understand the distinctive characteristics of each. This paper contributes in a second way by bringing greater clarity to these matters. Specifically, it identifies three properties that appear important for explicit treatments of substitution to support. It then surveys some of the prominent calculi in this realm through this prism. The suspension calculus that is developed in the earlier sections serves as a tool in understanding the various other systems. Through this process, a better grasp is also obtained of the capabilities of this specific notation.

The rest of the paper is structured as follows. In the next section we describe the new version of the suspension calculus. Section 3 then elucidates its properties: we show here the strong normalizability and confluence of the sub-calculus for treating substitutions and the confluence of the overall calculus even in the presence of graftable meta variables. Section 4 discusses other treatments of explicit substitutions and contrasts these with the one developed here. Section 5 concludes the paper.

## 2. THE SUSPENSION CALCULUS

The modified version of the suspension calculus of Nadathur and Wilson [1998] that we present in this section does not sacrifice any of the computational properties of the original calculus that are essential to its use in implementations. Rather, it em-

<sup>1</sup>Although this has not been made explicit previously, the original suspension calculus is confluent even in the presence of graftable meta variables.

bodies a view of it that is easier to reason about and to relate to other approaches to explicit substitutions. In the first two subsections below, we outline the intuitions underlying the suspension calculus and then substantiate this discussion through a precise description of its syntax and reduction rules. We then discuss the relationship of the version of the calculus we present here with the original version and also describe variants of it arising from the introduction of meta variables under two different interpretations.

## 2.1 Motivating the Encoding of Substitutions

We are interested in enhancing the syntax of the lambda calculus with a new category of expressions that is capable of encoding terms together with substitutions that have yet to be carried out on them. The kinds of substitutions that we wish to treat are those that arise from beta contraction steps being applied to lambda terms. Towards understanding what needs to be encoded in this context, we may consider a term with the following structure:

$$(\dots((\lambda \dots(\lambda \dots((\lambda \dots t \dots) s_1) \dots) \dots) s_2) \dots)$$

We assume here a de Bruijn representation for lambda terms, i.e., names are not used with abstractions and bound variable occurrences are replaced by indices that count abstractions back up to the one binding them [Bruijn 1972]. We have elided much of the detail in the term shown and have, in fact, focussed only on the following aspects: there is a beta redex in it (whose “argument” part is  $s_2$ ) that is embedded possibly under abstractions and that itself contains at least another embedded beta redex. Contracting the two beta redexes shown should produce a term of the form

$$(\dots(\dots(\lambda \dots(\dots t' \dots) \dots) \dots) \dots)$$

where  $t'$  is obtained from  $t$  by substituting  $s_2$  and (a modified form of)  $s_1$  for appropriate variables and adjusting the indices for other bound variables to account for the disappearance of two enclosing abstractions. Our goal is to represent  $t'$  as  $t$  coupled with the substitutions that are to be performed on it.

Towards developing a suitable encoding, it is useful to factor the variable references within  $t$  into two groups: those that are bound by abstractions inside the first beta redex that is contracted and those that are bound by abstractions enclosing this redex. Let us refer to the number of abstractions enclosing a term in a particular context as its embedding level relative to that context. For example, if we assume that every abstraction within the outer beta redex in the term considered above has been explicitly shown, then the embedding level of  $t$  in this context is 3. Rewriting a beta redex eliminates abstractions and therefore changes embedding levels. Thus, if the two beta redexes of interest are both contracted, the embedding level of  $t$  becomes 1. We shall call the embedding levels at a term before and after beta contractions the *old* and *new* embedding levels respectively. Simply recording these with a term is enough for encoding the change that needs to be made to the indices for variables bound by the “outer” group of abstractions; in particular, these indices must be decreased by the difference between the old and the new embedding levels.

Substitutions for the other group of variable references, i.e., those bound by abstractions within the first beta redex contracted, can be recorded explicitly in an environment. To suggest a concrete syntax, the term  $t'$  in the example considered may be represented by the expression  $\llbracket t, ol, nl, e \rrbracket$  where  $ol$  and  $nl$  are the old and new embedding levels, respectively, and  $e$  is the environment. Note that the number of entries in the environment must coincide with the old embedding level. It is convenient also to maintain the environment as a list or sequence of elements whose order is reverse that of the embedding level of the abstraction they correspond to; amongst other things, this allowed for an easy augmentation of the environment in a top-down traversal of the term. Now, one component of the entry for an abstraction that is contracted should obviously be the argument part of the relevant beta redex. For an abstraction not eliminated by a contraction, there is no new term to be substituted, but we can still correctly record the index corresponding to the first free variable as a pseudo substitution for it. In both these cases, we have also to pay attention to the following fact: the term in the environment may be substituted into a new context that has a larger number of enclosing abstractions and hence de Bruijn indices for free variables within it may have to be modified. To encode this renumbering, it suffices to record the (new) embedding level at the relevant abstraction with the environment entry. The difference between this and the (new) embedding level at the point of substitution determines the amount by which the free variable indices inside the term being substituted have to be changed. Thus, each environment entry has the form  $(t, l)$  where  $t$  is a term and  $l$  is a positive number. We refer to the second component of each such entry as its index and we observe that the indices for successive environment entries must form a non-increasing sequence at least for the simple form of environments we are presently considering.

Once we have permitted terms encoding substitutions into our syntax, it is possible for such terms to appear one inside another. A particular instance of this phenomenon is when they appear in juxtaposition as in the term

$$\llbracket \llbracket t, ol_1, nl_1, e_1 \rrbracket, ol_2, nl_2, e_2 \rrbracket.$$

This term corresponds to separately performing two sets of substitutions into  $t$ . It is useful to have a means for combining these into one set of substitutions, i.e., for rewriting the indicated term into one of the form  $\llbracket t, ol', nl', e' \rrbracket$ . In determining the shape of the new term, it is useful to note that  $e_1$  and  $e_2$  represent substitutions for overlapping sequences of abstractions within which  $t$  is embedded. The generation of the original term can, in fact, be visualized as follows: First, a walk is made over  $ol_1$  abstractions immediately enclosing  $t$ , possibly eliminating some of them via beta contractions, recording substitutions for all of them in  $e_1$  and eventually leaving behind  $nl_1$  enclosing abstractions. Then a similar walk is made over  $ol_2$  abstractions immediately enclosing the term  $\llbracket t, ol_1, nl_1, e_1 \rrbracket$ , recording substitutions for each of them in  $e_2$  and leaving behind  $nl_2$  abstractions. Notice that the  $ol_2$  abstractions scanned in the second walk are coextensive with some final segment of the  $nl_1$  abstractions left behind after the first walk and includes additional abstractions if  $ol_2 > nl_1$ .

Based on the image just evoked, it is not difficult to see what  $ol'$  in the term representing the combined form for the substitutions should be: this form represents

a walk over  $ol_1$  enclosing abstractions in the case that  $ol_2 \leq nl_1$  and  $ol_1 + (ol_2 - nl_1)$  abstractions otherwise and  $ol'$  should be the appropriate one of these values. Similarly, the number of abstractions eventually left behind is  $nl_2$  or  $nl_2 + (nl_1 - ol_2)$  depending on whether or not  $nl_1 \leq ol_2$ , and this determines the value of  $nl'$ . With regard to the environment  $e'$ , this should be composed of the elements of  $e_1$  modified by the substitutions encoded in  $e_2$  followed by a final segment of  $e_2$  in the case that  $ol_2 > nl_1$ . The modification to be effected on the elements of  $e_1$  may be understood as follows. Suppose  $e_1$  has as an element the pair  $(s, l)$ . Then  $s$  is affected by only that part of  $e_2$  that comes after the first  $nl_1 - l$  entries in it. Further, the index of the corresponding entry in the composite environment would have to be increased from  $l$  by an amount equal to  $ol_2 - nl_1$  in the case that  $ol_2 > nl_1$ . From these observations, it is clear that the merged environment can be generated completely from the components  $e_1, nl_1, ol_2$  and  $e_2$ . We correspondingly choose to encode this environment by the expression  $\{\{e_1, nl_1, ol_2, e_2\}\}$ .

Our focus here has been on motivating the new syntactic forms in the suspension calculus. However, implicit in this discussion has been a “meaning” for these new expressions in the sense of a translation into an underlying de Bruijn term. This informal semantics will be made precise in the next section through a collection of rewrite rules that can be used to incrementally “calculate” the intended encodings.

## 2.2 The Syntax of Terms and the Rewriting System

We now describe precisely the collections of expressions that constitute terms and environments in the suspension calculus. We assume that the lambda terms to be treated contain constant symbols drawn from a predetermined set. Letting  $c$  represent such constants, the  $t$  and  $e$  expressions given by the following rules define a “pre-syntax” for our terms and environments:

$$\begin{aligned} t & ::= c \mid \#i \mid (t \ t) \mid (\lambda t) \mid \llbracket t, n, n, e \rrbracket \\ e & ::= nil \mid ((t, n) :: e) \mid \{\{e, n, n, e\}\} \end{aligned}$$

In these rules,  $n$  corresponds to the category of natural numbers and  $i$  represents positive integers. Terms of the form  $(t_1 \ t_2)$  and  $(\lambda t)$  are, as usual, referred to as applications and abstractions. A term of the form  $\#i$ , known as a de Bruijn index, represents a variable bound by the  $i$ th abstraction looking outward from the point of its occurrence. Expressions of the form  $\llbracket t, ol, nl, e \rrbracket$  are called *suspensions*; these constitute a genuine extension to the syntax of lambda terms. The operator  $::$  provides the means for forming lists in environments. We use the conventions that application is left associative, that  $::$  is right associative and that application binds more tightly than abstraction to often omit parentheses in the expressions we write. We shall sometimes need to suppress the distinction between terms and environments and at these times we shall refer to them collectively as suspension expressions or, more simply, as expressions.

The reason we think of the rules above as defining only the pre-syntax is that we expect suspension expressions to also satisfy certain well-formedness constraints. In order to enunciate these constraints precisely, we need to associate the notions of *length* and *level* with environments. We do this through the following definitions. The symbol  $-$  used in these definitions denotes the subtraction operation on natural numbers.

*Definition 2.1.* The length of an environment  $e$  is denoted by  $len(e)$  and is defined by recursion on its structure as follows:

- (1)  $len(nil) = 0$
- (2)  $len((t, l) :: e) = 1 + len(e)$
- (3)  $len(\{\{e_1, nl_1, ol_2, e_2\}\}) = len(e_1) + (len(e_2) \dot{-} nl_1)$

*Definition 2.2.* The level of an environment  $e$ , denoted by  $lev(e)$ , is also given by recursion as follows:

- (1)  $lev(nil) = 0$
- (2)  $lev((t, l) :: e) = l$
- (3)  $lev(\{\{e_1, nl_1, ol_2, e_2\}\}) = lev(e_2) + (nl_1 \dot{-} ol_2)$

The legitimacy requirements that complement the syntax rules is now explicated as follows:

*Definition 2.3.* A suspension expression is considered well-formed just in case the following conditions hold of all its subexpressions:

- (1) If it is of the form  $\llbracket t, ol, nl, e \rrbracket$  then  $len(e) = ol$  and  $lev(e) \leq nl$ .
- (2) If it is of the form  $(t, l) :: e$  then  $l \geq lev(e)$ .
- (3) If it is of the form  $\{\{e_1, nl_1, ol_2, e_2\}\}$  then  $lev(e_1) \leq nl_1$  and  $len(e_2) = ol_2$ .

We henceforth consider only well-formed suspension expressions. We shall also sometimes restrict our attention to environments which have the structure of a list of bindings. We identify this class of environments below.

*Definition 2.4.* A simple environment is one of the form

$$(t_0, l_0) :: (t_1, l_1) :: \dots :: (t_{n-1}, l_{n-1}) :: nil$$

where by an abuse of notation, we allow  $n$  to be 0, in which case the environment in question is  $nil$ . For  $0 \leq i < n$ , we write  $e[i]$  to denote the environment element  $(t_i, l_i)$  and  $e\{i\}$  to denote  $(t_i, l_i) :: \dots :: (t_{n-1}, l_{n-1}) :: nil$ , i.e., the environment obtained from  $e$  by removing its first  $i$  elements. We extend the last notation by letting  $e\{i\}$  denote  $nil$  in the case that  $i \geq len(e)$  for any simple environment  $e$ .

The rewrite system associated with suspension expressions comprises three kinds of rules: the beta contraction rule that generates substitutions, the *reading rules* that distribute them over term structure and the *merging rules* that allow for the combination of substitutions generated by different beta contractions into a composite one. These three categories correspond to the rules in Figure 1 labelled  $(\beta_s)$ , (r1)-(r6) and (m1)-(m6), respectively. The application of several of these rules depends on arithmetic calculations on embedding levels and indices. We have been careful in the formal presentation to identify such calculations through side conditions on the rules. However, in the sequel, we will often assimilate such arithmetic operations into the rewrite rule itself with the understanding that they are to be “interpreted.” Using this approach, rule (r6) may have been written instead as

$$\llbracket (\lambda t), ol, nl, e \rrbracket \rightarrow (\lambda \llbracket t, ol + 1, nl + 1, (\#1, nl + 1) :: e \rrbracket).$$

- ( $\beta_s$ )  $((\lambda t_1) t_2) \rightarrow \llbracket t_1, 1, 0, (t_2, 0) :: nil \rrbracket$ .
- (r1)  $\llbracket c, ol, nl, e \rrbracket \rightarrow c$ , provided  $c$  is a constant.
- (r2)  $\llbracket \#i, 0, nl, nil \rrbracket \rightarrow \#j$ , where  $j = i + nl$ .
- (r3)  $\llbracket \#1, ol, nl, (t, l) :: e \rrbracket \rightarrow \llbracket t, 0, nl', nil \rrbracket$ , where  $nl' = nl - l$ .
- (r4)  $\llbracket \#i, ol, nl, (t, l) :: e \rrbracket \rightarrow \llbracket \#i', ol', nl, e \rrbracket$ ,  
where  $i' = i - 1$  and  $ol' = ol - 1$ , provided  $i > 1$ .
- (r5)  $\llbracket (t_1 t_2), ol, nl, e \rrbracket \rightarrow (\llbracket t_1, ol, nl, e \rrbracket \llbracket t_2, ol, nl, e \rrbracket)$ .
- (r6)  $\llbracket (\lambda t), ol, nl, e \rrbracket \rightarrow (\lambda \llbracket t, ol', nl', (\#1, nl') :: e \rrbracket)$ ,  
where  $ol' = ol + 1$  and  $nl' = nl + 1$ .
- (m1)  $\llbracket \llbracket t, ol_1, nl_1, e_1 \rrbracket, ol_2, nl_2, e_2 \rrbracket \rightarrow \llbracket t, ol', nl', \{\{e_1, nl_1, ol_2, e_2\}\} \rrbracket$ ,  
where  $ol' = ol_1 + (ol_2 - nl_1)$  and  $nl' = nl_2 + (nl_1 - ol_2)$ .
- (m2)  $\{\{e_1, nl_1, 0, nil\}\} \rightarrow e_1$ .
- (m3)  $\{\{nil, 0, ol_2, e_2\}\} \rightarrow e_2$ .
- (m4)  $\{\{nil, nl_1, ol_2, (t, l) :: e_2\}\} \rightarrow \{\{nil, nl'_1, ol'_2, e_2\}\}$ ,  
where  $nl'_1 = nl_1 - 1$  and  $ol'_2 = ol_2 - 1$ , provided  $nl_1 \geq 1$ .
- (m5)  $\{\{(t, n) :: e_1, nl_1, ol_2, (s, l) :: e_2\}\} \rightarrow \{\{(t, n) :: e_1, nl'_1, ol'_2, e_2\}\}$ ,  
where  $nl'_1 = nl_1 - 1$  and  $ol'_2 = ol_2 - 1$ , provided  $nl_1 > n$ .
- (m6)  $\{\{(t, n) :: e_1, n, ol_2, (s, l) :: e_2\}\} \rightarrow (\llbracket t, ol_2, l, (s, l) :: e_2 \rrbracket, m) :: \{\{e_1, n, ol_2, (s, l) :: e_2\}\}$ ,  
where  $m = l + (n - ol_2)$ .

Fig. 1. Rewrite Rules for the Suspension Calculus

*Definition 2.5.* We say that a suspension expression  $r$  is related to  $s$  by a  $\beta_s$ -contraction step, a reading step or a merging step if it is the result of applying the ( $\beta_s$ ) rule, one of the rules (r1)-(r6) or one of the rules (m1)-(m6), respectively, at any relevant subexpression of  $s$ . We denote these relations by writing  $s \triangleright_{\beta_s} r$ ,  $s \triangleright_r r$  and  $s \triangleright_m r$ , respectively. The union of the relations  $\triangleright_r$  and  $\triangleright_m$  will be denoted by  $\triangleright_{rm}$ , that of  $\triangleright_r$  and  $\triangleright_{\beta_s}$  by  $\triangleright_{r\beta_s}$  and, finally, that of all three relations by  $\triangleright_{rm\beta_s}$ . If  $R$  corresponds to any of these relations, we shall write  $R^*$  to denote its reflexive and transitive closure.

The following theorem shows that these various relations are well-defined.

**THEOREM 2.6.** *The relations  $\triangleright_{\beta_s}$ ,  $\triangleright_r$  and  $\triangleright_m$ , and, hence, any combination of them, preserve well-formedness of suspension expressions.*

**PROOF.** A somewhat stronger property can be proved for the rewriting relations of interest: (i) they leave the length of an environment unchanged, (ii) they never increase the level of an environment, and (iii) they preserve well-formedness. These facts are established simultaneously by induction on the structure of suspension expressions. The base case is verified by considering in turn each rewrite rule in Figure 1. The argument is then completed by considering each possibility for the structure of an expression and using the induction hypothesis. The details are entirely straightforward and hence omitted.  $\square$

We illustrate the rewrite rules by considering their use on the term

$$((\lambda (\lambda \lambda \#1 \#2 \#3) t_2) t_3),$$

where  $t_2$  and  $t_3$  are arbitrary terms. We trace a  $\triangleright_{rm\beta_s}$ -rewrite sequence for this term below:

$$\begin{aligned} & ((\lambda (\lambda \lambda \#1 \#2 \#3) t_2) t_3) \\ & \triangleright_{\beta_s}^* \llbracket \llbracket \lambda \#1 \#2 \#3, 1, 0, (t_2, 0) :: nil \rrbracket, 1, 0, (t_3, 0) :: nil \rrbracket \\ & \triangleright_m \llbracket \lambda \#1 \#2 \#3, 2, 0, \{ (t_2, 0) :: nil, 0, 1, (t_3, 0) :: nil \} \rrbracket \\ & \triangleright_m \llbracket \lambda \#1 \#2 \#3, 2, 0, (\llbracket t_2, 1, 0, (t_3, 0) :: nil \rrbracket, 0) :: \{ nil, 0, 1, (t_3, 0) :: nil \} \rrbracket \\ & \triangleright_m \llbracket \lambda \#1 \#2 \#3, 2, 0, (\llbracket t_2, 1, 0, (t_3, 0) :: nil \rrbracket, 0) :: (t_3, 0) :: nil \rrbracket. \end{aligned}$$

The last expression in this sequence is a term that represents, roughly, the “suspended” simultaneous substitution of  $t_2$ , modified by the substitution of  $t_3$  for its first free variable, and of  $t_3$  for the first two free variables in  $(\lambda \#1 \#2 \#3)$ . This suspension has been produced by contracting the two beta redexes in the original term and then using the merging rules to combine the two separate substitutions that are so generated. The combined environment can now be moved inside the abstraction, distributed over the applications and partially “evaluated” using the reading rules to yield

$$(\lambda \#1 \llbracket \llbracket t_2, 1, 0, (t_3, 0) :: nil \rrbracket, 0, 1, nil \rrbracket \llbracket t_3, 0, 1, nil \rrbracket \rrbracket).$$

This term manifests a structure that may be thought of as a generalization of head-normal forms to suspension terms. By applying reading and merging rules in accordance with the structure of  $t_2$  and  $t_3$ , we may further transform it into a head-normal form in the conventional sense.

The terms in the de Bruijn style presentation of the lambda calculus are a subset of the terms in the suspension calculus. In particular, they are exactly the terms in the present notation that do not contain any suspensions. Given a rewrite relation  $R$ , we shall say, as usual, that an expression is in  $R$ -normal form if it cannot be further transformed by the rules defining  $R$ . It is easily seen then that a suspension term is in de Bruijn form just in case it is in  $\triangleright_{rm}$ -normal form. We would, of course, be interested in knowing if any given suspension expression can be transformed into a normal form of this kind. We answer this question in the affirmative in the next section and subsequently relate the rewrite relations defined here with the usual notion of beta reduction over de Bruijn terms.

### 2.3 Relationship to the Original Suspension Calculus

The suspension calculus as we have described it here deviates from the original presentation in [Nadathur and Wilson 1998] in a few different ways. One distinction arises from the use in the earlier version of the calculus of a special form for the environment item that results from percolating a substitution under an abstraction. These items are written as  $@n$  where  $n$  is a natural number. The rule (r6) correspondingly has the form

$$\llbracket (\lambda t), ol, nl, e \rrbracket \rightarrow (\lambda \llbracket t, ol + 1, nl + 1, @nl :: e \rrbracket)$$

in that setting. This form was introduced into the syntax and treated in special ways by the rewrite rules in anticipation of an implementation optimization. It is, however, inessential at a theoretical level. In particular, the behaviour of a dummy

environment element of the form  $@n$  can be completely circumscribed by replacing it with  $(\#1, n + 1)^2$ . We assume the impact of this observation below.

Suspension expressions in the present setting constitute a subset of the expressions in the original calculus at a pre-syntax level. However, the well-formedness condition when restricted to these expressions is different in the two contexts. The earlier condition has a form that is identical to the one in Definition 2.3 except that the requirement on the levels of environments is replaced by one on their *indices*, a notion that is defined below.

*Definition 2.7.* Given a natural number  $i$ , the  $i$ -th index of an environment  $e$  is denoted by  $ind_i(e)$  and is defined as follows:

- (1) If  $e$  is *nil* then  $ind_i(e) = 0$ .
- (2) If  $e$  is  $(t, k) :: e'$  then  $ind_i(e)$  is  $k$  if  $i = 0$  and  $ind_{i-1}(e')$  otherwise.
- (3) If  $e$  is  $\{\{e_1, nl, ol, e_2\}\}$ , let  $m = (nl \dot{-} ind_i(e_1))^3$  and  $l = len(e_1)$ . Then

$$ind_i(e) = \begin{cases} ind_m(e_2) + (nl \dot{-} ol) & \text{if } i < l \text{ and } len(e_2) > m \\ ind_i(e_1) & \text{if } i < l \text{ and } len(e_2) \leq m \\ ind_{(i-l+nl)}(e_2) & \text{if } i \geq l. \end{cases}$$

The index of an environment, denoted by  $ind(e)$ , is  $ind_0(e)$ .

Any given environment expression  $e$  is expected to be reducible to a simple one of the form  $(t_0, l_0) :: \dots :: (t_{n-1}, l_{n-1}) :: nil$ . The  $i$ -th index of  $e$  is then precisely  $l_i$  if  $i < n$  and 0 otherwise. The level of  $e$ , in contrast, only *estimates* the 0-th index when  $e$  is reduced to this simple form while retaining information that is needed for interpreting intermediate expressions in the rewriting process. Nevertheless, we can observe the following:

*LEMMA 2.8.* *The well-formed expressions of the suspension calculus as described in this paper are a subset of the well-formed ones of the original presentation.*

*PROOF.* We prove the following by induction on the structure of a suspension expression that is well-formed under the criterion in this paper: (a) the expression is also well-formed under the earlier criterion and (b) if the expression is an environment  $e$ , then  $lev(e) \geq ind(e)$  and if  $i > j$  then  $ind_i(e) \geq ind_j(e)$ . These properties must be shown simultaneously: the induction hypothesis pertaining to (b) is needed for establishing (a) and we need to know that the expression is well-formed in the earlier sense in order to establish (b). The details are straightforward once these observations are made and hence we omit them here. The lemma is an immediate consequence of property (a).  $\square$

The final difference between the two versions of the suspension calculus is in the treatment of the composition of two environments. In the earlier presentation, the outer environment is distributed eagerly over the elements of the inner one. This is done by a rule of the form

<sup>2</sup>It should be noted, though, that the parsimony of the latter form is complemented by the introduction of more (perhaps unnecessary) possibilities for rewriting that considerably complicate the proof of termination for the reading and merging rules.

<sup>3</sup>The  $\dot{-}$  here can be replaced by  $-$  for well-formed expressions.

$$\{\!\{ et :: e_1, nl, ol, e_2 \}\!\} \rightarrow \langle\langle et, nl, ol, e_2 \rangle\rangle :: \{\!\{ e_1, nl, ol, e_2 \}\!\},$$

where  $\langle\langle et, nl, ol, e_2 \rangle\rangle$  represents an augmentation to the syntax of environment items for encoding the effect of transforming  $et$  by the relevant substitutions in  $e_2$ . The older version of the calculus has rules relating to expressions of the form  $\langle\langle et, nl, ol, e_2 \rangle\rangle$  that facilitate the pruning of  $e_2$  down to a part that really affects  $et$  and the subsequent generation of a suspension that captures its influence on the term component. By contrast, the present rendition of the calculus calculates the effect of  $e_2$  on  $et :: e_1$  by first pruning  $e_2$  down to a relevant part based on  $et$  and only later distributing the refined environment to  $e_1$ .

It follows naturally from the observations made above that the rules (m2), (m5) and (m6) do not appear in the original rendition of the suspension calculus. However, based on the discussions already in [Nadathur and Wilson 1998], it can be seen that each of these rules is admissible to the earlier version in the sense that their left and right hand sides can be rewritten to a common form in that setting. We can, in fact, make the following observation, a detailed proof of which appears in [Gacek 2006b]:

LEMMA 2.9. *Let  $x_1$  and  $x_2$  be suspension expressions such that  $x_1 \triangleright_{rm}^* x_2$ . Assume further that  $x_2$  is in  $\triangleright_{rm}$ -normal form. Then  $x_1$  also rewrites to  $x_2$  by virtue of the reading and merging rules in [Nadathur and Wilson 1998].*

Our focus up to this point has been on arguing that the suspension calculus as described here is a subsystem of sorts of the original presentation. It is important, of course, to also address the issue of why such a “subsystem” is of interest. There are several reasons for this, all arising out of the modified treatment of substitution composition. First, this treatment is a considerably simplified one and can, as a consequence, be used directly in practical applications. Second, it rectifies a problem with the original calculus that prevented certain interesting logical analyses over terms from being formulated: it is, for instance possible to describe a type assignment system now for terms [Gacek 2006b], something that was difficult to do with the original suspension calculus. Finally, this change is crucial to our ability to describe formal correspondences of the suspension calculus with other explicit substitution calculi later in this paper.

While there may be justifications for the modified suspension calculus, there is also a question about its adequacy. It is evident that this version can still treat substitutions explicitly and that it possesses the important capability of composing such substitutions. In the next section we see also that properties such as confluence and the ability to simulate the usual notion of beta reduction over lambda terms are preserved, thus settling any concern over adequacy.

## 2.4 Permitting Meta Variables In Suspension Terms

The syntax of suspension expressions does not presently allow for instantiatable variables. Such variables, also referred to as *meta variables*, are often used within lambda terms in situations such as those of higher-order theorem proving and symbolic manipulation of higher-order objects. In the former context, these variables arise naturally in attempts to prove existential statements: such proofs involve choosing instantiations for existential quantifiers and meta variables provide

a means for delaying actual choices till there is enough information for determining what they should be. In the latter context, instantiatable variables are instrumental in realizing structure recognition capabilities relative to the use of higher-order abstract syntax based representations of constructs whose structures involve binding notions. For example, consider the first-order formula  $\forall x((p\ x) \vee (q\ x))$ . Using an abstraction to capture the binding content of the quantifier, this formula can be rendered into the lambda term  $(all\ \lambda(or\ (p\ \#1)\ (q\ \#1)))$ , where *all* and *or* are constants chosen to encode universal quantification and disjunction in formulas. Given such representations, the lambda term  $(all\ \lambda(or\ (P\ \#1)\ (Q\ \#1)))$  in which *P* and *Q* are meta variables serves as a pattern for recognizing formulas that at the top-level have the structure of a disjunction embedded within a universal quantifier.

An important question concerning meta variables is that of how substitutions for them are to be treated. The logically correct interpretation of these variables requires that such substitutions respect the notion of scope. Thus, if *X* is an instantiatable variable that has an occurrence within an abstraction context, the term that is substituted for it cannot contain a bound variable that is captured by the enclosing abstraction. This view is one that also supports rather useful pattern matching capabilities. To understand this, we might reconsider the “template” we have described above for first-order formulas. Suppose that we want to refine this so that the formulas recognized by it are such that the right subpart of the disjunction does not depend on the top-level quantifier. If a treatment of meta variables in accordance with logical principles is used, then the following modified template achieves this purpose:  $(all\ \lambda(or\ (P\ \#1)\ Q))$ . The critical facet that ensures this behaviour is that no structure that is substituted for *Q* can have a variable occurrence in it that is captured by the abstraction corresponding to the quantifier.

An alternative possibility to the logical view of instantiatable variables is to treat them as placeholders against which any well-formed term can be grafted. This kind of “grafting” interpretation has been found useful in conjunction with explicit substitution notations in, for instance, realizing a new approach to unification in the context of lambda terms [Dowek et al. 2000]. The well-known procedure due to Huet [1975] calculates unifiers incrementally and requires the construction of a complicated term, the contraction of beta redexes and the calculation of their substitution effects all for the sole purpose of percolating dependency information to places where they can be used in later computation steps. By allowing meta variables to be substituted for by terms with variable occurrences that can be captured by enclosing abstractions, the dependencies can be transmitted by a much simpler process. Of course, treating instantiatable variables in this “graftable” way seems contradictory to their logical interpretation and also appears to fly in the face of pattern matching applications. However, a reconciliation is possible: variables can be interpreted initially in a logical way but then surrounded in an explicit substitution context so that a subsequent grafting treatment does not violate the required logical constraints. Thus, consider again the term  $(all\ \lambda(or\ (P\ \#1)\ Q))$ . This term may be transformed into  $(all\ \lambda(or\ ([P', 0, 1, nil]\ \#1)\ [Q', 0, 1, nil]))$ . By identifying *P* and *Q* with the terms  $[P', 0, 1, nil]$  and  $[Q', 0, 1, nil]$ , we insulate substitutions for them from a dependence on the external abstraction even under

a grafting interpretation of  $P'$  and  $Q'$ .

Either of the discussed views of meta variables can be built into the suspension notation. Towards this end, we first modify the syntax for terms to the following:

$$t ::= v \mid c \mid \#i \mid (t \ t) \mid (\lambda t) \mid \llbracket t, n, n, e \rrbracket,$$

where  $v$  represents the category of instantiatable variables. If we interpret these variables in the logical way, then they cannot be affected by substitutions generated by  $\beta$ -contractions. To support this view, therefore, we add the following to our reading rules:

$$(r7) \quad \llbracket v, ol, nl, e \rrbracket \rightarrow v, \text{ if } v \text{ is a meta variable.}$$

If, on the other hand, the grafting interpretation is chosen, then this rule is not acceptable and the original rewriting system, in fact, remains unchanged.

The choice of interpretation impact on the properties of the calculus in different ways. Under the logical view, meta variables behave like constants in that they may be replaced only by closed terms; this fact is explicitly manifest in the similarity of rule (r7) to (r1). Thus, all the properties of the calculus that includes them are already manifest in the subsystem described in Section 2.2. The situation is more intricate under the grafting view. For example, consider the term  $((\lambda((\lambda X) t_1)) t_2)$  in which  $X$  is an instantiatable variable and  $t_1$  and  $t_2$  are terms in  $\triangleright_{rm}$ -normal form. This term can be rewritten to

$$\llbracket \llbracket X, 1, 0, (t_1, 0) :: nil \rrbracket, 1, 0, (t_2, 0) :: nil \rrbracket$$

and also to

$$\llbracket \llbracket X, 2, 1, (\#1, 1) :: (t_2, 0) :: nil \rrbracket, 1, 0, (\llbracket t_1, 1, 0, (t_2, 0) :: nil \rrbracket, 0) :: nil \rrbracket,$$

amongst other terms. It is easy to see that these terms cannot now be rewritten to a common form using only the reading and  $(\beta_s)$  rules. The merging rules are essential to this ability. As we see in Section 3, these also suffice for this purpose.

We assume henceforth that the suspension calculus includes meta variables and that these are implicitly accorded the grafting interpretation. For reasons already mentioned, it is easy to see that the properties we establish for the resulting calculus will hold also under the logical interpretation.

### 3. PROPERTIES OF THE SUSPENSION CALCULUS

We now consider the coherence of the suspension calculus. Suspensions and the associated reading and merging rules are intended mainly to provide control and variability over substitution relative to the lambda calculus. In keeping with the finite nature of the substitution process, we would expect the reduction relations defined by these rules to be always terminating. We show this to be the case in the first subsection. There are evidently choices to be made in the application of the reading and merging rules. Regardless of how these choices are made, it is important that we produce the same normal form. We show that this confluence property holds in the second subsection below. We then digress briefly to establish an interesting structural property of the suspension calculus which relates two different methods for encoding the renumbering of bound variables; this property is used in the next section in relating the suspension calculus to the  $\lambda\sigma$ -calculus. Finally, we prove

that confluence continues to hold when the  $(\beta_s)$  rule is added to the collection and that this full system is also capable of simulating beta reduction over de Bruijn terms.

### 3.1 Strong Normalizability for Substitution Reductions

There are two steps to our argument that any sequence of rewritings based on the reading and merging rules must terminate. First we identify a collection of first-order terms over which we define a well-founded ordering using a variant of recursive path orderings [Dershowitz 1982; Ferreira and Zantema 1995]. We then describe a translation from suspension expressions to this collection of terms that is such that each of the relevant rewrite rules produces a smaller term relative to the defined order. The desired conclusion follows from these facts.

The terms that are intended to capture the essence of suspension expressions vis-a-vis termination are constructed using the following (infinite) vocabulary: the 0-ary function symbol  $*$ , the unary function symbol  $lam$ , and the binary function symbols  $app$ ,  $cons$  and, for each positive number  $i$ ,  $s_i$ . We denote this collection of terms by  $\mathcal{T}$ . We assume the following partial ordering  $\sqsupset$  on the signature underlying  $\mathcal{T}$ :  $s_i \sqsupset s_j$  if  $i > j$  and, for every  $i$ ,  $s_i \sqsupset app$ ,  $s_i \sqsupset lam$ ,  $s_i \sqsupset cons$  and  $s_i \sqsupset *$ . This ordering is now extended to the collection of terms.

*Definition 3.1.* The relation  $\succ$  on  $\mathcal{T}$  is inductively defined by the following property: Let  $s = f(s_1, \dots, s_m)$  and  $t = g(t_1, \dots, t_n)$ ; both  $s$  and  $t$  may be  $*$ , i.e., the number of arguments for either term may be 0. Then  $s \succ t$  if

- (1)  $f = g$  (in which case  $n = m$ ),  $(s_1, \dots, s_m) \succ_{lex} (t_1, \dots, t_n)$ , and,  $s \succ t_i$  for all  $i$  such that  $1 \leq i \leq n$ , or
- (2)  $f \sqsupset g$  and  $s \succ t_i$  for all  $i$  such that  $1 \leq i \leq n$ , or
- (3)  $s_i = t$  or  $s_i \succ t$  for some  $i$  such that  $1 \leq i \leq m$ .

Here  $\succ_{lex}$  denotes the lexicographic ordering induced by  $\succ$ .

In the terminology of [Ferreira and Zantema 1995],  $\succ$  is an instance of a recursive path ordering based on  $\sqsupset$ . It is easily seen that  $\sqsupset$  is a well-founded ordering on the signature underlying  $\mathcal{T}$ . The results in [Ferreira and Zantema 1995] then imply the following:

LEMMA 3.2.  $\succ$  is a well-founded partial order on  $\mathcal{T}$ .

We now consider the translation from suspension expressions to  $\mathcal{T}$ . The critical part of this mapping is the treatment of expressions of the form  $\llbracket t, ol, nl, e \rrbracket$  and  $\{\{e_1, nl, ol, e_2\}\}$ . Our translation ignores the embedding level components of these expressions and transforms them into terms whose top-level function symbol is  $s_i$  where  $i$  is a coarse measure of the remaining substitution work. In estimating this effort in a sufficiently fine-grained way relative to an abstraction, it is necessary to take cognizance of the following fact: rule (r6) creates a “dummy” substitution for the bound variable that is then adjusted by generating a “renumbering” suspension using rule (r3). To account for this additional work, we define a family of measures that relativizes the complexity of an expression to the number of enclosing suspensions. In calculating this quantity it is important to observe that the substitution

via rule (r3) of a term in an environment results in it being embedded in an additional suspension. We quantify the maximum such “internal embedding” below and then use this in estimating the substitution effort. In these definitions,  $max$  is the function that picks the larger of its two integer arguments.

*Definition 3.3.* The measure  $\mu$  that estimates the internal embedding potential of a suspension expression is defined as follows:

- (1) For a term  $t$ ,  $\mu(t)$  is 0 if  $t$  is a constant, a meta variable or a de Bruijn index,  $\mu(s)$  if  $t$  is  $(\lambda s)$ ,  $max(\mu(s_1), \mu(s_2))$  if  $t$  is  $(s_1 s_2)$ , and  $\mu(s) + \mu(e) + 1$  if  $t$  is  $\llbracket s, ol, nl, e \rrbracket$ .
- (2) For an environment  $e$ ,  $\mu(e)$  is 0 if  $e$  is nil,  $max(\mu(s), \mu(e_1))$  if  $e$  is  $(s, l) :: e_1$  and  $\mu(e_1) + \mu(e_2) + 1$  if  $e$  is  $\{\{e_1, nl, ol, e_2\}\}$ .

*Definition 3.4.* The measures  $\eta_i$  on terms and environments for each natural number  $i$  are defined simultaneously by recursion as follows:

- (1) For a term  $t$ ,  $\eta_i(t)$  is 1 if  $t$  is a constant, a meta variable or a de Bruijn index,  $\eta_i(s) + 1$  if  $t$  is  $(\lambda s)$ ,  $max(\eta_i(s_1), \eta_i(s_2)) + 1$  if  $t$  is  $(s_1 s_2)$ , and  $\eta_{i+1}(s) + \eta_{i+1+\mu(s)}(e) + 1$  if  $t$  is  $\llbracket s, ol, nl, e \rrbracket$ .
- (2) For an environment  $e$ ,  $\eta_i(e)$  is 0 if  $e$  is nil,  $max(\eta_i(s), \eta_i(e_1))$  if  $e$  is  $(s, l) :: e_1$  and  $\eta_{i+1}(e_1) + \eta_{i+1+\mu(e_1)}(e_2) + 1$  if  $e$  is  $\{\{e_1, nl, ol, e_2\}\}$ .

The measure  $\eta_0$  is meaningfully used only relative to suspensions. In this context, it estimates, in a sense, the maximum effort along any one path in the substitution process rather than the cumulative effort.

*Definition 3.5.* The translation  $\mathcal{E}$  of suspension expressions to  $\mathcal{T}$  is defined as follows:

- (1) For a term  $t$ ,  $\mathcal{E}(t)$  is  $*$  if  $t$  is a constant a meta variable or a de Bruijn index,  $app(\mathcal{E}(t_1), \mathcal{E}(t_2))$  if  $t$  is  $(t_1 t_2)$ ,  $lam(\mathcal{E}(t'))$  if  $t$  is  $(\lambda t')$  and  $s_i(\mathcal{E}(t'), \mathcal{E}(e'))$  where  $i = \eta_0(t)$  if  $t$  is  $\llbracket t', ol, nl, e' \rrbracket$ .
- (2) For an environment  $e$ ,  $\mathcal{E}(e)$  is  $*$  if  $e$  is nil,  $cons(\mathcal{E}(t'), \mathcal{E}(e'))$  if  $e$  is  $(t', l) :: e'$  and  $s_i(\mathcal{E}(e_1), \mathcal{E}(e_2))$  where  $i = \eta_0(e)$  if  $e$  is  $\{\{e_1, nl, ol, e_2\}\}$ .

We are now in a position to prove the strong normalizability of the substitution reduction relations.

**THEOREM 3.6.** *Every rewriting sequence based on the reading and merging rules terminates.*

**PROOF.** A tedious but straightforward inspection of each of the reading and merging rules verifies the following: If  $l \rightarrow r$  is an instance of these rules, then  $\mathcal{E}(l) \succ \mathcal{E}(r)$ ,  $\mu(l) \geq \mu(r)$ , and, for every natural number  $i$ ,  $\eta_i(l) \geq \eta_i(r)$ . Definition 3.1 ensures that  $\succ$  is monotonic, i.e., if  $v$  results from  $u$  by the replacement of a subpart  $x$  by  $y$  such that  $x \succ y$ , then  $u \succ v$ . Further, it is easily seen that if  $x$  and  $y$  are both either terms or environments such that  $\mu(x) \geq \mu(y)$  and  $\eta_i(x) \geq \eta_i(y)$  for each natural number  $i$  and if  $v$  is obtained from  $u$  by substituting  $y$  for  $x$ , then  $\eta_i(u) \geq \eta_i(v)$  for each natural number  $i$ . From these observations it follows easily that if  $t_1 \triangleright_{rm} t_2$  then  $\mathcal{E}(t_1) \succ \mathcal{E}(t_2)$ . The theorem is now a consequence of Lemma 3.2.  $\square$

As an interesting side note, we observe that the termination proof presented here has been formally verified using the *Coq* proof assistant [Gacek 2006a].

### 3.2 Confluence for the Substitution Calculus

Theorem 3.6 assures us that every suspension expression has a  $\triangleright_{rm}$ -normal form. From observations in Section 2 it follows therefore that every suspension term can be reduced to a de Bruijn term and every environment can be rewritten to one in a simple form using the reading and merging rules. We now desire to show that these normal forms are unique for any given expression. This would immediately be the case if we have the property of confluence, i.e., if for any  $s$ ,  $u$  and  $v$  such that  $s \triangleright_{rm}^* u$  and  $s \triangleright_{rm}^* v$  we know that there must be a  $t$  such that  $u \triangleright_{rm}^* t$  and  $v \triangleright_{rm}^* t$ . A well-known result, proved, for instance, in [Huet 1980], is that confluence follows from a weaker property known as local confluence for a reduction relation that is terminating. In our context this translates to it being sufficient to show for any suspension expression  $s$  that if  $s \triangleright_{rm} u$  and  $s \triangleright_{rm} v$  then there must be an expression  $t$  such that  $u \triangleright_{rm}^* t$  and  $v \triangleright_{rm}^* t$ . The usual method for proving local confluence for a rewrite system is to consider the different interfering ways in which pair of rules can be applied to a given term and to show that a common term can be produced in each of these cases. We use this approach in proving local confluence for the reading and merging rules here. The most involved part of the argument concerns the interference of rule (m1) with itself. We discuss this situation first and then use our analysis in proving the main result.

**3.2.1 An associativity property for environment composition.** The expression  $\llbracket \llbracket \llbracket t, ol_1, nl_1, e_1 \rrbracket, ol_2, nl_2, e_2 \rrbracket, ol_3, nl_3, e_3 \rrbracket$  can be transformed into a form corresponding to the term  $t$  under a substitution represented by a single environment in two different ways by using rule (m1). The composite environments in the two cases are given by the expressions

$$\{\{e_1, nl_1, ol_2, e_2\}, nl_2 + (nl_1 \dot{-} ol_2), ol_3, e_3\}$$

and

$$\{e_1, nl_1, ol_2 + (ol_3 \dot{-} nl_2), \{e_2, nl_2, ol_3, e_3\}\}.$$

Conceptually, these environments correspond to first composing  $e_1$  and  $e_2$  and then composing the result with  $e_3$  or, alternatively, to composing  $e_1$  with the result of composing  $e_2$  with  $e_3$ . An important requirement for local confluence is that these two environments can be made to converge to a common form, i.e., environment composition must, in a sense, be associative. We show this to be the case here. The argument we provide is inductive on the structures of the three environments and has the following broad outline: Based on the specific context, we consider the simplification of one of the two environments by relevant reading and merging rules. We then show that the other expression can also be rewritten, possibly by using the same rules, either to the same expression as the first or to an expression that is amenable to the use of the induction hypothesis.

We begin by noting some properties of the reading and merging rules that are useful in filling out the details of the proof. The first of these relates to the second environment displayed above and has the following content: At some point in the

reduction of this expression, it becomes possible to apply the rules relevant to evaluating the composition of  $e_2$  and  $e_3$ . Applying these rules immediately does not limit the normal forms that can be produced. This observation is contained in the next two lemmas.

LEMMA 3.7. *Let  $A$  be the environment  $\{\{e_1, nl_1, ol_1, \{\{e_2, nl_2, ol_3, e_3\}\}\}\}$  where  $e_3$  is a simple environment and  $e_2$  is of the form  $(t_2, n_2) :: e'_2$ . Further, for any positive number  $i$  such that  $i \leq nl_2 - n_2$  and  $i \leq ol_3$ , let  $B$  be the environment*

$$\{\{e_1, nl_1, ol_1, \{\{e_2, nl_2 - i, ol_3 - i, e_3\{i\}\}\}\}\}.$$

*If  $A \triangleright_{rm}^* C$  for any simple environment  $C$  then also  $B \triangleright_{rm}^* C$ .*

PROOF. It suffices to verify the claim when  $i = 1$ ; an easy induction on  $i$  then extends the result to the cases where  $i > 1$ . For the case of  $i = 1$ , the argument is by induction on the length of the reduction sequence from  $A$  to  $C$  with the essential part being a consideration of the first rule used. The details are straightforward and hence omitted.  $\square$

LEMMA 3.8. *Let  $A$  be the environment  $\{\{e_1, nl_1, ol_1, \{\{e_2, nl_2, ol_3, e_3\}\}\}\}$  where  $e_2$  and  $e_3$  are environments of the form  $(t_2, nl_2) :: e'_2$  and  $(t_3, n_3) :: e'_3$ , respectively. Further, let  $B$  be the environment*

$$\{\{e_1, nl_1, ol_1, ([t_2, ol_3, n_3, e_3], n_3 + (nl_2 \dot{-} ol_3)) :: \{\{e'_2, nl_2, ol_3, e_3\}\}\}\}.$$

*If  $A \triangleright_{rm}^* C$  for any simple environment  $C$  then also  $B \triangleright_{rm}^* C$ .*

PROOF. The proof is again by induction on the length of the reduction sequence from  $A$  to  $C$ . The first rule in this sequence either produces  $B$ , in which case the lemma follows immediately, or it can be used on  $B$  (perhaps at more than one place) to produce a form that is amenable to the application of the induction hypothesis.  $\square$

In evaluating the composition of  $e_2$  and  $e_3$ , it may be the case that some part of  $e_3$  is inconsequential. The last observation that we need is that this part can be “pruned” immediately in calculating the composition of the combination of  $e_1$  and  $e_2$  with  $e_3$ . The following lemma is consequential in establishing this fact.

LEMMA 3.9. *Let  $A$  be the environment  $\{\{e_1, nl_1, ol_2, e_2\}\}$  where  $e_2$  is a simple environment.*

- (1) *If  $ol_2 \leq nl_1 - lev(e_1)$  then  $A$  reduces to any simple environment that  $e_1$  reduces to.*
- (2) *For any positive number  $i$  such that  $i \leq nl_1 - lev(e_1)$  and  $i \leq ol_2$ ,  $A$  reduces to any simple environment that  $\{\{e_1, nl_1 - i, ol_2 - i, e_2\{i\}\}\}$  reduces to.*

PROOF. Let  $e_1$  be reducible to the simple environment  $e'_1$ . Then we may transform  $A$  to the form  $\{\{e'_1, nl_1, ol_2, e_2\}\}$ . Recalling that the level of an environment is never increased by rewriting, we have that  $lev(e'_1) \leq lev(e_1)$ . From this it follows that  $A$  can be rewritten to  $e'_1$  using rules (m5) and (m2) if  $ol_2 \leq nl_1 - lev(e_1)$ . This establishes the first part of the lemma.

The second part is nontrivial only if  $nl_1 - lev(e_1)$  and  $ol_2$  are both nonzero. Suppose this to be the case and let  $B$  be  $\{\{e_1, nl_1 - 1, ol_2 - 1, e_2\{1\}\}\}$ . The desired

result follows by an induction on  $i$  if we can show that  $A$  can be rewritten to any simple environment that  $B$  reduces to. We do this by an induction on the length of the reduction sequence from  $B$  to the simple environment. This sequence must evidently be of length at least one. If a proper subpart of  $B$  is rewritten by the first rule in this sequence, then the same rule can be applied to  $A$  as well and the induction hypothesis easily yields the desired conclusion. If  $B$  is rewritten by one of the rules (m3)-(m6), then it must be the case that  $A \triangleright_{rm}^* B$  via either rule (m4) or (m5) from which the claim follows immediately. Finally, if  $B$  is rewritten using rule (m2), then  $ol_2 \leq nl_1 - lev(e_1)$ . The second part of the lemma is now a consequence of the first part.  $\square$

We now prove the associativity property for environment composition:

LEMMA 3.10. *Let  $A$  and  $B$  be environments of the form*

$$\{\{e_1, nl_1, ol_2, e_2\}, nl_2 + (nl_1 \dot{-} ol_2), ol_3, e_3\}$$

and

$$\{e_1, nl_1, ol_2 + (ol_3 \dot{-} nl_2), \{e_2, nl_2, ol_3, e_3\}\},$$

respectively. Then there is a simple environment  $C$  such that  $A \triangleright_{rm}^* C$  and  $B \triangleright_{rm}^* C$ .

PROOF. We assume that  $e_1, e_2$  and  $e_3$  are simple environments; if this is not the case at the outset, then we may rewrite them to such a form in both  $A$  and  $B$  before commencing the proof we provide. Our argument is now based on an induction on the structure of  $e_3$  with possibly further inductions on the structures of  $e_2$  and  $e_1$ .

*Base case for first induction.* When  $e_3$  is *nil*, the lemma is seen to be true by observing that both  $A$  and  $B$  rewrite to  $\{e_1, nl_1, ol_2, e_2\}$  by virtue of rule (m2).

*Inductive step for first induction.* Let  $e_3 = (t_3, n_3) :: e'_3$ . We now proceed by an induction on the structure of  $e_2$ .

*Base case for second induction.* When  $e_2$  is *nil*, it can be seen that, by virtue of rules (m2), (m3) and either (m4) or (m5),  $A$  and  $B$  reduce to  $\{e_1, nl_1, ol_3 - nl_2, e_3\{nl_2\}\}$  when  $ol_3 > nl_2$  and to  $e_1$  otherwise. The truth of the lemma follows immediately from this.

*Inductive step for second induction.* Let  $e_2 = (t_2, n_2) :: e'_2$ . We consider first the situation where  $nl_1 > lev(e_1)$ . Suppose further that  $ol_3 \leq (nl_2 - n_2)$ . Using rules (m5) and (m2), we see then that

$$B \triangleright_{rm}^* \{e_1, nl_1, ol_2, e_2\}.$$

We also note that  $ol_3 \leq (nl_2 + (nl_1 \dot{-} ol_2)) - lev(\{e_1, nl_1, ol_2, e_2\})$  in this case. Lemma 3.9 assures us now that  $A$  can be rewritten to any simple environment that  $\{e_1, nl_1, ol_2, e_2\}$  reduces to and thereby verifies the lemma in this case.

It is possible, of course, that  $ol_3 > (nl_2 - n_2)$ . Here we see that

$$B \triangleright_{rm}^* \{\{e_1, nl_1 - 1, ol_2 + (ol_3 \dot{-} nl_2) - 1, \\ \{e'_2, n_2, ol_3 - (nl_2 - n_2), e_3\{nl_2 - n_2\}\}\}\}.$$

using rules (m5) and (m6). Using rule (m5), we also have that

$$A \triangleright_{rm}^* \{\{\{e_1, nl_1 - 1, ol_2 - 1, e'_2\}, nl_2 + (nl_1 \dot{-} ol_2), ol_3, e_3\}\}.$$

Invoking the induction hypothesis, it follows that  $A$  and

$$\{\{e_1, nl_1 - 1, ol_2 + (ol_3 \dot{-} nl_2) - 1, \{\{e'_2, nl_2, ol_3, e_3\}\}\}\}$$

reduce to a common simple environment. By Lemma 3.7 it follows that  $B$  must also reduce to this environment.

The only remaining situation to consider, then, is that when  $nl_1 = lev(e_1)$ . For this case we need the last induction, that on the structure of  $e_1$ .

*Base case for final induction.* If  $e_1$  is *nil*, then  $nl_1$  must be 0. It follows easily that both  $A$  and  $B$  reduce to  $\{\{e_2, nl_2, ol_3, e_3\}\}$  and that the lemma must therefore be true.

*Inductive step for final induction.* Here  $e_1$  must be of the form  $(t_1, nl_1) :: e'_1$ . We dispense first with the situation where  $n_2 < nl_2$ . In this case, by rule (m5)

$$B \triangleright_{rm}^* \{\{e_1, nl_1, ol_2 + (ol_3 \dot{-} nl_2), \{\{e_2, nl_2 - 1, ol_3 - 1, e'_3\}\}\}\}.$$

By the induction hypothesis used relative to  $e'_3$ ,  $B$  and the expression

$$\{\{e_1, nl_1, ol_2, e_2\}\}, nl_2 + (nl_1 \dot{-} ol_2) - 1, ol_3 - 1, e'_3\}$$

must reduce to a common simple environment. By Lemma 3.9,  $A$  must also reduce to this environment.

Thus, it only remains for us to consider the situation in which  $n_2 = nl_2$ . In this case by using rule (m1) twice we may transform  $A$  to the expression  $A_h :: A_t$  where

$$A_h = (\llbracket [t_1, ol_2, n_2, e_2], ol_3, n_3, e_3 \rrbracket, n_3 + ((nl_2 + (nl_1 \dot{-} ol_2)) \dot{-} ol_3))$$

and

$$A_t = \{\{e'_1, nl_1, ol_2, e_2\}\}, nl_2 + (nl_1 \dot{-} ol_2), ol_3, e_3\}.$$

Similarly,  $B$  may be rewritten to the expression  $B_h :: B_t$  where

$$B_h = (\llbracket [t_1, ol_2 + (ol_3 \dot{-} nl_2), n_3 + (nl_2 \dot{-} ol_3), \\ \llbracket [t_2, ol_3, n_3, e_3], n_3 + (nl_2 \dot{-} ol_3) \rrbracket :: \{\{e'_2, nl_2, ol_3, e_3\}\}, \\ n_3 + (nl_2 \dot{-} ol_3) + (nl_1 \dot{-} (ol_2 + (ol_3 \dot{-} nl_2))) \rrbracket$$

and

$$B_t = \{\{e'_1, nl_1, ol_2 + (ol_3 \dot{-} nl_2), \\ \llbracket [t_2, ol_3, n_3, e_3], n_3 + (nl_2 \dot{-} ol_3) \rrbracket :: \{\{e'_2, nl_2, ol_3, e_3\}\}\}.$$

Now, using straightforward arithmetic identities, it can be seen that the “index” components of  $A_h$  and  $B_h$  are equal. Further, the term component of  $A_h$  can be rewritten to a form identical to the term component of  $B_h$  by using the rules (m1) and (m6). Finally, by virtue of the induction hypothesis, it follows that  $A_t$  and the expression

$$\{\{e'_1, nl_1, ol_2 + (ol_3 \dot{-} nl_2), \{\{e_2, nl_2, ol_3, e_3\}\}\}\}$$

reduce to a common simple environment. Lemma 3.8 allows us to conclude that  $B_t$  can also be rewritten to this expression. Putting all these observations together it is seen that  $A$  and  $B$  can be reduced to a common simple environment in this case as well.  $\square$

3.2.2 *Uniqueness of Substitution Normal Forms.* We can now show that  $\triangleright_{rm}$  is a locally confluent reduction relation.

LEMMA 3.11. *For any expressions  $s, u$  and  $v$  such that  $s \triangleright_{rm} u$  and  $s \triangleright_{rm} v$  there must be an expression  $t$  such that  $u \triangleright_{rm}^* t$  and  $v \triangleright_{rm}^* t$ .*

PROOF. We recall the method of proof from [Huet 1980]. An expression  $t$  constitutes a nontrivial overlap of the rules  $R_1$  and  $R_2$  at a subexpression  $s$  if (a)  $t$  is an instance of the lefthand side of  $R_1$ , (b)  $s$  is an instance of the lefthand side of  $R_2$  and also does not occur within the instantiation of a variable on the lefthand side of  $R_1$  when this is matched with  $t$  and (c) either  $s$  is distinct from  $t$  or  $R_1$  is distinct from  $R_2$ . Let  $r_1$  be the expression that results from rewriting  $t$  using  $R_1$  and let  $r_2$  result from  $t$  by rewriting  $s$  using  $R_2$ . Then the pair  $\langle r_1, r_2 \rangle$  is called the conflict pair corresponding to the overlap in question. Relative to these notions, the lemma can be proved by establishing the following simpler property: for every conflict pair corresponding to the reading and merging rules, it is the case that the two terms can be rewritten to a common form using these rules.

In completing this line of argument, the nontrivial overlaps that we have to consider are those between (m1) and each of the rules (r1)-(r6), between (m1) and itself and between (m2) and (m3). The last of these cases is easily dealt with: the two expressions constituting the conflict pair are identical, both being *nil*. The overlap between (m1) and itself occurs over a term of the form  $[[[t, ol_1, nl_1, e_1], ol_2, nl_2, e_2], ol_3, nl_3, e_3]$ . By using rule (m1) once more on each of the terms in the conflict pair, these can be rewritten to expressions of the form  $[t, ol', nl', e']$  and  $[t, ol'', nl'', e'']$ , respectively, whence we can see that  $ol' = ol''$  and  $nl' = nl''$  by simple arithmetic reasoning and that  $e'$  and  $e''$  reduce to a common form using Lemma 3.10. The overlaps between (m1) and the reading rules are also easily dealt with. For instance consider the case of (m1) and (r1). Using rule (r1), the two terms in the conflict pair can be rewritten to the same constant. The other cases are similar even if a bit more tedious.  $\square$

As observed already, the main result of this subsection follows directly from Lemma 3.11 and Theorem 3.6.

THEOREM 3.12. *The relation  $\triangleright_{rm}$  is confluent.*

The uniqueness of  $\triangleright_{rm}$ -normal forms is an immediate consequence of Theorem 3.12. In the sequel, a notation for referring to such forms will be useful.

DEFINITION 3.13. The notation  $|t|$  denotes the  $\triangleright_{rm}^*$ -normal form of a suspension expression  $t$ .

It is easily seen that the  $\triangleright_{rm}$ -normal form for a term that does not contain meta variables is a term that is devoid of suspensions, i.e., a de Bruijn term. A further observation is that if all the environments appearing in the original term are simple, then just the reading rules suffice in reducing it to the de Bruijn term that is its unique  $\triangleright_{rm}$ -normal form.

### 3.3 An Equivalence Property Relating to Renumbering Substitutions

An important role for the subcalculus for substitutions is that of realizing the renumbering of de Bruijn indices necessitated by beta contractions. One mechanism

$$\begin{array}{c}
\overline{t \sim t} \qquad \qquad \qquad \overline{e \sim e} \\
\\
\frac{t_1 \sim t'_1 \quad t_2 \sim t'_2}{t_1 t_2 \sim t'_1 t'_2} \qquad \qquad \frac{t \sim t' \quad e \sim e'}{(t, n) :: e \sim (t', n) :: e'} \\
\\
\frac{t \sim t'}{\lambda t \sim \lambda t'} \qquad \qquad \frac{e_1 \sim e'_1 \quad e_2 \sim e'_2}{\{\{e_1, nl_1, ol_2, e_2\} \sim \{e'_1, nl_1, ol_2, e'_2\}\}} \\
\\
\frac{t \sim t' \quad e \sim e'}{\llbracket t, ol, nl, e \rrbracket \sim \llbracket t', ol, nl, e' \rrbracket} \qquad \frac{t \sim t'}{(t, n) \sim (t', n)} \\
\\
\frac{t \sim t' \quad r \sim r' \quad e \sim e'}{\llbracket t, ol, nl, r \rrbracket, nl + k :: e \sim \llbracket t', ol, nl', r' \rrbracket, nl' + k :: e'}
\end{array}$$

Fig. 2. The similarity relation,  $\sim$ 

for controlling such renumbering is the new embedding level in a suspension, i.e., the value chosen for  $nl$  in an expression of the form  $\llbracket t, ol, nl, e \rrbracket$ . Looking at the reading rule (r3), we see that another component that determines renumbering is the index of an environment term, i.e., the value chosen for  $n$  in an item of the form  $(t, n)$  in an environment. Now, these different mechanisms appear in juxtaposition in an environment item of the form  $(\llbracket t, ol, nl, e \rrbracket, n)$ . We observe here that  $\triangleright_{rm}$ -normal forms are invariant under a coordinated readjustment of the renumbering burden between the two devices in such an expression.

The permitted reapportionment is expressed formally through the notion of similarity defined below.

*Definition 3.14.* The similarity relation between (well-formed) terms and environments, respectively, is denoted by  $\sim$  and is given by the rules in Figure 2.

The property of interest is then the following:

**THEOREM 3.15.** *If  $t$  and  $t'$  are terms such that  $t \sim t'$ , then  $|t| = |t'|$ . If  $e$  and  $e'$  are environments such that  $e \sim e'$ , then they rewrite by reading and merging rules to similar simple environments.*

**PROOF.** Only a sketch is provided here; a detailed proof may be found in [Gacek 2006b]. Using the translation function from Definition 3.5, we define the relation  $\gg$  on suspension expressions as follows:  $u \gg v$  just in case  $\mathcal{E}(u) \succ \mathcal{E}(v)$ . Obviously  $\gg$  is a well-founded partial order. It is also easily seen that  $u \gg v$  if either  $v$  is a sub-expression of  $u$  or  $u \triangleright_{rm} v$ .

The argument is now an inductive one based on the ordering induced by  $\gg$  on pairs of expressions. In filling out the details, when considering two expressions  $u$  and  $v$  such that  $u \sim v$ , the additional properties of  $\gg$  and the induction hypothesis allow us to assume that any similar subparts of  $u$  and  $v$  that are terms are identical and that are environments are simple. We then consider the different cases for the structures of  $u$  and  $v$  and the rewriting rules that are applicable to them. The only nontrivial case when  $u$  and  $v$  are terms arises when these are suspensions to which

rule (r3) is applicable and the environment parts of these terms are similar but not identical. In this case we have

$$\begin{aligned}
 u &= \llbracket \#1, ol, nl, (\llbracket t_r, ol_r, nl_r, r \rrbracket, nl_r + k) :: e \rrbracket \\
 &\triangleright_{(r3)} \llbracket \llbracket t_r, ol_r, nl_r, r \rrbracket, 0, nl - (nl_r + k), nil \rrbracket \\
 &\triangleright_{(m1)} \llbracket t_r, ol_r, nl - (nl_r + k) + nl_r, \{\{r, nl_r, 0, nil\}\} \rrbracket \\
 &\triangleright_{(m2)} \llbracket t_r, ol_r, nl - k, r \rrbracket
 \end{aligned}$$

$$\begin{aligned}
 v &= \llbracket \#1, ol, nl, (\llbracket t_r, ol_r, nl'_r, r' \rrbracket, nl'_r + k) :: e' \rrbracket \\
 &\triangleright_{(r3)} \llbracket \llbracket t_r, ol_r, nl'_r, r' \rrbracket, 0, nl - (nl'_r + k), nil \rrbracket \\
 &\triangleright_{(m1)} \llbracket t_r, ol_r, nl - (nl'_r + k) + nl'_r, \{\{r', nl'_r, 0, nil\}\} \rrbracket \\
 &\triangleright_{(m2)} \llbracket t_r, ol_r, nl - k, r' \rrbracket
 \end{aligned}$$

By assumption,  $r \sim r'$ . Since  $u \gg \llbracket t_r, ol_r, nl - k, r \rrbracket$  and  $v \gg \llbracket t_r, ol_r, nl - k, r' \rrbracket$ , the induction hypothesis yields the desired conclusion. For environments, the nontrivial cases arise when  $u$  and  $v$  are of a form to which the rules (m5) or (m6) apply. The argument here is similar albeit more tedious.  $\square$

Theorem 3.15 casts an interesting light on rule (m6) of the suspension calculus. This rule has the form

$$\begin{aligned}
 \{\{t, n\} :: e_1, n, ol_2, (s, l) :: e_2\} \rightarrow \\
 (\llbracket t, ol_2, l, (s, l) :: e_2 \rrbracket, m) :: \{\{e_1, n, ol_2, (s, l) :: e_2\}\}
 \end{aligned}$$

where  $m = l + (n \dot{-} ol_2)$ . The righthand side of the rule has an environment item in which both an index and a new embedding level is chosen. Observe that a value larger than  $l$  could also be used for the new embedding level so long as the index is correspondingly modified and it remains consistent with the context in which the replacement is performed. Intuitively, this would correspond to eagerly relativizing  $\llbracket t, ol_2, l, (s, l) :: e_2 \rrbracket$  to a context with a larger number of enclosing abstractions and taking cognizance of this in its subsequent substitution.

### 3.4 Confluence for the Full Calculus

Now we turn to the confluence of the system given by the rules in Figure 1 that includes the  $(\beta_s)$  rule in addition to the ones for interpreting substitutions. In establishing this property, we adopt the method used in [Curien et al. 1996] to demonstrate that the  $\lambda\sigma$ -calculus is confluent. The following lemma, proved in [Curien et al. 1996], is a critical part of the argument.

**LEMMA 3.16.** *Let  $\mathcal{R}$  and  $\mathcal{S}$  be two reduction relations defined on a set  $X$  with  $\mathcal{R}$  being confluent and strongly normalizing and  $\mathcal{S}$  satisfying the property that for every  $t, u$  and  $v$  such that  $t \mathcal{S} u$  and  $t \mathcal{S} v$  there is an  $s$  such that  $u \mathcal{S} s$  and  $v \mathcal{S} s$ . Further suppose that for every  $t, u$  and  $v$  such that  $t \mathcal{S} u$  and  $t \mathcal{R} v$  there is an  $s$  such that  $u \mathcal{R}^* s$  and  $v (\mathcal{R}^* \cup \mathcal{S} \cup \mathcal{R}^*) s$ . Then the relation  $\mathcal{R}^* \cup \mathcal{S} \cup \mathcal{R}^*$  is confluent.*

In applying this lemma, we shall utilize the parallelization of  $\triangleright_{\beta_s}$  that is defined below.

**Definition 3.17.** The relation  $\triangleright_{\beta_s \parallel}$  on suspension expressions is defined by the rules in Figure 3.

$$\begin{array}{c}
\overline{t \rightarrow t} \\
\\
\frac{t_1 \rightarrow t'_1 \quad t_2 \rightarrow t'_2}{t_1 \ t_2 \rightarrow t'_1 \ t'_2} \\
\\
\frac{t \rightarrow t'}{\lambda t \rightarrow \lambda t'} \\
\\
\frac{t \rightarrow t' \quad e \rightarrow e'}{\llbracket t, ol, nl, e \rrbracket \rightarrow \llbracket t', ol, nl, e' \rrbracket} \\
\\
\frac{t_1 \rightarrow t'_1 \quad t_2 \rightarrow t'_2}{(\lambda t_1) \ t_2 \rightarrow \llbracket t'_1, 1, 0, (t'_2, 0) :: nil \rrbracket}
\end{array}
\qquad
\begin{array}{c}
\overline{e \rightarrow e} \\
\\
\frac{t \rightarrow t' \quad e \rightarrow e'}{(t, l) :: e \rightarrow (t', l) :: e'} \\
\\
\frac{e_1 \rightarrow e'_1 \quad e_2 \rightarrow e'_2}{\{\{e_1, nl_1, ol_2, e_2\}\} \rightarrow \{\{e'_1, nl_1, ol_2, e'_2\}\}}
\end{array}$$

Fig. 3. Rules defining  $\triangleright_{\beta_s \parallel}$ 

THEOREM 3.18. *The relation  $\triangleright_{rm\beta_s}$  is confluent.*

PROOF. Let  $\mathcal{R}$  be  $\triangleright_{rm}$  and let  $\mathcal{S}$  be  $\triangleright_{\beta_s \parallel}$ . We observe then that

$$\triangleright_{rm\beta_s} \subseteq (\mathcal{R}^* \cup \mathcal{S} \cup \mathcal{R}^*) \subseteq \triangleright_{rm\beta_s}^*.$$

Thus  $(\mathcal{R}^* \cup \mathcal{S} \cup \mathcal{R}^*)^* = \triangleright_{rm\beta_s}^*$  and hence  $\triangleright_{rm\beta_s}$  would be confluent if  $(\mathcal{R}^* \cup \mathcal{S} \cup \mathcal{R}^*)$  is.

To establish the latter we use Lemma 3.16, interpreting  $\mathcal{R}$  and  $\mathcal{S}$  as per the nomenclature of the lemma. We have already seen that  $\triangleright_{rm}$  is both confluent and strongly normalizing. To show that if  $t \triangleright_{\beta_s \parallel} u$  and  $t \triangleright_{\beta_s \parallel} v$  then there is an  $s$  such that  $u \triangleright_{\beta_s \parallel} s$  and  $v \triangleright_{\beta_s \parallel} s$ , we argue by induction on the structure of  $t$  and by considering the rules by which  $u$  and  $v$  are obtained. The only non-trivial case is that when  $t$  is the term  $(\lambda t_1) t_2$ , one of  $u$  and  $v$  is  $\llbracket t'_1, 1, 0, (t'_2, 0) :: nil \rrbracket$  and the other is  $(\lambda t''_1) t''_2$  where  $t_1 \triangleright_{\beta_s \parallel} t'_1$ ,  $t_1 \triangleright_{\beta_s \parallel} t''_1$ ,  $t_2 \triangleright_{\beta_s \parallel} t'_2$  and  $t_2 \triangleright_{\beta_s \parallel} t''_2$ . By the induction hypothesis, there exists an  $s_1$  such that  $t'_1 \triangleright_{\beta_s \parallel} s_1$  and  $t''_1 \triangleright_{\beta_s \parallel} s_1$  and an  $s_2$  such that  $t'_2 \triangleright_{\beta_s \parallel} s_2$  and  $t''_2 \triangleright_{\beta_s \parallel} s_2$ . We then pick  $s$  as  $\llbracket s_1, 1, 0, (s_2, 0) :: nil \rrbracket$ ; obviously  $u \triangleright_{\beta_s \parallel} s$  and  $v \triangleright_{\beta_s \parallel} s$ .

It only remains for us to show that for any  $t$ ,  $u$  and  $v$  such that  $t \triangleright_{\beta_s \parallel} u$  and  $t \triangleright_{rm} v$  there is an  $s$  such that  $u \triangleright_{rm}^* s$  and  $v (\triangleright_{rm}^* \cup \triangleright_{\beta_s \parallel} \cup \triangleright_{rm}^*) s$ . We do this again by induction on the structure of  $t$ . The argument is straightforward in all cases except perhaps when  $t$  is  $\llbracket (\lambda t_1) t_2, ol, nl, e \rrbracket$ ,  $v$  is  $\llbracket \lambda t_1, ol, nl, e \rrbracket \llbracket t_2, ol, nl, e \rrbracket$  and  $u$  is  $\llbracket \llbracket t'_1, 1, 0, (t'_2, 0) :: nil \rrbracket, ol, nl, e' \rrbracket$  where  $t_1 \triangleright_{\beta_s \parallel} t'_1$ ,  $t_2 \triangleright_{\beta_s \parallel} t'_2$  and  $e \triangleright_{\beta_s \parallel} e'$ . However, if we pick  $s$  to be

$$\llbracket t'_1, ol + 1, nl, (\llbracket t'_2, ol, nl, e' \rrbracket, nl) :: e' \rrbracket$$

we can easily show that it satisfies the requirements, thus completing the argument even in this case.  $\square$

Theorem 3.18 strengthens the confluence result established for the original suspension calculus in [Nadathur and Wilson 1998] in that it shows that this property

holds even when meta variables are permitted in terms. Although we have only shown this property to hold for the refinement of the suspension calculus presented here, our argument can be easily adapted to the original version.

### 3.5 Simulation of Beta Reduction

A fundamental requirement of any explicit substitution calculus is that it should allow for the simulation of beta reduction in the usual  $\lambda$ -calculus. In framing this requirement properly for the suspension calculus, it is necessary, first of all, to restrict attention to the situation where meta variables do not appear in terms. In this setting, as observed already, the lambda calculus terms under the de Bruijn notation are exactly those suspension terms that are devoid of suspensions. Moreover, beta contraction, denoted by  $\triangleright_\beta$ , is defined as follows:

*Definition 3.19.* Let  $t$  be a de Bruijn term and let  $s_1, s_2, s_3, \dots$  represent an infinite sequence of de Bruijn terms. Then the result of simultaneously substituting  $s_i$  for the  $i$ -th free variable in  $t$  for  $i \geq 1$  is denoted by  $S(t; s_1, s_2, s_3, \dots)$  and is defined recursively as follows:

- (1)  $S(c; s_1, s_2, s_3, \dots) = c$ , for any constant  $c$ ,
- (2)  $S(\#i; s_1, s_2, s_3, \dots) = s_i$  for any variable reference  $\#i$ ,
- (3)  $S((t_1 t_2); s_1, s_2, s_3, \dots) = (S(t_1; s_1, s_2, s_3, \dots) S(t_2; s_1, s_2, s_3, \dots))$ , and
- (4)  $S((\lambda t); s_1, s_2, s_3, \dots) = (\lambda S(t; \#1, s'_1, s'_2, s'_3, \dots))$  where, for  $i \geq 1$ ,  
 $s'_i = S(s_i; \#2, \#3, \#4, \dots)$ .

Using this substitution operation, the  $\beta$ -contraction rule is given by the following

$$((\lambda t_1) t_2) \rightarrow S(t_1; t_2, \#1, \#2, \dots).$$

A de Bruijn term  $t$  is related via  $\beta$ -contraction to  $s$  if  $s$  results from  $t$  by the application of this rule at an appropriate subterm. We denote this relationship by  $\triangleright_\beta$ . Beta reduction is the reflexive and transitive closure of  $\triangleright_\beta$ .

One part of the relationship between the suspension and lambda calculi that may also be viewed as the soundness of the  $(\beta_s)$  rule is the following:

**THEOREM 3.20.** *Let  $t$  and  $s$  be suspension terms such that  $t \triangleright_{\beta_s} s$ . Then  $|t| \triangleright_{\beta}^* |s|$ .*

**PROOF.** This theorem is proved for the original suspension calculus in [Nadathur and Wilson 1998]. The result carries over to the version of the calculus presented here by virtue of Lemma 2.9.  $\square$

The ability of the suspension calculus to simulate beta reduction is a suitably stated converse to the above theorem.

**THEOREM 3.21.** *Let  $t$  and  $s$  be de Bruijn terms such that  $t \triangleright_{\beta}^* s$ . Then  $t \triangleright_{rm\beta_s}^* s$ .*

**PROOF.** It has been shown in [Nadathur and Wilson 1998] for the original formulation of the suspension calculus that if  $t \triangleright_{\beta}^* s$  then  $t \triangleright_{r\beta_s}^* s$ . This observation carries over to the present version since the rules defining  $\triangleright_{r\beta_s}$  have essentially been preserved. The theorem obviously follows from this.  $\square$

#### 4. COMPARISON WITH OTHER EXPLICIT SUBSTITUTION CALCULI

We now survey some of the other explicit treatments of substitutions that have been proposed and contrast them with the suspension calculus. We restrict our attention in this study to calculi that utilize the de Bruijn scheme for representing bound variables. A good approach to understanding such calculi is to characterize them based on properties that are desired of them over and above their ability to encode substitutions. These are three such properties in our understanding: the ability to compose reduction substitutions, confluence in a situation where graftable meta variables are included and the preservation of strong normalizability for terms in the underlying lambda calculus. The first of these properties is central to combining substitution walks in normalization. Without it, for instance, the reduction of the term  $(\lambda \lambda t_1) t_2 t_3$  would require two separate traversals to be made over  $t_1$  for the purpose of substituting  $t_2$  and  $t_3$  for the relevant bound variables in it. The second property is important in developing algorithms that exploit the grafting view of meta variables. For example, confluence in the presence of such variables is a central requirement in realizing a new approach to higher-order unification [Dowek et al. 2000]. The final property has both a theoretical and a practical significance. At a theoretical level, it measures the coherence of the calculus. Explicit treatments of substitution are obtained usually by adding a terminating set of rules for carrying out the substitutions generated by beta contractions. The non-preservation of strong normalizability should, in this setting, be read as an undesirable interference between different parts of the overall rewrite system. At a practical level, this signifies that caution must be exercised in designing normalization procedures.

Of these various properties, the one that appears to be most important in practice is the ability to combine reduction substitutions: studies show that it is central to the efficient implementation of reduction [Liang et al. 2004], and, as indicated in Section 2, it also appears to be a natural way to realize confluence in the presence of graftable meta variables. Unfortunately, the majority of the explicit substitution calculi seem not to include this facility. Particular calculi sacrifice other properties as well. The  $\lambda v$ -calculus [Benaissa et al. 1996] preserves strong normalizability but does not permit graftable meta variables. The  $\lambda s_e$ -calculus permits such variables and is confluent even with this addition [Kamareddine and Ríos 1997] but does not preserve strong normalizability [Guillaume 2000]. The  $\lambda \zeta$ -calculus [Muñoz 1996] possesses both properties but obtains confluence by effectively requiring beta redexes to be contracted in an innermost fashion. Amongst the systems that do not permit the combination of substitutions, the  $\lambda_{ws_o}$ -calculus alone preserves strong normalizability and realizes confluence in the presence of graftable meta variables without artificially limiting reduction strategies [David and Guillaume 2001].

The only systems that permit the combination of reduction substitutions are, to our knowledge, the  $\lambda \sigma$ -calculus [Abadi et al. 1991], the closely related  $\Lambda \text{CCL}$  calculus [Field 1990] and the suspension calculus. The first two calculi are practically identical and, for this reason, we restrict our discussion of them to only the  $\lambda \sigma$ -calculus. The suspension and the  $\lambda \sigma$ -calculus both admit graftable meta variables without losing confluence and they are similar in many other respects as

well<sup>4</sup>. However, they have two important differences. One of these relates to the manner in which they represent substitutions. The  $\lambda\sigma$ -calculus encodes these as independent entities that can be separated from the term that they act on. This is a pleasant property at a formal level but it also leads to inefficiencies in the treatment of the renumbering of bound variables that is necessary when a substitution is moved under an abstraction. The second difference concerns the treatment of bound variables. In the  $\lambda\sigma$ -calculus, these are encoded as environment transforming operators in contrast to their representation directly as de Bruijn indices in the suspension calculus. The former representation is parsimonious in that rules that serve to compose substitutions can also be used to interpret bound variables. However, there are also disadvantages to such parsimony. It appears more difficult, for example, to separate out rules based on purpose and, hence, to identify simpler, yet complete, subsystems as has been done for the suspension calculus [Nadathur 1999]. The ambiguity in function also appears to play a role in the non-preservation of strong normalizability in the  $\lambda\sigma$ -calculus [Mellies 1995]: although the status of this property for the suspension calculus is as yet undetermined, a more focussed treatment of substitution composition disallows the known counterexample for the  $\lambda\sigma$ -calculus to be reproduced within it.

In the rest of this section we use the suspension calculus as a means for understanding the different treatments of explicit substitutions in more detail. We also attempt to substantiate the qualitative comparisons that we have provided above. Our approach to doing this is to describe translations between the suspension calculus and the other calculi that illuminate their differing characteristics. None of the calculi that we consider treat constants in terms and, for the sake of consistency, we assume these are missing also in suspension terms. We also do not include meta variables initially since these are not present in all calculi, but we bring them into consideration later as relevant. We divide our discussion of the other calculi into two subsections depending on whether or not they possess an ability to combine substitutions. As we shall see below, the calculi that do not have a combining capability correspond substantially to the suspension calculus without the merging rules.

#### 4.1 Calculi Without Substitution Composition

We discuss three calculi under this rubric: the  $\lambda\nu$ -calculus [Benaissa et al. 1996], the  $\lambda s$ -calculus [Kamareddine and Rios 1995], and the  $\lambda s_e$ -calculus [Kamareddine and Ríos 1997]. Qualitatively, these calculi provide an increasing sequence of capabilities. When the de Bruijn representation is used for lambda terms, the indices of externally bound variables in a term have to be incremented when it is substituted under an abstraction. The  $\lambda\nu$ -calculus requires such renumbering to be carried out in separate walks for each abstraction that the term is substituted under. The  $\lambda s$ -calculus improves on this situation by permitting all the renumbering walks to be combined into one although such a walk is still kept distinct from walks that realize substitutions arising out of beta contractions. The  $\lambda s_e$ -calculus extends the

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<sup>4</sup>To be accurate in spirit as well as in detail this statement needs a qualification: as we discuss later in the section, the original rewrite system of the  $\lambda\sigma$ -calculus needs to be extended slightly to obtain confluence in the presence of graftable meta variables.

|          |   |            |   |
|----------|---|------------|---|
| (B)      | $(\lambda a) b \rightarrow a[b/]$                   | (VarShift) | $\underline{n}[\uparrow] \rightarrow \underline{n+1}$                 |
| (App)    | $(a b)[s] \rightarrow a[s] b[s]$                    | (FVarLift) | $\underline{1}[\uparrow(s)] \rightarrow \underline{1}$                |
| (Lambda) | $(\lambda a)[s] \rightarrow \lambda a[\uparrow(s)]$ | (RVarLift) | $\underline{n+1}[\uparrow(s)] \rightarrow \underline{n}[s][\uparrow]$ |
| (FVar)   | $\underline{1}[a/] \rightarrow a$                   |            |   |
| (RVar)   | $\underline{n+1}[a/] \rightarrow \underline{n}$     |            |   |

Fig. 4. Rewrite rules for the  $\lambda v$ -calculus

$\lambda s$ -calculus by permitting graftable meta variables.

4.1.1 *The  $\lambda v$ -calculus.* The syntax of this calculus comprises two categories: *terms*, corresponding to lambda terms possibly encoding explicit substitutions, and *substitutions*.

*Definition 4.1.* The terms, denoted by  $a$  and  $b$  and the substitutions, denoted by  $s$ , of the  $\lambda v$ -calculus are given by the following syntax rules:

$$\begin{aligned} a &::= \underline{n} \mid a b \mid \lambda a \mid a[s] \\ s &::= a/ \mid \uparrow(s) \mid \uparrow \end{aligned}$$

The collection of expressions described may be understood intuitively as follows. The expression  $\underline{n}$  represents the  $n^{\text{th}}$  de Bruijn index, analogously to  $\#n$  in the suspension calculus. The binary operator  $[_]$ , referred to as a *closure*, introduces explicit substitutions into terms. The expression  $a/$ , created using the operator  $/$  called *slash*, represents the substitution of  $a$  for the first de Bruijn index and a shifting down of all other de Bruijn indices. The substitution  $\uparrow(s)$ , which uses the operator  $\uparrow$  called *lift*, provides a device for pushing substitutions underneath abstractions. Finally, the expression  $\uparrow$ , called *shift*, represents the effect of increasing the de Bruijn indices corresponding to externally bound variables by one.

The interpretations of the various syntactic devices are made explicit by the rules in Figure 4 that define the  $\lambda v$ -calculus. The rule labelled (B) in this collection emulates beta contraction by generating an explicit substitution. The rest of the rules, that constitute the sub-calculus  $v$ , serve to propagate such substitutions over the structure of a lambda term and to eventually evaluate them at the bound variable occurrences.

In relating the suspension and the  $\lambda v$ -calculus it is natural to identify the syntactic categories of terms in the two settings and to think of environments in the former framework as corresponding to substitutions in the latter. There is, however, an important difference in the view of the latter two entities. Substitutions in the  $\lambda v$ -calculus are self-contained objects that carry all the information needed for understanding them in context. In contrast, the interpretation of an environment requires also an associated old and new embedding level in the suspension calculus. This intuition underlies the following translation of  $\lambda v$  to suspension expressions.

*Definition 4.2.* The mappings  $T$  from terms in the  $\lambda v$ -calculus to terms in the suspension calculus and  $E$  from substitutions in the  $\lambda v$ -calculus to triples consisting of two natural numbers and a suspension environment are defined by recursion as follows:

- (1) For a term  $t$ ,  $T(t)$  is  $\#n$  if  $t$  is  $\underline{n}$ ,  $(T(a) T(b))$  if  $t$  is  $(a b)$ ,  $\lambda T(a)$  if  $t$  is  $\lambda a$ , and  $\llbracket T(a), ol, nl, e \rrbracket$  if  $t$  is  $a[s]$  and  $E(s) = (ol, nl, e)$ .
- (2) For a substitution  $s$ ,  $E(s)$  is  $(1, 0, (T(a), 0) :: nil)$  if  $s$  is  $a/$ ,  $(0, 1, nil)$  if  $s$  is  $\uparrow$ , and  $(ol + 1, nl + 1, (\#1, nl + 1) :: e)$  if  $s$  is  $\uparrow(s')$  and  $E(s') = (ol, nl, e)$ .

It is easy to see that  $T(a)$  must be a well-formed suspension term for every term  $a$  in the  $\lambda\nu$ -calculus. The difference in representation of bound variables in the two calculi is clearly only a cosmetic one and we shall ignore it in the discussion that follows. It is obvious then that  $T$  is a translation that preserves de Bruijn terms. It can also be easily verified that  $T$  and  $E$  are one-to-one mappings. There are, however, *many* suspension terms that are not the images under  $T$  of any term in the  $\lambda\nu$ -calculus: the set of substitutions that can be encoded in the latter calculus is quite limited. There are, in fact, only two forms that substitutions can take:  $\uparrow(\dots \uparrow(a/) \dots)$ , corresponding to preserving the first few de Bruijn indices, substituting  $a$  (with appropriate renumbering) for the next one and decreasing the remaining indices by one, and  $\uparrow(\dots \uparrow(\uparrow) \dots)$ , corresponding to preserving the first few de Bruijn indices and then incrementing the remaining ones by one. Thus, the  $\lambda\nu$ -calculus cannot encode an expression such as  $\llbracket t, 0, 2, nil \rrbracket$ , where  $t$  is a de Bruijn term, directly. This expression can be represented indirectly by  $t[\uparrow][\uparrow]$  that has the suspension term  $\llbracket \llbracket t, 0, 1, nil \rrbracket, 0, 1, nil \rrbracket$  as its image. This encoding highlights a problem with the manner in which the  $\lambda\nu$ -calculus treats renumbering of de Bruijn indices: incrementing by  $n$  has to be realized through  $n$  separate walks that each increment by 1. A more drastic example of the limitations of the  $\lambda\nu$ -calculus is that it possesses no simple way to encode the suspension term  $\llbracket t, 1, 2, (s, 2) :: nil \rrbracket$  that corresponds to substituting  $s$  for the first de Bruijn index in  $t$  and incrementing all the remaining indices by two. Finally, we note that only simple environments appear in terms that are in the image of  $T$ . This is, of course, to be expected since the  $\lambda\nu$ -calculus does not support the ability to compose substitutions.

At the level of rewriting, we would expect the  $\lambda\nu$ -calculus to translate into the subcalculus of the suspension calculus that excludes the merging rules. This is true for the most part: it is easily seen that if  $l \rightarrow r$  is an instance of any rule in Figure 4 other than (FVar) and (RVarLift), then  $T(l) \rightarrow T(r)$  is an instance of either the  $(\beta_s)$  rule or one of the reading rules in Figure 1. For the (FVar) rule, we observe first that the  $\llbracket t, 0, 0, nil \rrbracket \rightarrow t$  is an admissible rule in the suspension calculus in the absence of graftable meta variables. Now, this fact can be used to build a special case of (r3) into the rewrite system:

$$(r3') \quad \llbracket \#1, ol, 0, (t, 0) :: e \rrbracket \rightarrow t$$

The (FVar) rule corresponds directly to (r3') under the translation we have described.

The situation for the (RVarLift) rule is more involved. Any term that matches its lefthand side translates into a suspension term of the form

$$\llbracket \#(n + 1), ol + 1, nl + 1, (\#1, nl + 1) :: e \rrbracket$$

where either  $e$  is  $nil$ , in which case  $ol$  is 0 and  $nl$  is 1, or  $e$  has a first element of the form  $(t, nl)$ . In the suspension calculus, rule (r4) allows this term to be rewritten to the form

$$\llbracket \#n, ol, nl + 1, e \rrbracket.$$

In the case that  $e$  is  $nil$ , this suspension corresponds to incrementing the indices for externally bound variables in a de Bruijn term, constituted here by  $\#n$ , by 2. If  $e$  is of the form  $(t, nl) :: e'$  on the other hand, then the suspension represents a situation in which one or more terms are to be substituted into a context that includes more enclosing abstractions than were present in the context of their origin. The  $\lambda\nu$ -calculus is capable of representing neither situation directly but can encode both indirectly via a term that translates to

$$\llbracket \llbracket \#n, ol, nl, e \rrbracket, 0, 1, nil \rrbracket.$$

This is, in fact, the translation of the righthand side of the (RVarLift) rule. This term can be reduced to  $\llbracket \#n, ol, nl + 1, e \rrbracket$  by using the merging rules but represents the introduction of an extra renumbering walk in the absence of these rules.

The above discussion casts light on the efficiency with which beta reduction can be realized using the two calculi considered here. Normal forms for suspension expressions involving only simple environments are identical whether or not the merging rules are utilized. From this it follows easily that the normal forms produced by the two systems must be identical.

**4.1.2 The  $\lambda s$ -calculus.** The  $\lambda s$ -calculus also distinguishes between beta contraction and renumbering substitutions. However, it differs from the  $\lambda\nu$ -calculus in that it possesses a more general mechanism for renumbering de Bruijn indices and also has a more concise way of recording which de Bruijn indices are actually affected by beta contraction and renumbering substitutions. These devices are manifest in the syntax of terms.

*Definition 4.3.* The terms of the  $\lambda s$ -calculus, denoted by  $a$  and  $b$ , are given by the rules

$$a ::= n \mid a b \mid \lambda a \mid a \sigma^i b \mid \varphi_k^i a$$

where  $n$  and  $i$  range over positive integers and  $k$  ranges over non-negative integers.

Towards understanding this syntax, we observe first that de Bruijn terms are represented in the  $\lambda s$ -calculus exactly as they are in the suspension calculus with the cosmetic difference that the  $n^{\text{th}}$  de Bruijn index is denoted directly by  $n$  rather than  $\#n$ . Beyond this, there are two additional kinds of expressions that serve to make substitutions explicit. A term of the form  $a \sigma^i b$ , called a *closure* and intended to capture a beta contraction substitution, represents the substitution of a suitably renumbered version of  $b$  for the  $i^{\text{th}}$  de Bruijn index in  $a$  and a shifting down by one of all de Bruijn indices greater than  $i$  in  $a$ . A term of the form  $\varphi_k^i a$ , called an *update* and included to treat renumbering, represents an increase by  $i - 1$  of all de Bruijn indices greater than  $k$ . The purpose of these new kinds of expressions becomes clear from the rewriting rules for the  $\lambda s$ -calculus that are presented in Figure 5. The  $\sigma$ -generation rule is the counterpart of beta contraction in this collection. The remaining rules, referred to collectively as the  $s$  rules, serve to calculate substitutions introduced into terms by applications of the  $\sigma$ -generation rule.

Closures and updates can be understood as special forms of suspensions. This relationship is made precise by the following definition.

$$\begin{array}{ll}
 \sigma\text{-generation} & (\lambda a) b \rightarrow a \sigma^1 b \\
 \sigma\text{-}\lambda\text{-transition} & (\lambda a) \sigma^i b \rightarrow \lambda (a \sigma^{i+1} b) \\
 \sigma\text{-app-transition} & (a_1 a_2) \sigma^i b \rightarrow (a_1 \sigma^i b) (a_2 \sigma^i b) \\
 \\ 
 \sigma\text{-destruction} & n \sigma^i b \rightarrow \begin{cases} n-1 & \text{if } n > i \\ \varphi_0^i b & \text{if } n = i \\ n & \text{if } n < i \end{cases} \\
 \varphi\text{-}\lambda\text{-transition} & \varphi_k^i (\lambda a) \rightarrow \lambda (\varphi_{k+1}^i a) \\
 \varphi\text{-app-transition} & \varphi_k^i (a_1 a_2) \rightarrow (\varphi_k^i a_1) (\varphi_k^i a_2) \\
 \varphi\text{-destruction} & \varphi_k^i n \rightarrow \begin{cases} n+i-1 & \text{if } n > k \\ n & \text{if } n \leq k \end{cases}
 \end{array}$$

 Fig. 5. Rewrite rules for the  $\lambda s$ -calculus

*Definition 4.4.* The translation  $T$  of terms in the  $\lambda s$ -calculus to suspension terms is defined by recursion as follows:

$$T(t) = \begin{cases} \#n & \text{if } t \text{ is } n \\ T(a) T(b) & \text{if } t = (a b) \\ \lambda T(a) & \text{if } t = \lambda a \\ \llbracket T(a), i, i-1, (\#1, i-1) :: \\ \quad (\#1, i-2) :: \dots :: (\#1, 1) :: (T(b), 0) :: nil \rrbracket & \text{if } t = a \sigma^i b \text{ and} \\ \llbracket T(a), k, k+i-1, (\#1, k+i-1) :: \\ \quad (\#1, k+i-2) :: \dots :: (\#1, i) :: nil \rrbracket & \text{if } t = \varphi_k^i a. \end{cases}$$

The image of the translation function  $T$  is, once again, evidently a subset of the well-formed suspension terms. At a rewriting level, the  $\lambda s$ -calculus is, in a sense, contained within that fragment of the suspension calculus that excludes the merging rules. Towards making this comment precise, we observe first that the following is a derived rule of this fragment of the suspension calculus, assuming that  $e$  is a simple environment:

$$\llbracket \#n, ol, nl, e \rrbracket = \begin{cases} \#(n - ol + nl) & \text{if } n > ol, \\ \#(nl - l + 1) & \text{if } n \leq ol \text{ and } e[n] = (\#1, l), \text{ and} \\ \llbracket t, 0, nl - l, nil \rrbracket & \text{otherwise, assuming } e[n] = (t, l). \end{cases}$$

In particular, this rule embodies a sequence of applications of the rules (r2)-(r4) from Figure 1. Now, if we augment the reading rules to also include this rule, then the following theorem is easily proved:

**THEOREM 4.5.** *If  $a$  and  $b$  are terms of the  $\lambda s$ -calculus such that  $a$  rewrites to  $b$  in one step using the rules in Figure 5, then  $T(a) \triangleright_{r, \beta_s} T(b)$ .*

Noting that de Bruijn terms are preserved under the translation, we see then that any normalization sequence in the  $\lambda s$ -calculus can be mimicked in a one-to-one fashion within this fragment of the suspension calculus.

The comments above indicate a correspondence at a theoretical level but they gloss over issues relevant to the practical implementation of reduction. First, as

|                                     |   |                       |
|-------------------------------------|---|-----------------------|
| $\sigma$ - $\sigma$ -transition     | $(a \sigma^i b) \sigma^j c \rightarrow (a \sigma^{j+1} c) \sigma^i (b \sigma^{j-i+1} c)$    | if $i \leq j$         |
| $\sigma$ - $\varphi$ -transition 1  | $(\varphi_k^i a) \sigma^j b \rightarrow \varphi_k^{i-1} a$                                  | if $k < j < k + i$    |
| $\sigma$ - $\varphi$ -transition 2  | $(\varphi_k^i a) \sigma^j b \rightarrow \varphi_k^i (a \sigma^{j-i+1} b)$                   | if $k + i \leq j$     |
| $\varphi$ - $\sigma$ -transition    | $\varphi_k^i (a \sigma^j b) \rightarrow (\varphi_{k+1}^i a) \sigma^j (\varphi_{k+1-j}^i b)$ | if $j \leq k + 1$     |
| $\varphi$ - $\varphi$ -transition 1 | $\varphi_k^i (\varphi_l^j a) \rightarrow \varphi_l^j (\varphi_{k+1-j}^i a)$                 | if $l + j \leq k$     |
| $\varphi$ - $\varphi$ -transition 2 | $\varphi_k^i (\varphi_l^j a) \rightarrow \varphi_l^{j+i-1} a$                               | if $l \leq k < l + j$ |

Fig. 6. Additional rewrite rules for the  $\lambda s_e$ -calculus

the translation function indicates, the  $\lambda s$ -calculus provides a rather succinct encoding for the substitutions that arise when only the reading and the  $\beta_s$  rules are used. Second, the  $s$  rules utilize this representation to realize substitution rather efficiently in this context; observe, in this regard, that the derived reading rule actually embodies a possibly costly “look-up” operation that is necessary relative to the more elaborate encoding of substitutions used in the suspension calculus. However, this efficiency has an associated cost: closures in the  $\lambda s$ -calculus represent exactly *one* beta contraction substitution and, consequently, multiple such substitutions must be effected in separate walks. By contrast, even simple environments in the suspension calculus have the flexibility for encoding multiple beta contraction and arbitrary renumbering substitutions. Moreover, the merging rules are not needed in their full generality to exploit this capability: simple to implement derived rules can be described for this purpose [Nadathur 1999]. It has been observed that the ability to combine substitutions that is supported by the more general encoding for them leads to significantly greater efficiency in realizing reduction in practice than does the concise encoding facilitated by treating restricted forms of substitutions [Liang et al. 2004].

**4.1.3 The  $\lambda s_e$ -calculus and permutations of substitutions.** The  $\lambda s$ -calculus and the  $\lambda v$ -calculus lack confluence in the presence of graftable meta variables. In the absence of substitution composition, the only way to regain confluence is to permit permutations of substitutions<sup>5</sup>. In the context of the  $\lambda s$ -calculus, such permutability should apply to both the closure and the update forms of explicit substitutions. The  $\lambda s_e$ -calculus adds the rules in Figure 6 to those already present in the  $\lambda s$ -calculus in support of such permutability. There must, of course, be some kind of directionality to the permitted substitution reorderings to ensure termination and the side conditions on the new rules are intended to realize this. To understand the use of these rules and also the restrictions on permutations, we may consider the term  $((\lambda ((\lambda X) t_1)) t_2)$ . Mimicking in the  $\lambda s_e$ -calculus the two reduction paths seen for this term in Section 2.4, we get the terms  $(X \sigma^1 t_1) \sigma^1 t_2$  and  $(X \sigma^2 t_2) \sigma^1 (t_1 \sigma^1 t_2)$ . Notice now that the  $\sigma$ - $\sigma$ -transition rule is applicable only to the first of these terms. Thus, intuitively, this rule permits the permutation only of substitutions arising from the contraction of outer beta redexes over those arising from contracting inner ones. The effect of carrying out this rearrangement

<sup>5</sup>We note here that permutation and composition of substitutions are distinct notions although they seem sometimes to have been confused in the literature, e.g., see [Cosmo et al. 2003].

is to make the substitutions have the same form in both terms, as is desired.

The  $\lambda s_e$ -calculus has been shown to have an adequate mix of permutation rules to ensure confluence in the presence of meta variables [Kamareddine and Ríos 1997]. From the discussion of the  $\sigma$ - $\sigma$ -transition rule it might appear that it also restricts these rules sufficiently to preserve strong normalizability. Unfortunately, this is not the case: it has been shown that interactions between closures and updatings can give rise to nontermination even when the starting point is a lambda term that can be simply typed [Guillaume 2000]. The  $\lambda_{ws}$ -calculus [David and Guillaume 2001] provides a remedy to this situation by extending the syntax of de Bruijn terms (and hence the normal forms produced by reduction) to include terms with numeric labels that represent yet-to-be-computed renumbering substitutions.

## 4.2 Calculi with Substitution Composition

As we have noted, the main exemplars of this variety of treatment of explicit substitutions are the  $\lambda\sigma$ - and the suspension calculi. We discuss their relationship below. In contrast to the earlier situations, it is now relevant to consider mappings between these calculi in both directions.

**4.2.1 The  $\lambda\sigma$ -calculus.** The  $\lambda\sigma$ -calculus, like the  $\lambda\nu$ -calculus that is derived from it, treats substitutions as independent entities that can be meaningfully separated from the terms they act upon. Thus, its syntax is determined by *terms* and *substitutions*.

*Definition 4.6.* The following syntax rules in which  $a$  and  $b$  denote terms and  $s$  and  $t$  denote substitutions define the syntax of the  $\lambda\sigma$ -calculus:

$$\begin{aligned} a & ::= 1 \mid a \ b \mid \lambda a \mid a[s] \\ s & ::= id \mid a \cdot s \mid s \circ t \mid \uparrow \end{aligned}$$

A term of the form  $a[s]$  is called a *closure* and represents the term  $a$  with the substitution  $s$  to be applied to it. The substitution  $id$  is the identity substitution. The substitution  $a \cdot s$  is called *cons* and represents a term  $a$  to be substituted for the first de Bruijn index along with a substitution  $s$  for the remaining indices. The substitution  $s \circ t$  represents the composition of the substitution  $s$  with the substitution  $t$ . Finally, the substitution  $\uparrow$  is called *shift* and is intended to capture the increasing by 1 of all the de Bruijn indices corresponding to the externally bound variables in the term it is applied to. A form of substitution that has special significance is  $\uparrow \circ (\uparrow \circ \dots (\uparrow \circ \uparrow) \dots)$ . Assuming  $n$  occurrences of  $\uparrow$  in the expression, such a substitution represents an  $n$ -fold increment to the de Bruijn indices of the externally bound variables in the term it operates on. The shorthand  $\uparrow^n$  is used for such an expression and the notation is further extended by allowing  $\uparrow^0$  to denote  $id$ .

The reference to de Bruijn indices in the previous paragraph is accurate in spirit but not in detail. The  $\lambda\sigma$ -calculus represents abstracted variables as environment transforming operators rather than as indices. Specifically, only the first abstracted variable is represented directly by the index 1: for  $n > 1$ , the  $n$ -th such variable is represented by  $1[\uparrow^{n-1}]$ . When such a term is subjected to a substitution, the *shift* operators will play a role in determining the appropriate term to replace it with, as the rules of the calculus will elucidate. It will become clear then that composition

|           |  |             |  |
|-----------|--|-------------|--|
| (Beta)    | $(\lambda a) b \rightarrow a[b \cdot id]$                          |             |  |
| (App)     | $(a b)[s] \rightarrow a[s] b[s]$                                   | (Map)       | $(a \cdot s) \circ t \rightarrow a[t] \cdot (s \circ t)$ |
| (Abs)     | $(\lambda a)[s] \rightarrow \lambda a[1 \cdot (s \circ \uparrow)]$ | (Ass)       | $(s \circ t) \circ u \rightarrow s \circ (t \circ u)$    |
| (VarId)   | $1[id] \rightarrow 1$  | (IdL)       | $id \circ s \rightarrow s$                               |
| (VarCons) | $1[a \cdot s] \rightarrow a$                                       | (ShiftId)   | $\uparrow \circ id \rightarrow \uparrow$                 |
| (Clos)    | $a[s][t] \rightarrow a[s \circ t]$                                 | (ShiftCons) | $\uparrow \circ (a \cdot s) \rightarrow s$               |

Fig. 7. Rewrite rules for the  $\lambda\sigma$ -calculus

of substitutions is essential in this calculus even to the proper interpretation of variables bound by abstractions.

The rules that define the  $\lambda\sigma$ -calculus are presented in Figure 7. In this collection, the (Beta) rule serves to simulate beta contraction. The remaining rules, that define the subsystem  $\sigma$ , are meant to propagate substitutions generated by the (Beta) rule. The  $\sigma$  rules in the left column compute the effect of substitutions on terms. The (Clos) rule may generate a composition of substitutions in this process that the rules in the right column are useful in unravelling. Given two terms or two substitutions  $u$  and  $v$ , we write  $u \triangleright_{\lambda\sigma} v$  or  $u \triangleright_{\sigma} v$  to denote the fact that  $v$  results by replacing an appropriate subpart of  $u$  using any of these rules or only one of the  $\sigma$  rules, respectively. The reflexive and transitive closure of these relations is, as usual, denoted by  $\triangleright_{\lambda\sigma}^*$  and  $\triangleright_{\sigma}^*$ .

It is useful to understand the manner in which the rules of the  $\lambda\sigma$ -calculus function in the task of normalizing terms as a prelude to contrasting it with the suspension calculus. Towards this end, consider the lambda term given by  $(\lambda \lambda ((\lambda \lambda \lambda \#3) \#2))$  in the suspension calculus. This term is encoded by

$$(\lambda \lambda ((\lambda \lambda \lambda 1[\uparrow^2]) 1[\uparrow]))$$

in the  $\lambda\sigma$ -calculus. Applying the (Beta) rule to the only redex in this term we get

$$(\lambda \lambda ((\lambda \lambda 1[\uparrow^2])[1[\uparrow] \cdot id])).$$

The substitution generated by beta contraction can now be moved inside the two abstractions using the (Abs) rule to get the term

$$(\lambda \lambda \lambda \lambda (1[\uparrow^2][1 \cdot (1 \cdot ((1[\uparrow] \cdot id) \circ \uparrow) \circ \uparrow)]).$$

The substitution  $(1 \cdot ((1[\uparrow] \cdot id) \circ \uparrow) \circ \uparrow)$  that appears in this expression depicts the iterated adjustment of substitutions as they are pushed under abstractions in  $\lambda\sigma$ -calculus; by contrast, the suspension calculus captures the needed renumbering simply by a global adjustment to the new embedding level. The next conceptual step in the reduction is that of “looking up” the binding for the variable represented by  $1[\uparrow^2]$  in the substitution. This step requires the possible use of (ShiftCons) to prune off an initial portion of the substitution and an eventual use of (VarId) to select the desired term. However, the encoding of abstracted variables necessitates the use of the rules (Clos), (Ass) and (Map) to prepare the situation for applying these rules. The term that results at the end of this process is  $(\lambda \lambda \lambda \lambda 1[(\uparrow \circ \uparrow) \circ \uparrow])$ . The (Ass) rule can now be used to transform the term under all the abstractions into the form  $1[\uparrow \circ (\uparrow \circ \uparrow)]$  that is recognizable as the encoding of a de Bruijn index.

4.2.2 *Translating suspension expressions into  $\lambda\sigma$ -expressions.* The non-trivial part of this mapping concerns the treatment of environments in the suspension calculus. Intuitively, these must correspond to substitutions in the  $\lambda\sigma$ -calculus. However, environments obtain a meaning only relative to the new embedding level of the suspension terms they appear in. Moreover, to be well-formed, this embedding level must be at least as large as the level of the environment itself. Once this constraint is satisfied, the example just considered suggests the right translation to a “standalone” substitution.

*Definition 4.7.* The mappings  $S$  from suspension terms to  $\lambda\sigma$ -terms and  $R$  from pairs constituted by a suspension environment  $e$  and a natural number  $i$  such that  $\text{lev}(e) \leq i$  to  $\lambda\sigma$ -substitutions are defined simultaneously by recursion as follows:

$$(1) \quad S(\#1) = 1, \quad S(\#(n+1)) = 1[\uparrow^n] \text{ if } n > 0, \quad S(a \ b) = (S(a) \ S(b)), \quad S(\lambda a) = \lambda S(a) \\ \text{and } S(\llbracket t, ol, nl, e \rrbracket) = S(t)[R(e, nl)].$$

$$(2) \quad R(e, i) = \begin{cases} \overbrace{(\dots((id \circ \uparrow) \circ \uparrow) \dots) \circ \uparrow}^{i \text{ occurrences of } \uparrow} & \text{if } e = nil \\ \overbrace{(\dots(((S(t) \cdot R(e', n)) \circ \uparrow) \circ \uparrow) \dots) \circ \uparrow}^{i-n \text{ occurrences of } \uparrow} & \text{if } e = (t, n) :: e' \text{ and} \\ R(e_1, nl_1) \circ R(e_2, i - (nl_1 \dot{-} ol_2)) & \text{if } e = \{\{e_1, nl_1, ol_2, e_2\}\}. \end{cases}$$

The constraint on the pairs that  $R$  applies to raises a question concerning the well-definedness of  $R$ , and hence also of  $S$ . However, the well-formedness requirement on suspension expressions in Definition 2.3 ensures that these must be well-defined. Another fact that is easy to verify is that these mappings are both one-to-one; the critical observation in this regard is that Definition 4.7 is constructed so that  $R(e, i)$  is not equal to  $\uparrow^j$  for any  $e, i$  and  $j$ . Finally, we observe a correspondence also at the level of the rewriting:

**THEOREM 4.8.** *Let  $u$  and  $v$  be suspension expressions such that  $u \triangleright_{rm} v$  ( $u \triangleright_{rm\beta_s} v$ ). If  $u$  and  $v$  are terms, then there exists a  $\lambda\sigma$ -term  $w$  such that  $S(u) \triangleright_{\sigma}^* w$  and  $S(v) \triangleright_{\sigma}^* w$  (respectively,  $S(u) \triangleright_{\lambda\sigma}^* w$  and  $S(v) \triangleright_{\lambda\sigma}^* w$ ). If  $u$  and  $v$  are environments, then for any  $i$  such that  $\text{lev}(u) \leq i$ , there is a  $\lambda\sigma$ -substitution  $w$  such that  $R(u, i) \triangleright_{\sigma}^* w$  and  $R(v, i) \triangleright_{\sigma}^* w$  (respectively  $R(u, i) \triangleright_{\lambda\sigma}^* w$  and  $R(v, i) \triangleright_{\lambda\sigma}^* w$ ).*

**PROOF.** Applications of the rules  $(\beta_s)$ , (r5), (r6), (m1) and (m3) on suspension expressions map directly onto applications of (Beta), (App), (Abs), (Clos) and (IdL), respectively, on their translations. Rule (r2) that corresponds to renumbering a de Bruijn index translates into a sequence of uses of the (Map) and (Ass) rules in accordance with the representation of abstracted variables in the  $\lambda\sigma$ -calculus. Rule (r3) is similar to the rule (VarCons). However, the translation of the lefthand side must be “prepared” for the use of (VarCons) by a sequence of applications of (Map) and a peculiarity of the translation of the righthand side may require (IdL) to be used on it to produce a common form. In a similar sense, the rules (r4), (m4) and (m5) correspond to a “compiled form” of (ShiftCons) and (m6) corresponds to a compiled form of (Map). Finally, rule (m2) is similar to the use of (Ass) in producing a normal form.  $\square$

4.2.3 *Translating  $\lambda\sigma$ -expressions into suspension expressions.* Going in the reverse direction needs a decision on the range of the mapping for  $\lambda\sigma$ -substitutions. Considering a term of the form  $a[s]$  indicates what this might be. Such a term should translate into a suspension of the form  $\llbracket t, ol, nl, e \rrbracket$  where the triple  $(ol, nl, e)$  is obtained by “interpreting”  $s$ . In the case when every composition in  $s$  has a shift as its right operand, this triple can be arrived at in a natural way:  $e$  should reflect the substitution terms in  $s$ ,  $ol$  should be the number of such terms and  $nl$ , which counts the number of enclosing abstractions, should correspond to the length of the longest sequence of compositions with shifts at the top level in  $s$ . The intuition underlying the encoding of general substitution composition in the suspension calculus now allows this translation to be extended to arbitrary  $\lambda\sigma$ -substitutions.

*Definition 4.9.* The mapping  $T$  from  $\lambda\sigma$ -terms to suspension terms and the mapping  $E$  from  $\lambda\sigma$ -substitutions to triples of an old embedding level, a new embedding level, and a suspension environment are defined simultaneously by recursion as follows:

- (1)  $T(1) = \#1$ ,  $T(a b) = (T(a) T(b))$ ,  $T(\lambda a) = \lambda T(a)$  and  $T(a[s])$  is  $\#(n + 1)$  if  $a$  is 1 and  $s$  is  $\uparrow^n$  for  $n \geq 0$  and is  $\llbracket T(a), ol, nl, e \rrbracket$  where  $E(s) = (ol, nl, e)$  otherwise.
- (2)  $E(id) = (0, 0, nil)$ ,  $E(\uparrow) = (0, 1, nil)$ ,  $E(a \cdot s) = (ol + 1, nl, (T(a), nl) :: e)$  where  $E(s) = (ol, nl, e)$ , and  $E(s_1 \circ s_2)$  is  $(ol_1, nl_1 + 1, e_1)$  if  $s_2$  is  $\uparrow$  and is  $(ol_1 + (ol_2 \dot{-} nl_1), nl_2 + (nl_1 \dot{-} ol_2), \{\{e_1, nl_1, ol_2, e_2\}\})$  otherwise, assuming that  $E(s_1) = (ol_1, nl_1, e_1)$  and  $E(s_2) = (ol_2, nl_2, e_2)$ .

It is easily seen that, for any term  $a$  of the  $\lambda\sigma$ -calculus,  $T(a)$  is a well-formed suspension term. The translation treats a term of the form  $1[\uparrow^n]$  as a special case, reflecting its interpretation as the encoding of an abstracted variable. If this case were not singled out, the translation would produce the term  $\llbracket \#1, 0, nl, nil \rrbracket$  instead. This term can be rewritten to  $\#(n + 1)$  by the rule (r2). A similar observation applies to the translation of  $s \circ \uparrow$ . This case is treated as a special one to account for the manner in which a substitution is moved under an abstraction in the  $\lambda\sigma$ -calculus. If this issue were to be ignored, this substitution would translate to  $(ol, nl + 1, \{\{e, nl, 0, nil\}\})$  instead of  $(ol, nl + 1, e)$ , assuming that  $E(s) = (ol, nl, e)$ . The environment component of the former triple rewrites to that of the latter by the rule (m2).

The following theorem, whose proof is trivial, is evidence of the naturalness of our translations:

**THEOREM 4.10.** *For every suspension term  $t$ ,  $T(S(t)) = t$ .*

In order to state a correspondence between the rewrite systems, we need to extend the reduction relations on suspension expressions to triples of the form  $(ol, nl, e)$  that are the targets of the mapping  $E$ . We do this in the obvious way: a triple  $(ol, nl, e)$  is related to  $(ol, nl, e')$  by a rewriting relation just in case  $e$  is related to  $e'$  by that relation.

**THEOREM 4.11.** *If  $a$  and  $b$  are  $\lambda\sigma$ -terms such that  $a \triangleright_\sigma b$  ( $a \triangleright_{\lambda\sigma} b$ ), then there is a suspension-term  $u$  such that  $T(a) \triangleright_{rm}^* u$  ( $T(a) \triangleright_{rm\beta_s}^* u$ ) and  $T(b) \triangleright_{rm}^* u$  ( $T(b) \triangleright_{rm\beta_s}^* u$ ). If  $s$  and  $t$  are  $\lambda\sigma$ -substitutions such that  $s \triangleright_\sigma t$  ( $s \triangleright_{\lambda\sigma} t$ ) then there exist environments*

$e_1$  and  $e_2$  such that  $E(s) \triangleright_{rm}^*(ol, nl, e_1)$  ( $E(s) \triangleright_{rm\beta_s}^*(ol, nl, e_1)$ ),  $E(t) \triangleright_{rm}^*(ol, nl, e_2)$  ( $E(t) \triangleright_{rm}^*(ol, nl, e_2)$ ) and  $e_1 \sim e_2$ .

PROOF. The argument is by induction on the structure of  $\lambda\sigma$ -expressions. Theorem 3.15 permits us to focus on the situation where rewriting takes place at the root of the expression. Also, the observations about the “redundancy” of the special cases in the definitions of  $T$  and  $E$  allow us to ignore them in the proof.

Now, we can observe a relationship between several of the rules in the  $\lambda\sigma$ -calculus and rules in the suspension calculus: (Beta) corresponds to  $(\beta_s)$ , (App) to (r5), (Abs) to (r6), (VarId) to (a special case of) (r2), (VarCons) to (r3), (Clos) to (m1), (IdL) to (m3), (ShiftId) to (m2) and (ShiftCons) to (m4). In some cases the correspondence is precise in that the translation of the lefthand side rewrites exactly to the translation of the righthand side by the indicated rule. However, in most cases, some “adjustments” using other reading and merging rules are needed before or after the specific rule application to account for the peculiarities of the different calculi.

The two rules that remain are (Map) and (Ass). The former corresponds to (m6) but the correspondence is not quite the same as with the other rules. Suppose  $(a \cdot s_1) \circ s_2$  rewrites to  $a[s_2] \cdot (s_1 \circ s_2)$  by this rule. Let  $T(a) = t$ ,  $E(s_1) = (ol_1, nl_1, e_1)$ , and  $E(s_2) = (ol_2, nl_2, e_2)$ . The index components of  $E((a \cdot s_1) \circ s_2)$  and  $E(a[s_2] \cdot (s_1 \circ s_2))$  are quickly seen to be identical. The environment components are  $\{(t, nl_1) :: e_1, nl_1, ol_2, e_2\}$  and  $(\llbracket t, ol_2, nl_2, e_2 \rrbracket, nl_2 + (nl_1 \dot{-} ol_2)) :: \{e_1, nl_1, ol_2, e_2\}$ , respectively. These are like the left and right sides of rule (m6) with two differences. First,  $e_2$  might not have the form  $(s, l) :: e'_2$  that is needed by rule (m6). This can be “fixed” by rewriting  $e_2$  at the outset to such a form<sup>6</sup>. The second difference is that the index of the first environment term on the right side uses  $nl_2$  where rule (m6) uses  $l$ . However, this is not a problem because the two environments are claimed only to be similar, not identical.

Finally, turning to (Ass), we see that there is no rule in the suspension calculus that “simulates” it. Rather, this rule corresponds to a meta property of the calculus that was proved in Lemma 3.10.  $\square$

**4.2.4 Meta Variables and Preservation of Strong Normalizability.** Our presentation of the  $\lambda\sigma$ -calculus is true to its original description in [Abadi et al. 1991]. This rewrite system is not confluent when the syntax of terms is extended to include graftable meta variables. However, straightforward additions to the rule set suffice to regain this property [Curien et al. 1996]; see also [Dowek et al. 2000] for a system closer in form to the one discussed in this paper.

The  $\lambda\sigma$ -calculus does not preserve strong normalizability as we have already noted, although the substitution subsystem  $\sigma$  is strongly normalizing. The crux of the problem is that the (Beta) rule and the substitution rules can interact with each other to get a substitution to scope over its own subcomponents. To see how this might happen, consider the following reduction sequence adapted from [Mellies 1995]:

$$((\lambda a') b') [((\lambda a) b) \cdot id]$$

<sup>6</sup>For completeness, the case where  $e_2$  reduces to  $nil$  must also be discussed. (Map) in this case is related to (r2) and the argument is easier.

$$\begin{aligned}
& \triangleright_{\sigma}^* (\lambda (a'[1 \cdot (((\lambda a) b) \cdot id) \circ \uparrow)]) b'[((\lambda a) b) \cdot id] \\
& \triangleright_{Beta} a'[1 \cdot (((\lambda a) b) \cdot id) \circ \uparrow] [b'[((\lambda a) b) \cdot id] \cdot id] \\
& \triangleright_{\sigma}^* a'[b'[((\lambda a) b) \cdot id] \cdot (((\lambda a) b) \cdot id) \circ (\uparrow \circ (b'[((\lambda a) b) \cdot id] \cdot id))]
\end{aligned}$$

The substitution  $(\uparrow \circ (b'[((\lambda a) b) \cdot id] \cdot id))$  that appears as a subexpression of the last term in this sequence would be rewritten to  $id$  in a sensible progression to a normal form. However, it can also perversely be distributed over the preceding substitution using (Map) to produce the substitution subexpression

$$((\lambda a) b)[\uparrow \circ (b'[((\lambda a) b) \cdot id] \cdot id)] \cdot (id \circ (\uparrow \circ (b'[((\lambda a) b) \cdot id] \cdot id))).$$

Observe here that  $[(\lambda a) b] \cdot id$  has become a subpart of a substitution that stands over the term  $(\lambda a) b$  that originates from itself.

The preservation of strong normalizability is still an unsettled question with regard to the suspension calculus. However, Mellies' counterexample does not apply to this calculus because the kind of problem situation depicted above cannot be created within it. In particular, rule (m6) that corresponds to (Map) in the suspension calculus ensures that only relevant portions of an external environment are distributed over substitution terms.

## 5. CONCLUSION

This paper has presented a simplified and rationalized version of the suspension calculus. The new notation has several pleasing theoretical and practical properties some of which have been manifest here. This version also differs from the original presentation in that it preserves contextual information. This characteristic has been central to our ability to describe translations to the  $\lambda\sigma$ -calculus and has also been exploited elsewhere in defining a system for type assignment [Gacek 2006b]. This paper has also surveyed the world of explicit substitution calculi. It has attempted to do this in a top-down fashion, first elucidating properties that are important for such calculi to possess and then using these to categorize and to explain the motivations for the different proposed systems. In the process we have also distilled a better understanding of the capabilities of the suspension calculus.

This work can be extended in several ways. We mention two that we think are especially important. First, like the  $\lambda\sigma$ -calculus, the notation we have described here provides the basis for incorporating new treatments of higher-order unification that exploit graftable meta variables into practical systems. It is of interest to actually explicate such a treatment and to evaluate its benefits empirically. Second, the question of preservation of strong normalizability is still an open one for this calculus. This issue appears to be a non-trivial one to settle and an answer to it is likely to provide significant insights into the structure of the suspension calculus.

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